Table des matières du tome LXXIX, fascicule 2

Les FUNDAMENTA MATHEMATICAE publient, en langues des congrès internationaux, des travaux consacrés à la *Théorie des Ensembles*, *Topologie*, *Fondements de Mathématiques*, *Fonctions Réelles*, *Algèbre Abstraite*
Chaque volume paraît en 3 fascicules

Adresse de la Rédaction et de l'Échange:
FUNDAMENTA MATHEMATICAE, Śniadeckich 8, 00-950 Warszawa 1 (Pologne)

Tous les volumes sont à obtenir par l'intermédiaire de
ARS POLONA–RUCH, Krakowskie Przedmieście 7, 00-068 Warszawa 1 (Pologne)

DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

S-torsion free modules

by

M. A. Rauf Qureshi (Karachi)

Abstract. Let R be a ring with unity element, satisfying left Ore condition on a multiplicatively closed subset S of R. It is proved that there exists a fractional module (ψ_A, A') for an S-torsion free left R-module A, and, if S is contained in the set Q of all right regular elements of R, it follows that (ψ_R, R') exists. Imposing on S the conditions: (1) $S \subseteq Q$, (2) $\psi_R(R)$, S-divisible; it is shown that an S-module (ψ_A, A') exists uniquely for every left R-module A, and that A' is an exact covariant functor of A.

1. Introduction. Throughout this paper we shall suppose that R is a ring with $1 \neq 0$, satisfying the *left Ore property*

$$Rs \cap Sr \neq \emptyset \quad (r \in R,\ s \in S),$$

where S is a subsemigroup of R under multiplication and without zero. Also by a module we shall mean a left R-module.

If S satisfies the left semi-regular condition (i.e., for $r \in R$ and $s \in S$, $rs = 0$ implies that $\sigma r = 0$ for some $\sigma \in S$), then in [4] it was shown that an S-module (ψ_A, A') of a module A (see Sec. 3) exists, which is unique to within an isomorphism. In particular, the S-module (ψ_R, R') of R turned out to be the ring of fractions with denomenators in S, defined by Gabriel in [2] and [3].

The set

$$Q = \{\sigma \in R \mid r \in R,\ \sigma r = 0 \text{ implies } r = 0\}$$

of right regular elements of R is a semigroup, called the *right semigroup of* R. If S is a subsemigroup of Q, then it does not necessarily satisfy the left semi-regular condition, and consequently an S-module of A may not exist in general. However, we shall show that a fractional module of every S-torsion free module A with respect to S exists (see Sec. 3). This, in particular, gives a fractional module (ψ_R, R') of R with respect to S. Assuming that (ψ_R, R') is an S-module of R, we shall show that R' is flat as right R-module, and an S-module (ψ_A, A') exists for every module A, and that A' is isomorphic to $R' \otimes A$ $(\otimes = \otimes_R)$.

2. Preliminaries. Let A be a module. An element $a \in A$ is called *S-torsion free*, if $sa \neq 0$ for any $s \in S$, and a is said to be an *S-torsion element*, if it is not S-torsion free. If for $s \in S$ there exists an element

$b \in A$ such that $a = sb$, we say that a is *S-divisible*. The definitions of S-torsion free, S-torsion and S-divisible modules are clear. We shall write $T(A)$ for the set of all S-torsion elements of A.

It is clear that a *quotient of an S-divisible module is S-divisible*, and using Ore property of R it follows that $T(A)$ *is a submodule of* A, and $A/T(A)$ *is S-torsion free.*

A is called *S-injective*, if for any left ideal I of R with $I \cap S \neq \emptyset$ and for any homomorphism $f\colon I \to A$ there exists $a \in A$ such that $f(u) = ua$ for all $u \in I \cap S$. In view of Ore property it can easily be seen that *every S-torsion free and S-divisible module is S-injective.*

2.1. Proposition. *Let A be an S-divisible module, and*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

an exact sequence of modules. Then the sequence

$$0 \to A/T(A) \xrightarrow{\bar f} B/T(B) \xrightarrow{\bar g} C/T(C) \to 0$$

is exact, where $\bar f$ and $\bar g$ are defined in obvious way.

Proof. The fact that $B/T(B)$ and $C/T(C)$ are S-torsion free implies that $\bar f$ and $\bar g$ are well-defined. Clearly $\bar f$ and $\bar g$ are respectively monomorphism and epimorphism, and $\mathrm{Im}\bar f \subseteq \ker\bar g$.

If $\bar b \in \ker\bar g$, then there exist $s \in S$, $\bar a \in A$ such that $sb = f(a)$, and the S-divisibility of A implies that $a = sa'$ for some $a' \in A$. Hence $s\{b - f(a')\} = 0$, which gives $\bar b = \bar f(\bar a')$, i.e., $\ker\bar g \subseteq \mathrm{Im}\bar f$.

3. S-torsion free modules. A *fractional module of a module A with respect to S* is a pair (ψ, A') such that

(1) A' is S-torsion free and S-injective module,

(2) $\psi\colon A \to A'$ is an R-homomorphism, with $\ker\psi = T(A)$,

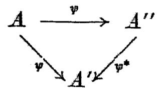
(3) for every $x \in A'$ there exists $s \in S$ such that $sx \in \psi(A)$.

If the pair (ψ, A') satisfies the conditions (2) and (3), with A' S-torsion free and S-divisible, we say that (ψ, A') is an *S-module of A*. Thus, in view of our earlier remark, *every S-module of A is also a fractional module of A with respect to S*. Two S-modules (ψ, A') and (φ, B) of A are called *isomorphic*, if there exists an isomorphism $\theta\colon A' \to B$ such that $\theta\psi = \varphi$.

3.1. Proposition. *If (ψ, A') satisfies* (1) *and* (2), *with A' S-divisible, then* (3) *is equivalent to the universal property: for every homomorphism $f\colon A \to B$, where B is S-torsion free and S-divisible, there exists a unique homomorphism $f^*\colon A' \to B$ such that $f^*\psi = f$.*

Proof. (i) Suppose that (3) holds. If $x \in A'$, then $sx = \psi(a)$ and $sb = f(a)$ for some $s \in S$, $a \in A$, $b \in B$. Define f^* by $f^*(x) = b$. Using Ore property and the condition $\ker\psi = T(A)$, it can be seen that f^* is well-defined homomorphism with $f = f^*\psi$. Also by (3) the uniqueness of f^* can be checked.

(ii) Let the universal property hold for the pair (ψ, A'). Then in view of Ore property $A'' = \{x \in A' \mid sx \in \psi(A) \text{ for some } s \in S\}$ is a submodule of A', and by (i) the diagram



commutes. In fact ψ^* is inclusion, and using the universal property of (ψ, A') and the above diagram it follows that ψ^* is isomorphism. Hence $A'' = A'$.

Thus an S-module of A, if it exists, is a universal object.

3.2. Proposition. *If A is an S-torsion free module, H the injective hull of A, and ψ_A the embedding homomorphism, then (ψ_A, A') is a fractional module of A with respect to S, where*

$$A' = \{x \in H \mid \text{there exists } s \in S \text{ with } sx \in \psi_A(A)\}.$$

Proof. It is clear that H is S-torsion free and A' is a submodule of H. Since every injective module is S-injective it follows that A' is S-injective.

Let Q be the right semigroup of R and $S \subseteq Q$. Then R is S-torsion free as left R-module, and it has a fractional module (ψ_R, R') with respect to S. If every element of $\psi_R(R)$ is S-divisible in R', then it is easy to see that R' is S-divisible and (ψ_R, R') is an S-module of R. *From now on we shall suppose that $S \subseteq Q$ and $\psi_R(R)$ is S-divisible in R'.*

Identify R in R' by ψ_R, so that every element of R' can be expressed as $s^{-1}a$ ($s \in S$, $a \in R$). Clearly R' is also a right R-module in obvious way.

3.3. Pr[illegible]ition. *R' is flat as right R-module, and*

$$T[illegible] \cong \mathrm{Tor}_1(K, A) \qquad (\mathrm{Tor} = \mathrm{Tor}^R \text{ and } K = R'/R)$$

for any module A.

Proof. We employ the modified arguments of Cartan and Eilenberg in [1, Ch. VII, p. 130].

If $s \in S$, then $D_s = s^{-1}R$ is free and it follows that $R' = \varinjlim(D_s)$ and R' is flat.

Now the exact sequence $0 \to R \to R' \to K \to 0$ of right R-modules yields the exact sequence $0 \to \mathrm{Tor}_1(K, A) \to A \xrightarrow{f} R' \otimes A$. The map $f_s\colon A \to D_s \otimes A$ ($f_s(a) = 1 \otimes a$, $a \in A$) is the combined map

$$A \xrightarrow{f_s} D_s \otimes A \xrightarrow{g} R \otimes A \xrightarrow{h} A\,,$$

where h is canonical isomorphism and $g(s^{-1}r\otimes a) = r\otimes a$. Hence $\ker f_s = \ker(hgf_s) = \{a \in A \mid sa = 0\} \subseteq T(A)$, and evidently, $\bigcup_{s\in S} \ker f_s = T(A)$. Thus

$$T(A) = \varinjlim(\ker f_s) \cong \ker f \cong \operatorname{Tor}_1(K, A).$$

3.4. Proposition. *For every module A there exists an S-module.*

Proof. Write $A_s = R'\otimes A$ and consider the map $\varphi\colon A \to A_s$ ($\varphi(a) = 1\otimes a$, $a \in A$). Since A_s is S-divisible, therefore $A' = A_s/T(A_s)$ is S-torsion free and S-divisible.

If ψ_A is the combined map $A \xrightarrow{\varphi} A_s \xrightarrow{g} A'$, where g is the natural homomorphism, then $\psi_A(a) = 0$ implies that $\varphi(a) \in T(A_s)$, i.e., $\sigma\varphi(a) = 0$ for some $\sigma \in S$. Hence by 3.3 $\sigma a \in T(A)$, and it is clear that $\ker\psi_A = T(A)$.

Now every element $x \in A'$ can be written as $g(s^{-1}\otimes a)$, so that $sx = \psi_A(a)$. This completes the proof.

It follows that, *if A is S-torsion module, then its S-module is zero.*

3.5. Proposition. *If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ is an exact sequence of modules, then the sequence*

$$0 \to A' \to B' \to C' \to 0$$

is exact, where (ψ_A, A'), (ψ_B, B'), and (ψ_C, C') are S-modules of A, B, and C respectively.

Proof. The exactness of the given sequence implies the exactness of

$$0 \to A_s \xrightarrow{f^*} B_s \xrightarrow{g^*} C_s \to 0 \qquad (f^* = 1_{R'}\otimes f, \text{ and } g^* \text{ similarly}).$$

Since A_s is S-divisible, therefore by 2.1 we obtain the exact sequence $0 \to A' \xrightarrow{\overline{f^*}} B' \xrightarrow{\overline{g^*}} C' \to 0$.

In view of 3.5 and the remark at the end of 3.4 it follows that $A' \cong (A/T(A))'$ for any module A. Furthermore, it is evident that A' is a covariant exact functor of A.

References

[1] H. Cartan and S. Eilenberg, *Homological Algebra*, 1956, Ch. VII.
[2] N. Bourbaki, *Algèbre commutative*, Paris 1961, pp. 157–168.
[3] P. Gabriel, *Des catégories abéliennes*, Bull. Soc. Math. France 90 (1962), pp. 323–448.
[4] M. A. Rauf Qureshi, *S-modules and local rings*, J. Nat. Sci. & Math. 9 (1969), pp. 259–270.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF KARACHI
At present: UNIVERSITY OF LIBYA

Reçu par la Rédaction le 24. 11. 1970

A note on the Levitzki radical of a semiring

by

Dwight M. Olson (*) (Lawton, Okla.) and **Terry L. Jenkins** (Laramie, Wyo.)

Abstract. In this paper the authors prove that the Levitzki radical of an arbitrary semiring S is necessarily a k-ideal of S. A preliminary lemma states that if I is a locally nilpotent ideal of a semiring S then the closure of I is also a locally nilpotent ideal of S. These results strengthen certain of those obtained by E. Barbut [1].

1. A set S with two binary operations $+$ and $\cdot$ is called a *semiring* if $(R, +)$ is a commutative semigroup with zero, $(R, \cdot)$ is a semigroup, and both the left and right distributive laws hold for multiplication over addition. It is also required that $0\cdot x = x\cdot 0 = 0$ for all $x \in S$. A non-empty subsemiring I is called a *right ideal of* S if for all $x \in I$ and $r \in S$, $xr \in I$. Left ideals and (two-sided) ideals are defined in a similar manner. An ideal I of S is called a *k-ideal of* S [5] if $x+y \in I$ and $y \in I$ implies $x \in I$ for each $x, y \in S$.

E. Barbut [1] defined the Levitzki radical of a semiring and could prove many results concerning this radical providing the Levitzki radical is a k-ideal. In this note we prove that this radical is necessarily a k-ideal which strengthens many of Barbut's results.

2. If I is an ideal of the semiring S, the quotient semiring S/I is the one defined by S. Bourne [2] where for $a, b \in S$,

$$a \equiv b (\operatorname{mod} I) \text{ iff there exists } i_1, i_2 \in I \text{ such that } a+i_1 = b+i_2.$$

Definition 1. A semiring S is called *locally nilpotent* if every finite subset F of S generates a nilpotent subsemiring of S, or equivalently, if for each finite subset F of S there exists a positive integer N_F such that every product of N_F elements from F is zero.

Definition 2. [1] The *Levitzki radical* $L(S)$ of a semiring S is the sum of all locally nilpotent ideals of S.

E. Barbut [1] has shown that $L(S)$ is a locally nilpotent ideal of S which contains every locally nilpotent right or left ideal of S.

(*) This paper is part of Dwight M. Olson's Ph. D. dissertation prepared under the direction of Professor Terry L. Jenkins at the University of Wyoming, Laramie.