Variants of the axiom of choice in set theory with atoms

by

Ulrich Felgner (Heidelberg) and **Thomas J. Jech** (Amherst, N.Y.) (*)

Abstract. BG is the axiomatic set theory of Bernays–Gödel, BG_0 denotes the modification which allows urelements or atoms. (AC) is the axiom of choice. The paper deals with several statements of BG_0, denoted (Inj), (Proj), (MC), (A), (LW), (PW) and it is shown that in BG_0,

$$(\mathrm{Inj}) \to (\mathrm{Proj}) \to (\mathrm{AC}) \to (\mathrm{MC}) \to (\mathrm{A}) \to (\mathrm{LW}) \to (\mathrm{PW}),$$

while none of the converse implications is provable. In BG however all these statements are equivalent. The independence proofs use permutation models of Mostowski–Specker. Some of the independencies have been known before.

There are two methods of consistency proofs dealing with the axiom of choice (AC) and its variants. The method of Fraenkel and Mostowski involves models of a set theory BG_0 with atoms (cf. [11], [16]). The more Cohen's method of forcing [2] enables to construct models of the ordinary Bernays-Gödel set theory BG without atoms.

Using a certain similarity between the permutation models of Fraenkel and Mostowski and the generic models of Cohen, many results were obtained for BG which had previously known for BG_0. As a matter of fact, one can apply the embedding theorem of Jech-Sochor [6], [7] and the refinement of D. Pincus [13], [14] to transfer several of the consistency results for BG_0 to BG. However not all of them can be transferred. The statements discussed below are examples of nontransferable results. Consider the following statements:

(MC) Axiom of multiple choice. For every family S of non-empty sets there exists a function f such that $f(X)$ is a finite non-empty subset of X, for each $X \in S$.

(A) Antichain principle. Every partially ordered set has a maximal antichain (i.e. a maximal subset of mutually incomparable elements).

(LW) Every linearly orderable set is well-orderable.

(PW) The power-set of every well-orderable set is well-orderable.

(*) The second author acknowledges support from NSF grant PO-34191-XOO.

Notice that (MC) is the $Z(\infty)$ of [10] and the FS(1) of [1]. (A) appears in [9]. In contrast to these statements which follow in BG_0 from (AC), there are two statements which easily imply (AC) in the system BG_0:

(Inj) INJECTION PRINCIPLE. For every proper class C and every set s there is a one-to-one mapping from s into C.

(Proj) PROJECTION PRINCIPLE. For every proper class C and every set s there is a mapping from C onto s.

The statements (Inj) and (Proj) are formulated in the Rubins' book [15], p. 71, where it is shown that both are equivalents of the axiom of choice (AC). However, the axiom of foundation is used in the proof and the question is raised whether they are equivalent to (AC) in the system BG_0.

BG is the set theory whose axioms are A, B, C and D of Gödel [4]. The axioms of BG_0 are A, B, and C of [4]. Hence BG results from BG_0 by adding the axiom of foundation D. We are considering the Bernays–Gödel set theory with classes rather than Zermelo–Fraenkel set theory ZF since the principle (Proj) cannot be formulated in the language of ZF.

THEOREM. *In* BG_0, *the Bernays–Gödel set theory without the axiom of foundation, the following implications are provable, while none of the converse implications is provable*:

$$(\mathrm{Inj})\to(\mathrm{Proj})\to(\mathrm{AC})\to(\mathrm{MC})\to(\mathrm{A})\to(\mathrm{LW})\to(\mathrm{PW})$$

In BG, *the Bernays–Gödel set theory with the axiom of foundation, all these statements are mutually equivalent.*

Some of the implications and nonimplications are trivial, some can be found in the literature and some are proved here.

I. The implications. (a) Obviously (Inj) implies (Proj) and (AC) implies (MC).

(b) (Proj)$\to$(AC): Every set is an image of the proper class *On* of all ordinals, and thus well-orderable.

(c) (MC)$\to$(A): Let $(P, <)$ be a partially ordered set. By (MC) there is a function F such that for each nonempty $X \subseteq P$, $F(X)$ is finite and $0 \neq F(X) \subseteq X$. Let G be the following function:

$$G(X) = \text{the set of all } <\text{-minimal elements of } F(X)$$

$G(X)$ is finite and nonempty, and an antichain. Now we can get a maximal antichain A in P by transfinite recursion: $A = \bigcup_\alpha A_\alpha$, where $A_\alpha = G(X)$, $X = \{x \in P\colon\ x \text{ is imcomparable with all } a \in \bigcup_{\beta<\alpha} A_\beta\}$.

(d) (A)$\to$(LW). Let $(L, <)$ be linearly ordered. Let

$$P = \{(X, x)\colon\ 0 \neq X \subseteq L \text{ and } x \in X\},$$

$$(X, x) \precnsim (Y, y) \text{ iff } X = Y \text{ and } x < y.$$

By (A), $(P, \precnsim)$ has a maximal antichain A; A defines a choice function on the power set of L and so L can be well ordered.

(e) (LW)$\to$(PW): Let s be a well-ordered set. Then

$$u \precnsim v \quad \text{iff} \quad \mathrm{Min}[(u \cup v) - (u \cap v)] \in v$$

is a linear ordering on the power-set of s and by (LW) there is a well-ordering on it.

This shows that all the implications listed in the theorem hold in the system BG_0. It is known that, if one adds the axiom of foundation D to BG_0, then in $BG = BG_0 + D$ also (PW)$\to$(AC) and (AC)$\to$(Inj) hold (see [15], p. 71 and p. 77). Hence in BG all these statements are equivalent. We shall show that this is not the case in the axiom system BG_0.

II. The nonimplications. We use permutation models with an infinite number of reflexive sets $x = \{x\}$ (or *urelements*, *atoms*, cf. [11] and [16]). First, we deal with consequences of (AC). Here we use permutation models with finite supports. Let A be a countably infinite set of atoms and let G be a group of permutations of A. Every permutation π of A extends to an $\in$-automorphism of the universe. For a finite $E \subseteq A$ put $G(E) = \{\pi \in G;\ \pi \text{ leaves } E \text{ pointwise fixed}\}$. A class C is *symmetric* (with respect to G) if there is a finite subset E of A such that $\pi(C) = C$ for all $\pi \in G(E)$. E is said to be a *support* of C. The collection of all sets which are *hereditarily symmetric* with respect to G is a model of BG_0. The classes of the model are the hereditarily symmetric classes.

(a) (MC)$\nrightarrow$(AC): This was proved by Levy in [10]. By the way, one does not need Levy's construction (used for a stronger result) to get (MC)$\nrightarrow$(AC); it holds also in Fraenkel's model where $A = \bigcup_{n=0}^{\infty}\{a_n, b_n\}$, and G is the group generated by the transpositions of $\{a_n, b_n\}$.

(b) (A)$\nrightarrow$(MC): Let G be the group of all permutations of the set A of atoms. Halpern proved in his thesis that (A) holds in the model (cf. [5] or [3]). To prove that (MC) fails in the model, we show that the power-set of A does not have a multiple choice function. For, if F is a function on $\mathfrak{P}(A)$ and E is a support of F, then $Z = F(A - E)$ cannot be a nonempty finite subset of $A - E$. Otherwise, there is $\pi \in G$ such that $\pi E = E$, $\pi F = F$ and $\pi Z \neq Z$, a contradiction.

(c) (PW)$\nrightarrow$(LW): In Mostowski's model [11], (PW) holds and (LW) fails. As a matter of fact, (PW) holds in every permutation model. On the other hand, in Mostowski's model every set can be linearly ordered but not every set can be well-ordered.

(d) (LW)$\leftrightarrow$(A): We use the construction employed first by Mathias; some properties of the model are described in Pincus' Thesis. Let $(A,<)$ be a countably universal homogeneous partial ordering, and let G be the group of all automorphisms of $(A,<)$. We shall need the following properties of $(A,<)$, cf. [8].

(i) Let $(P,\precsim)$ be a finite partially ordered set, let $P_0 \subset P$ and let e_0 be an embedding of $(P_0,\precsim)$ in $(A,<)$. Then there is an embedding e of $(P,\precsim)$ in $(A,<)$ such that $e_0 \subset e$.

(ii) If E_1, E_2 are finite subsets of A and if i is an isomorphism of E_1 and E_2 then there is an automorphism π of $(A,<)$ such that $i \subset \pi$.

First we show that $(A,<)$ does not have a maximal antichain in the model. By (i), every finite antichain in A can be extended and so a maximal antichain would have to be infinite. On the other hand, if X is antichain and E is a support of X then $X \subseteq E$. For, if $x \notin E$ then by (i) and (ii) there is $\pi \in G$ and $y \in A$ such that $x < y$, $\pi x = y$ and π leaves E pointwise fixed; thus $x \in X$ would imply that $y \in X$.

To show that (LW) holds in the model, we utilize the following property of supports:

(*) If both E_1 and E_2 are supports of X then $E_1 \cap E_2$ is also a support of X.

(For proof of (*) we refer the reader either to [12] or to [11] where (*) is proved and used for the group of automorphisms of the rationals.) As a consequence, each symmetric X has a least support $s(X)$, and the function s is in the model. For each finite $E \subset A$, let $D(E)$ be the class of all sets X of the model such that $s(X) = E$. Any function mapping $D(E)$ into the ordinals is in the model; actually, it is supported by E. Thus $D(E)$ can be well ordered in the model.

Now, let $(L,\precsim)$ be a linearly ordered set in the model. Let $S = \{s(x)\colon x \in L\}$. We claim that S can be linearly ordered; $\precsim$ induces an ordering of S. If $E \in S$, let $L_E = D(E) \cap L$. Divide S into equivalence classes such that E_1, E_2 are in the same class if and only if $(E_1,<), (E_2,<)$ are isomorphic. For each equivalence class C, choose (in the universe) an $E \in C$ and a one-to-one function F_C of L_E into ordinals; then let $W = \{\pi F_C\colon \pi \in G,\ C \text{ an equivalence class}\}$. The set W is symmetric and thus in the model, we have assigned to each $E \in S$ a finite set of well-orderings of L_E, namely

$$\mathfrak{F}_E = \{\pi F_C\colon F_C \text{ is defined on } L_{E'},\ \text{and } \pi \text{ is an isomorphism of } E' \text{ and } E\}.$$

Now, for each $E \in S$, let Q_E be the finite set

$$Q_E = \{a_F\colon F \in \mathfrak{F}_E \text{ and } a_F \text{ is the } F\text{-least element of } L_E\}.$$

Now let g_E be the $\precsim$-least element of Q_E. The correspondence $E \to g_E$ constitutes a one-to-one mapping (in the model) of S into L, and so S can be linearly ordered.

Thus the set $S = \{s(x)\colon x \in L\}$ is linearly ordered. Let E_1 be a support of the linear ordering of S. We show that each $E \in S$ is a subset of E_1. Since $L = \bigcup_{E \in S} L_E$ and each L_E can be well ordered, it will follow that L can be well ordered. If E is not a subset of E_1, then by (i) and (ii), there exists $\pi \in G$ and $E' \neq E$ such that $\pi E = E'$, $\pi E' = E$ and π leaves E_1 pointwise fixed; $E \in S$ would clearly contradict the fact that π preserves the linear ordering of S.

To handle the statements (Inj) and (Proj) we use permutation models with a proper class of atoms. Let A be a proper class of atoms. By a *permutation* of A we mean a *set*, a one-to-one mapping of some set $S \subseteq A$ onto itself. The product of two permutations is defined in the obvious way: the domain of $\varrho \cdot \sigma$ is the union $\operatorname{dom}(\varrho) \cup \operatorname{dom}(\sigma)$ and each permutation is supposed to be the identity outside its domain. Every permutation π of A extends to an ϵ-automorphism of the universe, which in this case is $\bigcup_{S \subset A} \bigcup_{\alpha \in On} \mathfrak{P}^\alpha(S)$, where S ranges over subsets of A and $\mathfrak{P}^\alpha(S)$ is the relative von Neumann hierarchy: $\mathfrak{P}^0(S) = S$, $\mathfrak{P}^{\alpha+1}(S) = \mathfrak{P}(\mathfrak{P}^\alpha(S))$ and $\mathfrak{P}^\alpha(S) = \bigcup_{\beta<\alpha} \mathfrak{P}^\beta(S)$ for a limit α.

(e) (AC)$\leftrightarrow$(Proj). Let G be the group of all permutations of A. A class C is *symmetric* if there is a finite subset E of A and that $\pi(C) = C$ for all π such that π leaves E pointwise fixed. We consider the model $\mathfrak{M}$ consisting of all hereditarily symmetric sets; the classes of the model are the hereditarily symmetric classes. It is easy to see that only finite subsets of A are symmetric and consequently, if $X \in \mathfrak{M}$ then $TC(X) \cap A$ is finite ($TC(X)$ is the transitive closure of X). It follows that every $X \in \mathfrak{M}$ can be mapped one-to-one into the ordinals (in $\mathfrak{M}$) and so $\mathfrak{M}$ satisfies the axiom of choice.

To show that (Proj) fails in $\mathfrak{M}$, suppose that there is a function F in $\mathfrak{M}$ which maps A onto ω. For each $n \in \omega$, let $B_n = \{a \in A\colon F(a) = n\}$. There is a finite set E of atoms such that $\pi(F) = F$ whenever $\pi a = a$ for all $a \in E$. Pick $m, n \in \omega$, $m \neq n$, and $x \in B_m$, $y \in B_n$ such that $B_m \cap E = B_n \cap E = 0$. Let σ be a permutation of A which interchanges x and y and leaves E pointwise fixed. Then $\sigma(F) = F$ while $F(\sigma(X)) \neq F(X)$, a contradiction.

(f) (Proj)$\leftrightarrow$(Inj): Let A be a proper class of atoms and let On^ω be the class of all functions from ω into the class On of all ordinals. We enumerate A by the class of these functions: $A = \{a_f;\ f \in On^\omega\}$. For a function $f\colon \omega \to On$ and an integer n let $f|n$ be the sequence of the first n values of f, i.e. $f \upharpoonright n = \langle f(0), f(1), \ldots, f(n-1)\rangle$. Define, for each n,

$$R_n = \{(a_f, a_g)\colon f \upharpoonright n = g \upharpoonright n\}.$$

Each R_n is an equivalence relation on A. A permutation π of A is called *compatible* with R_n if for every $a, b \in A$, $(a, b) \in R_n$ implies $(\pi a, \pi b) \in R_n$. If $m < n$ and π is compatible with R_n then π is also compatible with R_m. A class C is *symmetric* if there is a finite set E and an integer n such that $\pi C = C$ for every π which leaves E pointwise fixed and is compatible with R_n. We consider the model $\mathfrak{M}$ consisting of all hereditarily symmetric sets; the classes of $\mathfrak{M}$ are the hereditarily symmetric classes. The usual proofs show that $\mathfrak{M}$ is a model of BG_0. The relations R_n are proper classes of $\mathfrak{M}$. Every equivalence class of R_n is itself a proper class in $\mathfrak{M}$ and R_n has a proper class of equivalence classes. Hence a symmetry argument shows that if s is a set of atoms in $\mathfrak{M}$, then s is a finite. Thus there is no one-to-one mapping of ω into A in $\mathfrak{M}$. This shows that (Inj) does not hold in $\mathfrak{M}$. On the other hand, every set can be well-ordered in $\mathfrak{M}$ and since On can be mapped onto every well-ordered set, it suffices to show that every class of $\mathfrak{M}$ can be mapped onto On. The class A has a projection F onto On, namely $F(a_f) = f(0)$ and since R_1 is a class of $\mathfrak{M}$, F is a class of $\mathfrak{M}$ too. If γ is a finite sequence of ordinals of length m, say, then similarly $A_\gamma = \{a_f;\ f \upharpoonright m = \gamma\}$ has a symmetric map onto On too, namely $F(a_f) = f(m)$. If C is an arbitrary proper class of $\mathfrak{M}$, then consider the class $D = \{TC(x) \cap A;\ x \in C\}$. Since for every set $x \in \mathfrak{M}$, $TC(x) \cap A$ is a set of atoms in $\mathfrak{M}$, it is finite for each set x of $\mathfrak{M}$. As $\bigcup D \subseteq A$, $\bigcup D$ is either finite or a proper class. If $E = \bigcup D$ is finite, then C contains sets of arbitrary large rank (in the relative von Neumann hierarchy). Hence C can be projected onto On. If $\bigcup D$ is a proper class, then for some n a whole equivalence class of R_n is contained in $\bigcup D$. Hence there is a sequence γ of ordinals of length n such that $A_\gamma \subseteq \bigcup D$. It follows that $\bigcup D$ and therefore also C can be projected onto On by a map in the model. This shows that (Proj) holds in $\mathfrak{M}$.

References

[1] M. N. Bleicher, *Multiple choice axioms and axiom of choice for finite sets*, Fund. Math. 57 (1965), pp. 247–252.
[2] P. J. Cohen, *Set Theory and the Continuum Hypothesis*, New York 1966.
[3] U. Felgner, *Models of ZF-set theory*, Lecture Notes in Mathematics 233 (1971).
[4] K. Gödel, *The consistency of the continuum hypothesis*, Ann. Math. Studies 3 (1940).
[5] J. D. Halpern, *Ph. D. Thesis*, Berkeley 1962.

[6] T. Jech and A. Sochor, *On θ-model of the set theory*, Bull. Acad. Polon. Sci., Ser. Math. 14 (1966), pp. 297–303.
[7] — — *Applications of the θ-model*, Bull. Acad. Polon. Sci., Ser. Math. 14 (1966), pp. 351–355.
[8] B. Jonsson, *Homogeneous universal relational structures*, Math. Scand. 8 (1960), pp. 137–142.
[9] G. Kurepa, *Über das Auswahlaxiom*, Math. Ann. 126 (1953), pp. 381–384.
[10] A. Levy, *Axioms of multiple choice*, Fund. Math. 50 (1962), pp. 475–483.
[11] A. Mostowski, *Über die Unabhängigkeit des Wohlordnungssatzes vom Ordnungsprinzip*, Fund. Math. 32 (1939), pp. 201–252.
[12] D. Pincus, *Ph. D. Thesis*, Harvard University 1969.
[13] — *Support structures for the axiom of choice*, J. Symb. Logic 36 (1971), pp. 28–38.
[14] — *ZF consistency results by FM methods*, J. Symb. Logic 37 (1972), pp. 721–743.
[15] H. Rubin and J. Rubin, *Equivalents of the Axiom of Choice*, Amsterdam 1963.
[16] E. Specker, *Zur Axiomatik der Mengenlehre*, Z. Math. Logik Grundlagen d. Math. 3 (1957), pp. 173–210.

MATH. INST. DER UNIVERSITÄT HEIDELBERG
and
STATE UNIVERSITY OF NEW YORK AT BUFFALO

Reçu par la Rédaction le 14. 12. 1971