Variants of the axiom of choice in set theory with atoms

by

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Abstract. BG is the axiomatic set theory of Bernays-Gödel. BG₀ denotes the modification which allows urelements or atoms. (AC) is the axiom of choice. The paper deals with several statements of BG₀, denoted (Inj), (Proj), (MC), (A), (LW), (PW) and it is shown that in BG₀,

\[
\text{Inj} \rightarrow \text{Proj} \rightarrow \text{AC} \rightarrow \text{MC} \rightarrow \text{A} \rightarrow \text{LW} \rightarrow \text{PW},
\]

while none of the converse implications is provable. In BG however all these statements are equivalent. The independence proofs use permutation models of Mostowski-Specker. Some of the independencies have been known before.

There are two methods of consistency proofs dealing with the axiom of choice (AC) and its variants. The method of Fraenkel and Mostowski involves models of a set theory BG₀ with atoms (cf. [11], [16]). The more Cohen's method of forcing [2] enables to construct models of the ordinary Bernays-Gödel set theory BG without atoms.

Using a certain similarity between the permutation models of Fraenkel and Mostowski and the generic models of Cohen, many results were obtained for BG which had previously known for BG₀. As a matter of fact, one can apply the embedding theorem of Jech-Sochor [6], [7] and the refinement of D. Pincus [13], [14] to transfer several of the consistency results for BG₀ to BG. However not all of them can be transferred. The statements discussed below are examples of nontransferable results. Consider the following statements:

(MC) **Axiom of Multiple Choice.** For every family \( S \) of non-empty sets there exists a function \( f \) such that \( f(X) \) is a finite non-empty subset of \( X \), for each \( X \in S \).

(A) **Antichain Principle.** Every partially ordered set has a maximal antichain (i.e. a maximal subset of mutually incomparable elements).

(LW) Every linearly orderable set is well-orderable.

(PW) The power-set of every well-orderable set is well-orderable.

(*) The second author acknowledges support from NSF grant P0-34191-XG0.
Notice that (MC) is the $Z(\omega)$ of [10] and the FS(1) of [11] is implied in [9]. In contrast to these statements which follow in $BG_3$ from (AC), there are two statements which easily imply (AC) in the system $BG_3$:

\begin{itemize}
  \item \textbf{(Inj) INJECTION PRINCIPLE.} For every proper class $C$ and every set $s$ there is a one-to-one mapping from $s$ into $C$.
  
  \item \textbf{(Proj) PROJECTION PRINCIPLE.} For every proper class $C$ and every set $s$ there is a mapping from $C$ onto $s$.
\end{itemize}

The statements (Inj) and (Proj) are formulated in the Rubin's book [15], p. 71, where it is shown that both are equivalents of the axiom of choice (AC). However, the axiom of foundation is used in the proof and the question is raised whether they are equivalent to (AC) in the system $BG_3$.

$BG_3$ is the set theory whose axioms are $A, B, C$ and $D$ of Gödel [4].

The axioms of $BG_3$ are $A, B, C$ and $D$ of [4]. Hence $BG_3$ results from $BG_3$ by adding the axiom of foundation $D$. We are considering the Bernays-Gödel set theory with classes rather than Zermelo-Fraenkel set theory $ZF$ since the principle (Proj) cannot be formulated in the language of $ZF$.

\textbf{Theorem.} In $BG_3$, the Bernays-Gödel set theory without the axiom of foundation, the following implications are provable, while none of the converse implications is provable:

\begin{itemize}
  \item \textbf{(Inj)}$\rightarrow$(\textbf{Proj})$\rightarrow$(\textbf{AC})$\rightarrow$(\textbf{MC})$\rightarrow$(\textbf{A})$\rightarrow$(\textbf{LW})$\rightarrow$(\textbf{FW})$
\end{itemize}

In $BG_3$, the Bernays-Gödel set theory with the axiom of foundation, all these statements are mutually equivalent.

Some of the implications and nonimplications are trivial, some can be found in the literature and some are proved here.

\textbf{I. The implications.} (a) Obviously (Inj) implies (Proj) and (AC) implies (MC).

(b) (Proj) $\rightarrow$ (AC): Every set is an image of the proper class $On$ of all ordinals, and thus well-orderable.

(c) (MC) $\rightarrow$ (A): Let $(P, <)$ be a partially ordered set. By (MC) there is a function $F$ such that for each nonempty $X \subseteq P$, $F(X)$ is finite and $0 \neq F(P) \subseteq P$. Let $G$ be the following function:

$G(X) = \text{the set of all } \prec\text{-minimal elements of } F(X)$

$G(X)$ is finite and nonempty, and an antichain. Now we can get a maximal antichain $A$ in $P$ by transfinite recursion: $A = \bigcup A_s$, where $A_s = G(X)$, $X = \{x \in P: x \text{ is incomparable with all } a \in A_s\}$.

\textbf{(d) (A) $\rightarrow$ (LW).} Let $(L, \prec)$ be linearly ordered. Let

$P = \{(X, x): 0 \neq X \subseteq L \land x \in X\}$,

$(X, x) \preceq (Y, y) \iff X = Y \land x < y$.

By (A), $(P, \preceq)$ has a maximal antichain $A$; $A$ defines a choice function on the power set of $L$ and so $L$ can be well ordered.

(e) (LW) $\rightarrow$ (FW): Let $s$ be a well-ordered set. Then

$u \preceq v \iff \min\{u \cup v\} = (u \cap v) \cup e \in v$

is a linear ordering on the power-set of $s$ and by (LW) there is a well-ordering on it.

This shows that all the implications listed in the theorem hold in the system $BG_3$. It is known that, if one adds the axiom of foundation $D$ to $BG_3$, then in $BG_3 + D$ also (FW) $\rightarrow$ (AC) and (AC) $\rightarrow$ (Inj) hold (see [16], p. 71 and p. 77). Hence in $BG_3$ all these statements are equivalent. We shall show that this is not the case in the axiom system $BG_3$.

\textbf{II. The nonimplications.} We use permutation models with an infinite number of reflexive sets $\{x\}$ (or urelements, atoms, cf. [11] and [16]). First, we deal with consequences of (AC). Here we use permutation models with finite supports. Let $A$ be a countably infinite set of atoms and let $G$ be a group of permutations of $A$. Every permutation $\pi$ of $A$ extends to an $e$-automorphism of the universe. For a finite $E \subseteq A$ put $G(E) = \{\pi \in G: \pi$ leaves $E$ pointwise fixed$\}$. A class $C$ is symmetric (with respect to $G$) if there is a finite subset $E$ of $A$ such that $\forall \pi \in G(E), \pi C = C$. The collection of all sets which are hereditarily symmetric with respect to $G$ is a model of $BG_3$. The classes of the model are the hereditarily symmetric classes.

(a) (MC) $\leftrightarrow$ (AC): This was proved by Levy in [10]. By the way, one does not need Levy's construction (used for a stronger result) to get (MC) $\leftrightarrow$ (AC); it holds also in Fraenkel's model where $A = \bigcup_{n=0}^{\infty} (a_n, b_n)$, and $G$ is the group generated by the transpositions of $(a_n, b_n)$.

(b) (A) $\leftrightarrow$ (MC): Let $G$ be the group of all permutations of the set $A$ of atoms. Halpern proved in his thesis that (A) holds in the model (cf. [5] or [3]). To prove that (MC) fails in the model, we show that the power-set of $A$ does not have a multiple choice function. For, if $F$ is a function on $f(A)$ and $E$ is a support of $F$, then $Z = F(A \setminus E)$ cannot be a nonempty finite subset of $A \setminus E$. Otherwise, there is $\pi \in G$ such that $\pi Z = E$, $\pi F = F$ and $\pi Z \neq Z$, a contradiction.

(c) (FW) $\leftrightarrow$ (LW): In Mostowski's model [11], (FW) holds and (LW) fails. As a matter of fact, (FW) holds in every permutation model. On
the other hand, in Moteikowski’s model every set can be linearly ordered but not every set can be well-ordered.

(d) (MW) $= \langle A, \ll \rangle$: We use the construction employed first by Mathias; some properties of the model are described in Pincus’s Thesis. Let $\langle A, \ll \rangle$ be a countably universal homogeneous partial ordering, and let $G$ be the group of all automorphisms of $\langle A, \ll \rangle$. We shall need the following properties of $\langle A, \ll \rangle$, cf. [8].

(i) Let $\langle P, \ll \rangle$ be a finite partially ordered set, let $P_0 \subseteq P$ and let $\varphi_0$ be an embedding of $\langle P_0, \ll \rangle$ in $\langle A, \ll \rangle$. Then there is an embedding $\varphi$ of $\langle P, \ll \rangle$ in $\langle A, \ll \rangle$ such that $\varphi_0 \subseteq \varphi$.

(ii) If $E_1, E_2$ are finite subsets of $A$ and if $\varphi$ is an isomorphism of $E_1$ and $E_2$, then there is an automorphism $\pi$ of $\langle A, \ll \rangle$ such that $\varphi = \pi \varphi_0 \pi^{-1}$.

First we show that $\langle A, \ll \rangle$ does not have a maximal antichain in the model. By (i), every finite antichain in $A$ can be extended and so a maximal antichain would have to be infinite. On the other hand, if $X$ is antichain and $E$ is a support of $X$ then $X \subseteq E$. For, if $x \notin E$ then by (i) and (ii) there is $\pi \in \theta$ and $\pi \in \theta$ such that $\pi < y$, $\pi x = y$ and $\pi$ leaves $E$ pointwise fixed; thus $x \notin X$ would imply that $y \notin X$.

To show that (LM) holds in the model, we utilize the following property of supports:

(*) If both $E_1$ and $E_2$ are supports of $X$ then $E_1 \cap E_2$ is also a support of $X$.

(For proof of (*) we refer the reader either to [12] or to [11] where (*) is proved and used for the group of automorphisms of the rationals.) As a consequence, each symmetric $X$ has a least support $s(X)$, and the function $s$ is in the model. For each finite $E \in A$, let $D(E)$ be the class of all sets $X$ of the model such that $s(X) = E$. Any function mapping $D(E)$ into the ordinals is in the model; actually, it is supported by $E$. Thus $D(E)$ can be well ordered in the model.

Now, let $\langle L, \ll \rangle$ be a linearly ordered set in the model. Let $S = \langle s(E), x \ll E \rangle$. We claim that $S$ can be linearly ordered; $\ll$ induces an ordering of $S$. If $E, F \in S$, let $L_{EF} = D(E) \cap D(F)$. Divide $S$ into equivalence classes such that $E, F$ are in the same class if and only if $E, F$ are isomorphic. For each equivalence class $C$, choose (in the universe) an $E \in C$ and a one-to-one function $F_0$ of $L_{EF}$ into ordinals; then let $W = \langle s(F): \pi \in \theta, C \rangle$ an equivalence class. The set $W$ is symmetric and thus in the model, we have assigned to each $E \in S$ a finite set of well-orderings of $L_E$, namely

$$\tau_E = \langle s(F): F \text{ is defined on } L_E, \quad \text{and } \pi \text{ is an isomorphism of } F' \text{ and } F_0 \rangle.$$
enumerate $A$ by the class of these functions: $A = \{a; f \in On^x\}$. For a function $f: \omega \to On$ and an integer $n$ let $f \upharpoonright n$ be the sequence of the first $n$ values of $f$; i.e., $f \upharpoonright n = \langle f(0), f(1), \ldots, f(n-1) \rangle$. Define, for each $n$,

$$R_n = \{(a, a_\varepsilon); f \upharpoonright n = g \upharpoonright n\}.$$ 

Each $R_n$ is an equivalence relation on $A$. A permutation $\pi$ of $A$ is called compatible with $R_n$ if for every $a, b \in A$, $(a, b) \in R_n$ implies $(\pi a, \pi b) \in R_n$.

If $m < n$ and $\pi$ is compatible with $R_m$ then $\pi$ is also compatible with $R_n$.

A class $C$ is symmetric if there is a finite set $E$ and an integer $n$ such that $\pi C = C$ for every $\pi$ which leaves $E$ pointwise fixed and is compatible with $R_n$. We consider the model $\mathfrak{M}$ consisting of all hereditarily symmetric sets; the classes of $\mathfrak{M}$ are the hereditarily symmetric classes.

The usual proofs show that $\mathfrak{M}$ is a model of $\text{BCH}$. The relations $R_n$ are proper classes of $\mathfrak{M}$.

Every equivalence class of $R_n$ is itself a proper class in $\mathfrak{M}$ and $R_n$ has a proper class of equivalence classes. Hence a symmetry argument shows that if $s$ is a set of atoms in $\mathfrak{M}$, then $s$ is a finite. Thus there is no one-to-one mapping of $\omega$ into $A$ in $\mathfrak{M}$. This shows that (Inj) does not hold in $\mathfrak{M}$. On the other hand, every set can be well-ordered in $\mathfrak{M}$ and every set can be mapped onto every well-ordered set, it suffices to show that every class of $\mathfrak{M}$ can be mapped onto $\mathfrak{M}$. The class $A$ has a projection $F$ onto $\mathfrak{M}$, namely $F(a) = f(0)$ and since $E$ is a class of $\mathfrak{M}$, $F$ is a class of $\mathfrak{M}$ too. If $\gamma$ is a finite sequence of ordinals of length $m$, say, then similarly $A_\gamma = \{a; f \upharpoonright \gamma = \gamma\}$ has a symmetric map onto $\mathfrak{M}$ too, namely $F(a) = f(m)$. If $C$ is an arbitrary proper class of $\mathfrak{M}$, then consider the class $D = \{T(x) \land \gamma; x \in C\}$. Since for every set $x \in \mathfrak{M}$, $\text{TC}(x) \cap \gamma$ is a set of atoms in $\mathfrak{M}$, it is finite for each set $x \in C$ as $(\bigcup D) \subseteq A_\gamma \cup D$ is either finite or a proper class. If $E = \bigcup D$ is finite, then $C$ contains sets of arbitrary large rank (in the relative von Neumann hierarchy). Hence $C$ can be projected onto $\mathfrak{M}$. If $\bigcup D$ is a proper class, then for some $n$ a whole equivalence class of $\mathfrak{M}$ is contained in $\bigcup D$. Hence there is a sequence $\gamma$ of ordinals of length $n$ such that $A_\gamma \subseteq \bigcup D$. It follows that $\bigcup D$ and therefore also $C$ can be projected onto $\mathfrak{M}$ by a map in the model. This shows that (Proj) holds in $\mathfrak{M}$.

References