The remainder of the table is computed in much the same fashion making use of the lattice to relate computations at one stage with those at another, e.g., the extension question when $n = 7 \bmod 8$ above.

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On the scarcity of tame disks in certain wild cells

by

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Abstract. If K is a k-cell topologically embedded in Euclidean n-space E^n and P is a $(k-1)$-dimensional polyhedron topologically embedded in K, does there exist a re-embedding h of P in K such that $h(P)$ is locally tame relative to E^n and h is close to the inclusion map? In case $k \leqslant 2$ the answer is known to be affirmative. This paper aims to provide relatively simple examples, each cell K being locally tame modulo a Cantor set, to indicate that the question has a negative answer whenever $3 \leqslant k < n$.

Introduction. R. H. Bing proved that each disk in Euclidean 3-space E^3 contains many tame arcs [5], and using this, Martin showed that the disk contains tame arcs that pass through certain boundary points [7]. Seebeck [10] proved a similar theorem for disks in E^n $(n \geqslant 5)$, after which Sher established the analogue for disks in E^4 [11]. Their results are summarized in the following statement: if D is a disk in E^n $(n \geqslant 3)$, A an arc in D such that $A \cap \partial D$ is contained in ∂A, and $\varepsilon > 0$, then there exists an ε-homeomorphism h of D onto itself such that $h(A)$ is a tame arc. It also follows from Bing's work that for each 3-cell C in E^3 and each disk D in C such that $D \cap \partial C = \partial D$ and D is locally tame at each point of $\operatorname{Int} D$, there exist arbitrarily small homeomorphisms h of C onto itself such that $h(D)$ is a tame disk.

Our purpose here is to indicate, by exhibiting peculiar embeddings of cells in E^n, that a generalization of these results is false. For $3 \leqslant k \leqslant n$ and $n \geqslant 4$, we find (see Theorems 5.2 and 5.4) a k-cell K in E^n and a disk D in K such that for any sufficiently small homeomorphism h of K to itself, $h(D)$ is wildly embedded.

The cells constructed to satisfy Theorems 5.2 and 5.4 appear somewhat simple, each being locally tame modulo a Cantor set. According to the results of Section 6, such a Cantor set, whether viewed as a subset of the Euclidean space or of the embedded cell, must be wildly embedded, and most wild Cantor sets found in the literature lack the complications required for occurring in these examples.

In a sense the essence of the work here consists of the identification of the suitable complications, which are implicitly prescribed by the definition of *special defining sequence*, found in Section 4. The significance of this property is exposed by the combination of two results, Lemma 3.1 and Lemma 4.2. The first of these provides a particularly nice homeomorphism between two Cantor sets, one in a k-cell B^k, the other in E^n, that have compatible special defining sequences. For any extension of this homeomorphism to an embedding of B^k in E^n, the second of these reveals that many of the subcells of B^k are embedded with non-simply connected complements.

1. Definitions and notation. An *n-manifold* is a separable metric space, every point of which has a neighborhood homeomorphic to Euclidean n-space E^n. Similarly, an *n-manifold-with-boundary* is a separable metric space, every point of which has a neighborhood whose closure is an n-cell. The *interior* of an n-manifold-with-boundary M^n, denoted $\operatorname{Int} M^n$, is defined as the subset of M^n consisting of points having a neighborhood homeomorphic to E^n, and the *boundary* of M^n, denoted ∂M^n, is defined as $M^n - \operatorname{Int} M^n$.

We use B^n to denote the set of all points p in E^n such that the distance from p to the origin is no larger than 1. For $m < n$ we consider B^m to be included naturally in B^n.

Suppose f and g are maps of a space X into a space Y that has a metric ϱ. The symbol $\varrho(f, g) < \varepsilon$ means that $\varrho\big(f(x), g(x)\big) < \varepsilon$ for each point x in X.

For any set A in a metric space Y and any positive number ε, $N_\varepsilon(A)$ denotes the set of points in Y whose distance from A is less than ε. Following [4] we say that a homeomorphism h of Y onto itself is an *ε-push* of (Y, A) iff there exists an isotopy h_t $(0 \leqslant t \leqslant 1)$ of Y such that $h_0 =$ identity, $h_t | Y - N_\varepsilon(A) =$ identity $(0 \leqslant t \leqslant 1)$, $\varrho(h_s, h_t) < \varepsilon$ $(0 \leqslant s \leqslant t \leqslant 1)$, and $h_1 = h$. In addition, we say that a map f of Y into itself is an *ε-map* iff $\varrho(f, \text{identity}) < \varepsilon$.

A *polyhedron* is the underlying set of a finite simplicial complex. Given a polyhedron P topologically embedded in a PL n-manifold M^n, we say that P is *tamely embedded*, or *tame*, iff there exist a homeomorphism h of M^n onto itself and a homeomorphism f of P onto itself such that hf is piecewise linear (PL). Similarly, we say that P is *locally tame at a point* p of P iff, for some triangulation T of P, hf is PL with respect to T on some neighborhood of p.

We use the word "tame" in another sense: a Cantor set X in a PL n-manifold M is said to be *tame* iff X is contained in a tame arc in M.

Let A denote a subset of a metric space Y and p a limit point of A. We say that A is *locally simply connected at* p, written 1-LC at p, iff for each $\varepsilon > 0$ there is a $\delta > 0$ such that each map of ∂B^2 into $A \cap N_\delta(p)$ can be extended to a map of B^2 into $A \cap N_\varepsilon(p)$. Furthermore, we say that A is *uniformly locally simply connected*, written 1-ULC, if for each $\varepsilon > 0$ there is a $\delta > 0$ such that each map of ∂B^2 into a δ-subset of A can be extended to a map of B^2 into an ε-subset of A.

2. Extension of embeddings of Cantor sets to embeddings of cells. Let Z denote a wild Cantor set in E^n. It is well known that for $k \leqslant n$ there exist embeddings H of B^k in E^n such that $H(B^k)$ contains Z, $H(B^k)$ is locally tame modulo Z, and $H^{-1}(Z)$ is a nice Cantor set in B^k. In this section we indicate how to obtain such an embedding H so that the inverse image of Z is a preassigned, possibly wild Cantor set X in B^k, the only restriction being that X be a subset of ∂B^k in case $k = n$. The author is indebted to Charles D. Bass, whose valuable comments prompted the approach used here.

LEMMA 2.1. *Let X denote a Cantor set in B^k, where X is contained in ∂B^k in case $k = n$, and h an embedding of X onto a tame Cantor set in ∂B^n $(k \leqslant n)$. Then there exists an embedding H of B^k in B^n such that* (1) $H|X = h$, (2) $H(B^k) \cap \partial B^n = H(X)$, (3) *$H(B^k)$ is locally tame modulo $H(X)$.*

Proof. In case $k \leqslant n-2$, the proof is straightforward. Let θ' denote an embedding of B^k into a tame $(n-2)$-sphere in ∂B^n. Push $\theta'(B^k)$ slightly towards $\operatorname{Int} B^n$, keeping $\theta'(X)$ fixed, thereby defining an embedding θ of B^k in B^n such that $\theta|X = \theta'|X$, $\theta(B^k) \cap \partial B^n = \theta(X)$, and $\theta(B^k)$ is locally tame modulo $\theta(X)$. Since $\theta(X)$ and $h(X)$ are each tame subsets of B^n, there exists a homeomorphism ψ of B^n onto itself such that $\psi\theta|X = h$. Define the embedding H as $\psi\theta$.

In any case the crux of the proof is the method for locating an embedding θ of B^k in B^n such that $\theta(B^k) \cap \partial B^n = \theta(X)$, $\theta(X)$ is tame in ∂B^n, and $\theta(B^k)$ is locally tame modulo $\theta(X)$. One way to obtain this embedding is first to move B^k onto a tame subset of $\operatorname{Int} B^n$ and then to move the image of X onto a tame Cantor set in ∂B^n carefully enough that the composition extends to an embedding θ as required. The details are easy to handle and are left to the reader.

THEOREM 2.2. *Let X denote a Cantor set in B^k, where X is contained in ∂B^k in case $k = n$, and f an embedding of X onto a Cantor set in a connected n-manifold M^n $(k \leqslant n)$. Then there exists an embedding F of B^k in M^n such that $F|X = f$ and $F(B^k)$ is locally tame modulo $F(X)$.*

Proof. Let Y denote a tame Cantor set in ∂B^n. By standard techniques, used in [1] and [3, Th. 3F], among other places, and stated formally in [9], there exists an embedding G of B^n into M^n such that $G(Y) = f(X)$. By Lemma 2.1 there exists an embedding H of B^k in B^n such that $H|X = G^{-1}f$, $H(X) \cap \partial B^n = Y$, and $H(B^k)$ is locally tame modulo Y. Then GH defines the required embedding F.

3. A homeomorphism mixing the admissible subsets of the Cantor sets. Let X denote a Cantor set. A sequence $\{\mathcal{M}_i |\ i = 1, 2, \ldots\}$ is called an *abstract defining sequence for* X iff (1) each $\mathcal{M}_i$ is a finite set, the elements of which are pairwise disjoint, non-void, closed subsets of X, (2) the union of the elements of $\mathcal{M}_i$ equals X, (3) each element of $\mathcal{M}_{i+1}$ is a subset of some element of $\mathcal{M}_i$, and (4) if d_i denotes the diameter of the largest element of $\mathcal{M}_i$, then $d_i \to 0$ as $i \to \infty$. We say that $\{\mathcal{M}_i\}$ is a *special abstract defining sequence for* X iff (i) $\{\mathcal{M}_i\}$ is an abstract defining sequence for X, (ii) the cardinality of $\mathcal{M}_1$ is an integer $k(0) > 1$, and each element of $\mathcal{M}_1$ contains exactly $k(0)$ elements of $\mathcal{M}_2$, and (iii) for each even positive integer n there exists an integer $k(n) > 1$ such that each element of $\mathcal{M}_n$ contains exactly $k(n)$ elements of $\mathcal{M}_{n+1}$ and each element of $\mathcal{M}_{n+1}$ contains exactly $k(n)$ elements of $\mathcal{M}_{n+2}$.

Two special abstract defining sequences $\{\mathcal{M}_i\}$ and $\{\mathcal{N}_i\}$ for the same or different Cantor sets are said to be *compatible* iff $k(n) = k'(n)$ for $n = 0, 2, 4, \ldots$, where $k(n)$ and $k'(n)$ are those integers used to indicate that $\{\mathcal{M}_i\}$ and $\{\mathcal{N}_i\}$, respectively, satisfy the definition of special abstract defining sequence.

Suppose $\{\mathcal{M}_i\}$ is a special abstract defining sequence for a Cantor set X. A subset C of X is said to be *admissible with respect to* $\{\mathcal{M}_i\}$ iff C is non-void and compact and whenever (a) i is an odd positive integer, (b) M is an element of $\mathcal{M}_i$ such that $M \cap C \neq \emptyset$, and (c) M' is an element of $\mathcal{M}_{i+1}$ such that $M' \subset M$, then $M' \cap C \neq \emptyset$. Usually the context supplies just one special abstract defining sequence for X, and we simply call such a set C an *admissible subset of* X.

The purpose of this section is to show that the admissible subsets of two Cantor sets can be thoroughly intermixed by a homeomorphism between the Cantor sets.

LEMMA 3.1. *Suppose X and Z are Cantor sets, and $\{\mathcal{M}_i\}$ and $\{\mathcal{N}_i\}$ are compatible special abstract defining sequences for X and Z, respectively. Then there exists a homeomorphism h of X onto Z such that for each admissible subset C of X and each admissible subset C' of Z, $h(C) \cap C' \neq \emptyset$.*

Proof. We shall construct a sequence $h_1, h_3, \ldots, h_{2k+1}, \ldots$ of homeomorphisms of X onto Z satisfying

a) $h_1(M_{j,1}) \cap N_{k,1} \neq \emptyset$,

b) $h_{i+2}(M_{j,i+1}) = h_i(M_{j,i+1}) \in \mathcal{N}_{i+1}$,

c) if $M_{k,i+2} \subset M_{j,i+1}$ and $N_{r,i+2} \subset h_i(M_{j,i+1})$, then $h_{i+2}(M_{k,i+2}) \cap N_{r,i+2} \neq \emptyset$.

In the conditions above $M_{j,i}(N_{j,i})$ represents an arbitrary element of $\mathcal{M}_i(\mathcal{N}_i)$.

To begin, let $k(0)$ denote the cardinality of $\mathcal{M}_1$. Since $\{\mathcal{M}_i\}$ is a special abstract defining sequence, the cardinality of $\mathcal{M}_2$ is $(k(0))^2$, and each element of $\mathcal{M}_1$ contains exactly $k(0)$ elements of $\mathcal{M}_2$. Since $\{\mathcal{M}_i\}$ and $\{\mathcal{N}_i\}$ are compatible, similar properties hold for $\mathcal{N}_1$ and its elements. Index the elements of $\mathcal{M}_2$ as m_{rs}, where $1 \leqslant r \leqslant k(0)$ and $1 \leqslant s \leqslant k(0)$ (this indexing differs from the notation of the preceding paragraph, which requires s to equal 2), such that each m_{rs} is contained in $M_{r,1} \in \mathcal{M}_1$, and index the elements of $\mathcal{N}_2$ as n_{rs} such that each n_{rs} is contained in $N_{r,1} \in \mathcal{N}_1$. Define a homeomorphism h_1 of X onto Z such that $h_1(m_{rs}) = n_{sr}$ $(1 \leqslant r \leqslant k(0)$ and $1 \leqslant s \leqslant k(0))$. This homeomorphism satisfies Condition a, and $h_1(M_{i,2}) \in \mathcal{N}_2$ for each $M_{i,2} \in \mathcal{M}_2$.

The homeomorphism h_3 is defined in similar fashion. The hypothesis that $\{\mathcal{M}_i\}$ and $\{\mathcal{N}_i\}$ are compatible implies that there exists an integer $k(2) > 1$ such that each element of $\mathcal{M}_2(\mathcal{N}_2)$ contains $k(2)$ elements of $\mathcal{M}_3(\mathcal{N}_3)$ and each element of $\mathcal{M}_3(\mathcal{N}_3)$ contains $k(2)$ elements of $\mathcal{M}_4(\mathcal{N}_4)$. For each element $M_{j,2} \in \mathcal{M}_2$, use the procedure above to define a homeomorphism h_3^* of $M_{j,2}$ onto $h_1(M_{j,2})$ such that

$$h_3^*(M_{j,3}) \cap N_{k,3} \neq \emptyset,$$

for each $M_{j,3} \in \mathcal{M}_3$ that is contained in $M_{j,2}$ and each $N_{k,3} \in \mathcal{N}_3$ that is contained in $h_1(M_{j,2})$, and such that

$$h_3^*(M_{j,4}) \in \mathcal{N}_4$$

for each $M_{j,4} \in \mathcal{M}_4$ that is contained in $M_{j,2}$. By combining such homeomorphisms we obtain the required homeomorphism h_3.

We continue this process in defining $h_5, h_7, \ldots$

Next we shall prove that the limit of the h_i's, which we call h, is a homeomorphism of X onto Z. Condition 4 in the definition of *abstract defining sequence* and Condition b above together imply that $\{h_i\}$ is a uniformly convergent sequence. As a result, h is continuous and onto. To see that h is 1–1, consider two points x_1 and x_2 of X $(x_1 \neq x_2)$: there exists an odd positive integer i such that

$$x_1 \in M_{j,i} \in \mathcal{M}_i \quad \text{and} \quad x_2 \notin M_{j,i},$$

and Condition b implies that $h(x_1) \in h_i(M_{j,i})$ and $h(x_2) \notin h_i(M_{j,i})$.

Finally, we shall show that h satisfies the conclusion of this lemma. Consider two admissible subsets C of X and C' of Z. We prove inductively that for each odd positive integer i there exist indices $r(i)$ and $s(i)$ such that

$$h_i(M_{r(i),i}) \cap N_{s(i),i} \neq \emptyset, \tag{1}$$

$$M_{r(i),i} \cap C \neq \emptyset, \tag{2}$$

$$N_{s(i),i} \cap C' \neq \emptyset. \tag{3}$$

To do this, note first that Condition a makes this obvious for the case $i = 1$. Assume this holds for the (odd) integer k, and let $M_{r(k),k} \in \mathcal{M}_k$

and $N_{s(k),k} \in \mathcal{N}_k$ denote the sets satisfying Conditions 1, 2, and 3. Use Condition b to find indices $r(k+1)$ and $s(k+1)$ such that

$$(4) \qquad h_k(M_{r(k+1),k+1}) = N_{s(k+1),k+1} \in \mathcal{N}_{k+1},$$

where $M_{r(k+1),k+1} \subset M_{r(k),k}$ and $N_{s(k+1),k+1} \subset N_{s(k),k}$. Now the crucial step: it follows from the admissibility of C and C' that $C \cap M_{r(k+1),k+1} \neq \emptyset$ and $C' \cap N_{s(k+1),k+1} \neq \emptyset$. Accordingly, choose indices $r(k+2)$ and $s(k+2)$ such that

$$(2') \qquad C \cap M_{r(k+2),k+2} \neq \emptyset \quad \text{and} \quad M_{r(k+2),k+2} \subset M_{r(k+1),k+1},$$

$$(3') \qquad C' \cap N_{s(k+2),k+1} \neq \emptyset \quad \text{and} \quad N_{s(k+2),k+2} \subset N_{s(k+1),k+1}.$$

Then Condition c implies

$$(1') \qquad h_{k+2}(M_{r(k+2),k+2}) \cap N_{s(k+2),k+2} \neq \emptyset.$$

The induction argument is complete.

Translating the property established in the preceding paragraph into epsilontics, one can show quite easily that for each $\varepsilon > 0$ there exists an odd positive integer k such that $\varrho(h_k(C), C') < \varepsilon$. Using the definition of h, one can show then that $\varrho(h(C), C') = 0$, or, equivalently, since C and C' are compact, $h(C) \cap C' \neq \emptyset$.

4. Special defining sequences for Cantor sets. Given a Cantor set X embedded in an n-manifold Q, we want to consider defining sequences for X that reflect properties of this embedding rather than the abstract defining sequences discussed in the preceding section. Let $\{\mathcal{M}_i\}$ denote a sequence such that each $\mathcal{M}_i$ is a finite set of compact, connected manifolds-with-boundary contained in Q, no two of which intersect, and let $|\mathcal{M}_i|$ denote $\bigcup \{M| \ M \in \mathcal{M}_i\}$. Such a sequence is called a *defining sequence for* X iff each element of $\mathcal{M}_i$ contains a point of X, $|\mathcal{M}_{i+1}| \subset \operatorname{Int}|\mathcal{M}_i|$, and $\bigcap |\mathcal{M}_i| = X$. Associated with any defining sequence $\{\mathcal{M}_i\}$ for X is an abstract defining sequence $\{\mathcal{A}_i\}$ for X given by

$$\mathcal{A}_i = \{M \cap X| \ M \in \mathcal{M}_i\}.$$

A defining sequence $\{\mathcal{M}_i\}$ for X is called a *special defining sequence for* X iff (1) for each positive integer i and each element M of $\mathcal{M}_i$, M is homeomorphic to the Cartesian product of B^2 and some $(n-2)$-manifold and (2) the abstract defining sequence $\{\mathcal{A}_i\}$ for X associated with $\{\mathcal{M}_i\}$ is special. Two special defining sequences for Cantor sets X and Z embedded in manifolds are said to be *compatible* if and only if the associated abstract defining sequences for X and Z are compatible.

Given two Cantor sets X_1 and Z_1 embedded in manifolds and defining sequences for each, we shall indicate how one can construct related Cantor sets X_2 and Z_2 that have compatible defining sequences. The methods employed in proving this fact are as important as the result itself, for later we shall be concerned with some properties of the original pair of Cantor sets and shall need to know that each of the ones constructed shares these properties.

LEMMA 4.1. *Suppose X_1 and Z_1 are Cantor sets embedded in a k-manifold Q^k and an n-manifold Q^n, respectively, $\{\mathcal{M}_i\}$ a defining sequence for X_1 such that for each index i and element M of $\mathcal{M}_i$ there exists a $(k-2)$-manifold S such that M is homeomorphic to $B^2 \times S$, and $\{\mathcal{N}_i\}$ a defining sequence for Z_1 such that for each index i and element N of $\mathcal{N}_i$ there exists an $(n-2)$-manifold T such that N is homeomorphic to $B^2 \times T$. Then there exist Cantor sets X_2 and Z_2 in* $\operatorname{Int}|\mathcal{M}_1|$ *and* $\operatorname{Int}|\mathcal{N}_1|$, *respectively, that have compatible special defining sequences.*

Proof. Step 1. First we modify the Cantor sets X_1 and Z_1 and their defining sequences so that for $i = 0, 1, 2, \ldots$ there exists an integer $k(i)$ such that $\mathcal{M}_1$ and $\mathcal{N}_1$ each contain exactly $k(0)$ elements and for $i > 0$ each element of $\mathcal{M}_i(\mathcal{N}_i)$ contains exactly $k(i)$ elements of $\mathcal{M}_{i+1}(\mathcal{N}_{i+1})$.

To do this, it suffices to describe how to add exactly one element of $\mathcal{M}_{i+1}$ inside a preassigned element M' of $\mathcal{M}_i$ without changing $\mathcal{M}_j$ $(j \leqslant i)$. Choose some element M of $\mathcal{M}_{i+1}$ such that $M \subset M'$. By hypothesis M can be topologically identified with $B^2 \times S$, where S is a compact $(k-2)$-manifold. Select two disjoint disks B_1 and B_2 in B^2, and define

$$\mathcal{M}^*_{i+1} = \{M^* \in \mathcal{M}_{i+1}| \ M^* \neq M\} \cup \{B_1 \times S\} \cup \{B_2 \times S\}.$$

Let $h_e\colon B^2 \times S \to B_e \times S$ $(e = 1, 2)$ be a homeomorphism that preserves the second coordinates. Then for $j > i+1$ define

$$\mathcal{M}^*_j = \{M^* \in \mathcal{M}_j| \ M^* \cap M = \emptyset\} \cup \{h_e(M^*)| \ M^* \in \mathcal{M}_j,\ M^* \subset M, \text{ and } e = 1, 2\},$$

and for $j \leqslant i$ define $\mathcal{M}^*_j = \mathcal{M}_j$. It is easy to show that $\{\mathcal{M}^*_i\}$ is a defining sequence for a Cantor set $X^* = \bigcap |\mathcal{M}^*_i|$. Using this device we can obtain defining sequences $\{\mathcal{M}^*_i\}$ and $\{\mathcal{N}^*_i\}$ for Cantor sets in $\operatorname{Int}|\mathcal{M}_1|$ and $\operatorname{Int}|\mathcal{N}_1|$, respectively, for which there exist integers $k(i)$ $(i = 0, 1, 2, \ldots)$ satisfying the property stated above.

Step 2. Suppose now that $\{\mathcal{M}^*_i\}$ is the defining sequence for a Cantor set X^* in Q^k and $\{k(i)\}$ the sequence of integers constructed in Step 1. We shall construct a special defining sequence $\{\mathcal{R}_i\}$ for a Cantor set X_2 in $\operatorname{Int}|\mathcal{M}^*_1|$, controlling this operation with the sequence $\{k(i)\}$ so that the analogous construction in Q^n produces a special defining sequence $\{\mathcal{S}_i\}$ compatible with $\{\mathcal{R}_i\}$.

First, define $\mathcal{R}_1 = \mathcal{M}^*_1$. For each $M \in \mathcal{M}^*_1$ there exists a compact $(k-2)$-manifold S such that M can be topologically identified with $B^2 \times S$. As before, select $k(0)$ pairwise disjoint disks $B_1, \ldots, B_{k(0)}$ in B^2, and let $\{B_e\times S| \ e=1, \ldots, k(0)\}$ be the elements of $\mathfrak{R}_2$ that are contained in M. Thus, $\mathfrak{R}_1$ consists of $k(0)$ elements, each of which contains exactly $k(0)$ elements of $\mathfrak{R}_2$. Note that if $R'\in\mathfrak{R}_2$ and $R\in\mathfrak{R}_1$ with $R'\subset R$, then $R-\operatorname{Int}R'$ is homeomorphic to $\partial R\times B^1$.

Define the elements of $\mathfrak{R}_3$ that are contained in $B_e\times S$, where $B_e\times S$ is associated with $M\in\mathcal{M}_1^*=\mathfrak{R}_1$, to look exactly like the elements of $\mathcal{M}_2^*$ that are subsets of M. Consequently, each element of $\mathfrak{R}_2$ contains exactly $k(1)$ elements of $\mathfrak{R}_3$. The method used in the preceding paragraph can be repeated in defining $\mathfrak{R}_4$ so that each element of $\mathfrak{R}_3$ contains exactly $k(1)$ elements of $\mathfrak{R}_4$, and if $R'\in\mathfrak{R}_4$ and $R\in\mathfrak{R}_3$ with $R'\subset R$, $R-\operatorname{Int}R'$ is homeomorphic to $\partial R\times B^1$.

Continuing this process we obtain a defining sequence $\{\mathfrak{R}_i\}$ for a Cantor set X_2 such that, for each positive integer i, each element of $\mathfrak{R}_{2i}$ contains exactly $k(i)$ elements of $\mathfrak{R}_{2i+1}$, each element of $\mathfrak{R}_{2i+1}$ contains exactly $k(i)$ elements of $\mathfrak{R}_{2i+2}$, and, if $R'\in\mathfrak{R}_{2i+2}$ and $R\in\mathfrak{R}_{2i+1}$ with $R'\subset R$, $R\text{-}\operatorname{Int}R'$ is homeomorphic to $\partial R\times B^1$.

As a result, the sequence $\{\mathfrak{R}_i\}$ satisfies Condition 2 in the definition of special defining sequence. Because of the way the elements of the $\mathfrak{R}_i$'s are defined, the sequence satisfies Condition 1 as well. Obviously, application of the same procedure to the defining sequence $\{\mathcal{N}_i^*\}$ in Q^n yields a special defining sequence $\{\mathcal{S}_i\}$ compatible with $\{\mathfrak{R}_i\}$.

Remark. Ultimately the modifications introduced in Step 1 serve mostly for notational convenience, but those introduced in Step 2 have much more value, because they interject complications in the original Cantor sets and thereby produce examples that are the unions of uncountably many Cantor sets. In fact, assuming X_1 to be wild, we can prove X_2 to be homeomorphic with $X_1\times X_1$ in such a way that, for each q in X_1, $\{q\}\times X_1$ corresponds to a wild subset of Q^k. The necessity for interjecting such complications is explained further in Section 6.

ADDENDUM TO LEMMA 4.1. *Suppose each element M of $\mathcal{M}_i$ $(i=1,2,\ldots)$ has the following property: the (inclusion induced) homomorphism*

$$j_*\colon \pi_1(\partial M)\to\pi_1(Q^k-|\mathcal{M}_{i+1}|)$$

is an injection. Then the special defining sequence $\{\mathfrak{R}_i\}$ for X_2 constructed in Lemma 4.1 satisfies the following conditions:

(1) *for each element R of $\mathfrak{R}_i$ $(i=1,2,\ldots)$ the homomorphism $j_*\colon \pi_1(\partial R)\to\pi_1(Q^k-|\mathfrak{R}_{i+1}|)$ is an injection,*

(2) *for each odd positive integer i, each $R\in\mathfrak{R}_i$, and each $R'\in\mathfrak{R}_{i+1}$ such that $R'\subset R$, the homomorphisms $j_*\colon \pi_1(\partial R)\to\pi_1(R-R')$ and $j'_*\colon \pi_1(\partial R')\to\pi_1(R\text{-}\operatorname{Int}R')$ are injections.*

Proof. The modifications introduced in Step 1 preserve the property of $\{\mathcal{M}_i\}$, because for each element M^* of $\mathcal{M}_i^*$, there exist an element M of $\mathcal{M}_i$, a subset A^* of $|\mathcal{M}_{i+1}|$ (A^* is a union of some, but not necessarily all, elements of $\mathcal{M}_{i+1}^*$), and a homeomorphism of $Q^k-|\mathcal{M}_{i+1}|$ onto Q^k-A^* that carries ∂M onto ∂M^*.

That the modifications introduced in Step 2 also preserve this property follows by the same argument (except that when i is odd A^* is a union of elements of $\mathfrak{R}_{i+2}$), and this is equivalent to Condition 1. Condition 2 is an obvious consequence of the construction.

LEMMA 4.2. *Suppose X is a Cantor set in a k-manifold Q^k, $\{\mathfrak{R}_i\}$ a special defining sequence for X that satisfies the conclusions of the Addendum to Lemma 4.1, and J a simple closed curve contained in $Q^k-|\mathfrak{R}_1|$ such that J is contractible in Q^k but not in $Q^k-|\mathfrak{R}_1|$. Then for each map f of B^2 into Q^k that sends ∂B^2 homeomorphically onto J, $f(B^2)$ contains an admissible (with respect to $\{\mathfrak{R}_i\}$) subset of X.*

Proof. Using general position techniques we approximate f by a map g such that $g|f^{-1}(X)\cup\partial B^2=f|f^{-1}(X)\cup\partial B^2$ and for each i $g^{-1}(\partial|\mathfrak{R}_i|)$ is a finite collection of pairwise disjoint simple closed curves. If for any such curve J the map $g|J\colon J\to\partial|\mathfrak{R}_i|$ is null homotopic, we use this fact to redefine g on the disk D in B^2 bounded by J so that $g(D)\subset\partial|\mathfrak{R}_i|$ and then adjust slightly so that $g(D)\cap\partial|\mathfrak{R}_i|=\emptyset$. Of course, we perform this operation in stages, first for $i=1$, then for $i=2$, and so on, and the resulting sequence of maps converges to a map h because the diameter of the largest element of $\mathfrak{R}_i$ goes to zero as i goes to infinity.

Note that $h(B^2)\cap X\subset g(B^2)\cap X\subset f(B^2)\cap X$. We complete the argument by proving that $h(B^2)\cap X$ contains an admissible subset of X.

Let C_1 denote the union of all those disks D in B^2 such that $h(D)\subset|\mathfrak{R}_1|$ and $h(\partial D)\subset\partial|\mathfrak{R}_1|$. Let C_2 denote the union of all those disks D in B^2 such that $D\subset C_1$, $h(D)\subset|\mathfrak{R}_2|$, and $h(\partial D)\subset\partial|\mathfrak{R}_2|$. After C_i has been defined, let C_{i+1} denote the union of all those disks D in B^2 such that $D\subset C_i$, $h(D)\subset|\mathfrak{R}_{i+1}|$, and $h(\partial D)\subset\partial|\mathfrak{R}_{i+1}|$.

Now we show that $C=h(\bigcap C_i)$ is an admissible subset of X. First we claim that for $i=1,2,\ldots$, C_i is non-void and each component D_i of C_i contains a component D_{i+1} of C_{i+1}. Since J is not contractible in $Q-|\mathfrak{R}_1|$, there exists a simple closed curve J_1 in $h^{-1}(\partial|\mathfrak{R}_1|)$ such that $h(\operatorname{Int}D_1)\cap\partial|\mathfrak{R}_1|=\emptyset$, where D_1 denotes the disk in B^2 bounded by J_1. By our construction of h, $h|J_1$ defines a nontrivial loop in $h(\partial|\mathfrak{R}_1|)$ and the Addendum to Lemma 4.1 implies that $h(D_1)\subset|\mathfrak{R}_1|$.

Consequently C_1 is non-void. Assume inductively that C_i is non-void, and let D_i denote a component of C_i. Then, since $h|\partial D_i$ defines a nontrivial loop in $\partial|\mathfrak{R}_i|$, Condition 1 of the Addendum implies that $h(D_i)$ intersects $|\mathfrak{R}_{i+1}|$. Let J_{i+1} be a simple closed curve in D_i such that $J_{i+1}\subset h^{-1}(\partial|\mathfrak{R}_{i+1}|)$ and $h(\operatorname{Int}D_{i+1})\cap\partial|\mathfrak{R}_{i+1}|=\emptyset$, where D_{i+1} denotes the subdisk of D_i bounded by J_{i+1}, and the Addendum again implies that $h(D_{i+1})\subset|\mathfrak{R}_{i+1}|$. Thus, D_{i+1} is a component of C_{i+1}, and the claim is established.

Next we claim that if i is an odd positive integer, R an element of $\mathcal{R}_i$ such that $h(C_i) \cap R \neq \emptyset$, and R' an element of $\mathcal{R}_{i+1}$ such that $R' \subset R$, then $h(C_{i+1}) \cap R' \neq \emptyset$. This claim follows almost immediately from Condition 2 of the Addendum, which implies that $h(C_i) \cap R' \neq \emptyset$. The argument of the previous paragraph can be applied to obtain a component D_{i+1} of C_{i+1} such that $h(D_{i+1}) \subset R'$.

Since $\{C_i\}$ is a nested sequence of compact, non-void sets, C is compact and non-void, and the two claims above imply that C is an admissible subset of X.

5. The main results.

LEMMA 5.1. *Suppose Z is a Cantor set in E^n $(n \geqslant 4)$ such that for some loop L in $E^n - Z$, the intersection of Z with the image of each contraction of L contains an admissible subset of Z, X a Cantor set in B^k $(3 \leqslant k < n)$, and e an embedding of B^k in E^n such that for each admissible subset C of X and each admissible subset C' of Z, $e(C) \cap C' \neq \emptyset$. Then, for any complex P in B^k that contains an admissible subset of X, $e(P)$ is wildly embedded.*

Proof. This is trivial, for if $e(P)$ were tame, then $e(P \cap X)$ would be tame, and this, in turn, means that $Z \cap e(P \cap X)$ would be tame. By hypothesis, however, the image of each contraction of L meets $Z \cap e(P \cap X)$; therefore, $E^n - (Z \cap e(P \cap X))$ fails to be simply connected, and $Z \cap e(P \cap X)$ must be wild.

Now the various parts of this paper are ready to be assembled.

THEOREM 5.2. *Suppose k and n are positive integers such that $3 \leqslant k < n$. There exist an embedding e of B^k in E^n and a positive number δ such that (1) $e(B^k)$ is locally tame modulo a Cantor set and (2) for any δ-push θ of B^k onto itself, the disk $e\theta(B^2)$ is wildly embedded (in fact, $E^n - e\theta(B^2)$ fails to be simply connected).*

Proof. Antoine's construction [2] (for the case $k = 3$) or Blankenship's construction [3] (for the case $k > 3$) produces an example of a wild Cantor set X_1 in $\operatorname{Int} B^k$ and a defining sequence $\{\mathcal{M}_i\}$ for X_1 such that for $i = 1, 2, \ldots$ and each element M of $\mathcal{M}_i$, the inclusion induced homomorphism

$$j_*\colon \pi_1(\partial M) \to \pi_1(B^k - |\mathcal{M}_{i+1}|)$$

is injective. There exist a wild Cantor set Z_1 in E^n and a defining sequence $\{\mathcal{N}_i\}$ for Z_1 with the analogous property. Either of these constructions requires each $M \in \mathcal{M}_i$ $(i = 1, 2, \ldots)$ to be homeomorphic to the Cartesian product of B^2 with $k-2$ circles, and each $N \in \mathcal{N}_i$ $(i = 1, 2, \ldots)$ to the Cartesian product of B^2 with $n-2$ circles.

By extending some disk of the form $B^2 \times$ (a point of each circle), we can produce a tame disk D in $\operatorname{Int} B^k$ such that $\partial D \cap |\mathcal{M}_1| = \emptyset$ and ∂D is not contractible in $B^k - |\mathcal{M}_1|$ (hence, not in $B^k - X_1$), and without loss of generality we can assume that $D \subset B^2$. Similarly, there exists a simple closed curve L in $E^n - |\mathcal{N}_1|$ that is not contractible in $E^n - Z_1$.

Using Lemma 4.1 we find Cantor sets X and Z in $\operatorname{Int} B^k$ and E^n, respectively, and compatible special defining sequences $\{\mathcal{R}_i\}$ and $\{\mathcal{S}_i\}$ for X and Z, respectively, that satisfy the conclusions of the Addendum to Lemma 4.1. Furthermore, $|\mathcal{R}_1| \subset |\mathcal{M}_1|$ and $|\mathcal{S}_1| \subset |\mathcal{M}_1|$.

Applying Lemma 3.1 we obtain a homeomorphism h of X onto Z such that for each admissible subset C of X and each admissible subset C' of Z, $h(C) \cap C' \neq \emptyset$. According to Theorem 2.2 there exists an embedding e of B^k in E^n such that $e|X = h$ and $e(B^k)$ is locally tame modulo Z. We shall prove that this embedding satisfies the conclusion of Theorem 5.2.

Let δ denote the distance in B^k from ∂D to X. For any δ-push θ of B^k, $\theta(D)$ contains an admissible subset of X for the following reason: one can easily define a construction of ∂D in $\theta(D)$ plus the image of ∂D under the push; by Lemma 4.2 this set contains an admissible subset of X, and since the choice of δ keeps X from intersecting the image of ∂D under the push, $\theta(D)$ contains an admissible subset of X.

Note that Lemma 4.2 also implies that Z intersects the image of any contraction of L in an admissible subset of Z. Invoking our assumption that $D \subset B^2$, we find that $\theta(B^2)$ contains an admissible subset of X. Thus, we appeal to Lemma 5.1, considering the complex P to be $\theta(B^2)$, to determine that $e\theta(B^2)$ is wild.

In fact, $e\theta(B^2)$ cannot have simply connected complement. Since $n \geqslant 4$, the curve L can be deformed slightly in $E^n - Z$ so that the resulting loop L' misses $e\theta(B^2)$. Then L' cannot be shrunk to a point in $E^n - e\theta(B^2)$, for the image of any such contraction of L' must contain an admissible subset C' of Z, and C' must intersect $e\theta(B^2)$.

COROLLARY 5.3. *Suppose k and n are integers such that $3 \leqslant k < n$. There exists an embedding e' of B^k in E^n such that (1) $e'(B^k)$ is locally tame mod a Cantor set Z, (2) $e'(\partial B^2) \cap Z = \emptyset$, and (3) for any embedding θ of B^2 in B^k such that $\theta(\partial B^2) = \partial B^2$, $e'\theta(B^2)$ is wildly embedded.*

Proof. Let K be a k-cell in $\operatorname{Int} B^k$ such that there exists a homeomorphism h of B^k onto K such that $h(B^2) = D$, where D denotes the disk mentioned in the proof of Theorem 5.2, and let e denote the embedding of B^k in E^n promised by Theorem 5.2. Then $e' = eh$ is the embedding required by Corollary 5.3, and the result follows as before because for each homeomorphism θ of B^2 in B^k such that $\theta(\partial B^2) = \partial B^2$, the disk $h\theta(B^2)$ is contained in K and has $h\theta(\partial B^2) = h(\partial B^2) = \partial D$ as its boundary, which implies that $h\theta(B^2)$ contains an admissible subset of that Cantor set X for which $e(B^k)$ is locally tame modulo $e(X)$.

THEOREM 5.4. *For $n \geqslant 4$ there exist an embedding e of B^n in E^n and a positive number δ such that* (1) *$e(B^n)$ is locally tame modulo a Cantor set and* (2) *for any δ-push θ of B^n onto itself, $e\theta(\partial B^3)$ is wildly embedded.*

Proof. The argument parallels that of Theorem 5.2, except that X_1, X, and D are chosen in ∂B^n, and D is assumed to be contained in ∂B^3.

6. Conditions implying a disk in an embedded cell can be approximated by tame disks. In this section we explain why for certain Cantor sets X in B^k and certain others Z in E^n there is no embedding h of B^k in E^n satisfying Theorem 5.2. We make use of some additional terminology: a 0-dimensional F_σ subset F of a PL manifold (without boundary) M is said to be *tame* iff F can be expressed as a countable union of closed, tame subsets of M; similarly, a 0-dimensional F_σ-subset F of a PL manifold-with-boundary N is said to be *tame* iff $F \cap \partial N$ and $F \cap \operatorname{Int} N$ are tame subsets of ∂N and $\operatorname{Int} N$, respectively.

THEOREM 6.1. *If h is an embedding of B^k (or any other compact k-manifold) in E^n $(3 \leqslant k \leqslant n-2)$, then there exists a 0-dimensional F_σ-subset F of B^k such that $h(F)$ is a tame subset of E^n and $E^n - h(B^k - F)$ is* 1-ULC.

Proof. Refer to the proof of Theorem 3 in [10], which contains almost the entire argument required for Theorem 6.1. The set $h(F)$ is contained in the union of the 2-skeletons of a sequence of curvilinear triangulations of E^n, which forces $h(F)$ to be tame.

If, instead of $h(F)$ being a tame subset of E^n, F itself is tame relative to B^k, then any $k-1$ dimensional polyhedron in $\operatorname{Int} B^k$ can be pushed a little so that its image under h is tame.

THEOREM 6.2. *Suppose that h is an embedding of B^k in E^n $(3 \leqslant k \leqslant n-2)$ and F a 0-dimensional F_σ subset of B^k such that F is a tame subset of B^k and $E^n - h(B^k - F)$ is* 1-ULC, *P a polyhedron of dimension less than k in B^k such that $P \cap \partial B^k$ has dimension less than $k-1$, and $\varepsilon > 0$. Then there exists an ε-push θ of B^k onto itself such that $h\theta(P)$ is tame.*

Proof. Use the tameness of F to obtain an $\frac{1}{2}\varepsilon$-push θ_1 of B^k that pushes $P \cap \partial B^k$ off $F \cap \partial B^k$. Then obtain another $\frac{1}{2}\varepsilon$-push that moves $\theta_1(P) \cap \operatorname{Int} B^k$ off $F \cap \operatorname{Int} B^k$ and keeps ∂B^k fixed. The composition of these two pushes is an ε-push θ such that $E^n - h\theta(P)$ is 1-ULC. Then Theorem 1 of [4] implies that $h\theta(P)$ is tame.

COROLLARY 6.3. *Suppose Z is a Cantor set in E^n, C a countable dense subset of Z such that $(E^n - Z) \cup C$ is* 1-ULC, *and h an embedding of B^k $(3 \leqslant k \leqslant n-2)$ in E^n such that $h(B^k)$ contains Z and is locally tame modulo Z. Then, for each $\varepsilon > 0$, there exists an ε-push θ of B^k onto itself such that the disk $h\theta(B^2)$ is tame.*

Proof. This follows immediately from Theorem 6.2 and the observation that $\big(E^n - h(B^k)\big) \cup C$ is 1-ULC.

If h is an embedding of B^k in E^n $(3 \leqslant k \leqslant n-2)$ satisfying Theorem 5.2 and $h(B^k)$ is locally tame modulo a Cantor set Z, then Corollary 6.3 provides information about this Cantor set. The following result provides similar information about the corresponding Cantor set in B^k.

THEOREM 6.4. *Suppose X is a Cantor set in B^k $(k = 3$ or $k \geqslant 5)$, C a countable subset of X such that $(B^k - X) \cup C$ is* 1-ULC, *and h an embedding of B^k in E^n $(3 \leqslant k \leqslant n-2)$ such that $h(B^k)$ is locally tame modulo $h(X)$. Then, for each $\varepsilon > 0$, there exists an ε-push θ of (B^k, X) such that the disk $h\theta(B^2)$ is tame.*

Proof. It is relatively easy to show that for any compact subset Y of $X - C$, $B^k - Y$ is 1-ULC. Thus, either Theorem 5.1 of [6] or, generally, the main result of [8] implies that Y is tame relative to E^k.

We assume that the 0-dimensional F_σ subset F of B^k that satisfies the conclusions of Theorem 6.1 is contained in $X - C$, for the proof allows us to obtain such an F missing any preassigned countable subset of B^k. By the preceding paragraph, $F \cap \operatorname{Int} B^k$ is a tame subset of $\operatorname{Int} B^k$. (Note: we do not claim $F \cap \partial B^k$ to be tame relative to ∂B^k). The desired ε-push θ is obtained by first pushing $\operatorname{Bd} B^2$ off $F \cap \operatorname{Bd} B^k$ and then pushing the resulting image of B^2 off $F \cap \operatorname{Int} B^k$. Then $E^n - h\theta(B^2)$ is 1-ULC, and, again by [4, Th. 1], $h\theta(B^2)$ is tame.

W. T. Eaton first mentioned to the author that for any countable dense C of Antoine's necklace X in E^3 [2], $(E^3 - X) \cup C$ is 1-ULC. In verifying this remark one is led to a more general version such as the following.

LEMMA 6.5. *Suppose $\{\mathcal{M}_i\}$ is a special defining sequence for a Cantor set X in E^n $(n \geqslant 3)$ such that for each positive integer i, each element M of $\mathcal{M}_i$, and each element M' of $\mathcal{M}_{i+1}$ satisfying $M' \subset M$, every loop in ∂M that is null homotopic in M is also null homotopic in $(M - |\mathcal{M}_{i+1}|) \cup M'$. Then, for each countable dense subset C of X, $(E^n - X) \cup C$ is* 1-ULC.

Proof. Let C denote a countable dense subset of X. Suppose f is a map of B^2 into E^n such that $f(\partial B^2) \subset E^n - X$. We shall indicate how to find another map f^* close to f such that $f^*(B^2) \cap X \subset C$.

Let ε denote a positive number. Choose an integer i such that the diameter of each element of $\mathcal{M}_i$ is less than ε and $f(\partial B^2)$ misses $|\mathcal{M}_i|$, and choose a finite subset C^* of C such that C^* contains a point of each element of $\mathcal{M}_i$.

Write f as f_i. We find a map g_{i+1} close to f_i such that

(1) $f_i(p) = g_{i+1}(p)$ for each p in $B^2 - f_i^{-1}(|\mathcal{M}_i|)$,

(2) each component of $B^2 - g_{i+1}^{-1}(|\mathcal{M}_{i+1}|)$ is a disk.

All that needs to be done here is to put $f_i(B^2)$ in general position with respect to spines of the elements of $\mathcal{M}_{i+1}$ and then to blow up appropriately chosen regular neighborhoods of these spines to the elements themselves. For each $K \in \mathcal{M}_i$ such that $g_{i+1}(B^2) \cap K \neq \emptyset$, choose $K' \in \mathcal{M}_{i+1}$ such that $K' \subset K$ and $C^* \cap K' \neq \emptyset$. Then, using the algebraic hypothesis of this lemma, we find a map f_{i+1} of B^2 into E^n such that

(A) $f_{i+1}(p) = f_i(p)$ for each p in $B^2 - f_i^{-1}(|\mathcal{M}_i|)$,

(B) if $K' \in \mathcal{M}_{i+1}$ such that $f_{i+1}(B^2) \cap K' \neq \emptyset$, then $C^* \cap K' \neq \emptyset$.

By continuing the process we obtain a sequence $\{f_k|\ k > i\}$ of maps of B^2 into E^n such that

(A_k) $f_k(p) = f_{k-1}(p)$ for each p in $B^2 - f_{k-1}^{-1}(|\mathcal{M}_{k-1}|)$,

(B_k) if $K' \in \mathcal{M}_k$ such that $f_k(B^2) \cap K' \neq \emptyset$, then $C^* \cap K' \neq \emptyset$.

Once f_k is obtained, then for each K in $\mathcal{M}_k$ such that $f_k(B^2) \cap K \neq \emptyset$, the key step is to choose K' in $\mathcal{M}_{k+1}$ such that $C^* \cap K' \neq \emptyset$ and to modify f_k, by applying the hypothesis of this lemma, so that

$$f_{k+1}(B^2) \cap (K \cap |\mathcal{M}_{k+1}|) \subset K' .$$

It follows that $f^* = \lim f_k$ is a continuous function of B^2 into E^n, $f^*|\partial B^2 = f|\partial B^2$, $f^*(B^2) \cap X \subset C^*$, and $\varrho(f, f^*) < \varepsilon$. This essentially proves that $(E^n - X) \cup C$ is 1-ULC.

COROLLARY 6.6 (Eaton). *If X denotes Antoine's necklace in E^3 and C a countable dense subset of X, then* $(E^3 - X) \cup C$ *is* 1-ULC.

COROLLARY 6.7. *If X denotes a wild Cantor set in E^n ($n \geqslant 4$) described by Blankenship* [3] *and C a countable dense subset of X, then* $(E^n - X) \cup C$ *is* 1-ULC.

To prove either of the corollaries above we observe that the defining sequences given in [2] and [3] to describe the Cantor sets satisfy the hypotheses of Lemma 6.5.

Addendum. In another paper we expand these techniques to embed B^k in E^n $(3 \leqslant k < n)$ so that no disk in the image is tame.

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