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Homotopy sequences of fibrations

by

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Abstract. The homotopy sequence of a fibration is generalized to include pairs, triads and squares of fibrations. In accomplishing this the (three dimensional) homotopy lattice of a cube is described and is used to define an associated lattice for a fibration. The standard exact sequences are briefly described. Finally, a potpourri of examples is presented, including some calculations concerning the effect of Lundell's non-stable Bott map on the non-stable homotopy of $U(n)$, with the intention of indicating the breadth of relevance and the usefulness of this method.

Introduction. A map of pairs $i\colon (F, F') \to (E, E') \to (B, B')\colon p$ is said to be a *fibration* if both $i\colon F \to E \to B\colon p$ and $i\colon F' \to E' \to B'\colon p$ are (Serre) fibrations. Hilton [3] has described a homotopy sequence for such fibrations. We recover this sequence from the homotopy sequence of a triad [1]. This approach is then extended, via the homotopy sequences of triads, squares and cubes to provide a functorial lattice which is commutative, up to sign, and which relates the various homotopy sequences of a square of fibrations.

In the first section the basic properties of squares of fibrations are described, while, in the second, the homotopy of cubes is described and employed in defining the homotopy lattice of the fibration. We note that objects described here are special cases of a very general phenomenon. In the third section we present several examples, as well as results concerning the effect of the Bott map b_n', [6], on the non-stable homotopy of $U(n)$.

FIBRATIONS. By a fibration, $i\colon F \to E \to B\colon p$, we mean a fiber space in the sense of Serre, i.e., $F = p^{-1}(b_0)$ and for any CW complex, K, and commutative diagram,

$$\begin{array}{rccc} H\colon & K\times o & \to & E \\ & \downarrow & & \downarrow{\scriptstyle p} \\ F\colon & K\times I & \to & B \end{array}$$

there is an extension of H to $K\times I$ so that the diagram is commutative. The natural extension of this property to pairs, triads and squares provides an equivalent definition of fibration to that requiring that each restriction is a fibration. A square (triad) is a quadruple $(B; B_1, B_2; B_3)$ such that

$$\begin{array}{ccc} B_3 & \subset & B_1 \\ \cap & & \cap \\ B_2 & \subset & B \end{array}$$

(and $B_3 = B_1 \cap B_2$).

PROPOSITION 1. $i\colon (F; F_1, F_2; F_3) \to (E; E_1, E_2; E_3) \to (B; B_1, B_2; B_3)\colon p$ *is a fibration if and only if for any* CW *space*, $(K; K_1, K_2; K_3)$ *and commutative diagram*,

$$\begin{array}{ccc} H\colon (K; K_1, K_2; K_3) \times o & \to & (E; E_1, E_2; E_3) \\ \downarrow & & \downarrow p \\ F\colon (K; K_1, K_2; K_3) \times I & \to & (B; B_1, B_2; B_3) \end{array}$$

there is an extension of H *to* $(K; K_1, K_2; K_3) \times I$ *so that the diagram is commutative.*

Proof. The covering homotopy for the square implies, by restriction, that each is a fibration.

Conversely, if each is a fibration, the covering homotopy property for the square is regained by first covering the homotopy into B_3, next using the CW structure to extend this to K_1 from K_3 in the fibration $i\colon F_1 \to E_1 \to B_1\colon p$, then, similarly, from $K_1 \cap K_2$ to K_2 via the fibration $i\colon F_2 \to E_2 \to B_2\colon p$, and finally from $K_1 \cup K_2$ to K using the fibration $i\colon F \to E \to B\colon p$.

Thus basic definitions and propositions given for fibrations extend easily to this situation. For example, two elementary fibrations are the product fibration $i\colon (F; F_1, F_2; F_3) \to (B \times F; B_1 \times F_1, B_2 \times F_2; B_3 \times F_3) \to (B; B_1, B_2; B_3)\colon p$ and the restriction of a fibration $i\colon F \to E \to B\colon p$ to the square $(B; B_1, B_2; B_3)$, $i\colon (F; F, F; F) \to (B; p^{-1}(B_1), p^{-1}(B_2); p^{-1}(B_3)) \to (B; B_1, B_2; B_3)\colon p$. In the third section we shall consider several more interesting examples of this phenomena.

If $(\xi; \xi_1, \xi_2; \xi_3)$ and $(\eta; \eta_1, \eta_2; \eta_3)$ are fibrations, a map of fibrations is a pair of maps, $(f, \bar{f})$, such that the diagram

$$\begin{array}{ccc} \bar{f}\colon (E\xi; E\xi_1, E\xi_2; E\xi_3) & \to & (E\eta; E\eta_1, E\eta_2; E\eta_3) \\ p\xi \downarrow & & p\eta \downarrow \\ f\colon (B\xi; B\xi_1, B\xi_2; B\xi_3) & \to & (B\eta; B\eta_1, B\eta_2; B\eta_3) \end{array}$$

is commutative. In this way a section to a fibration $(\xi; \xi_1, \xi_2; \xi_3)$ is a map $\sigma\colon (B\xi; B\xi_1, B\xi_2; B\xi_3) \to (E\xi; E\xi_1, E\xi_2; E\xi_3)$ such that $p\xi \circ \sigma = 1$.

Via the construction of the induced fibration for a single fibration, the induced fibration for a map of squares,

$$f\colon (B; B_1, B_2; B_3) \to (B\xi; B\xi_1, B\xi_2; B\xi_3)\,,$$

is defined by

$$f^*(\xi; \xi_1, \xi_2; \xi_3) = \big(f^*\xi; (f|B_1)^*\xi_1, (f|B_2)^*\xi_2; (f|B_3)^*\xi_3\big)\,.$$

Many other standard constructions carry over to this situation. For example if we wish to replace a map of squares $f\colon (X; X_1, X_2; X_3) \to (Y; Y_1, Y_2; Y_3)$ by a fibration we merely consider, as usual, the space of paths in $(Y; Y_1, Y_2; Y_3)$ which begin in $f(X; X_1, X_2; X_3)$ and end in $(Y; Y_1, Y_2; Y_3)$. The projection is, of course, evaluation at the end of the path. A construction which we shall find useful later associates to a fibration the fibration of loop spaces. Specifically, consider $(\xi; \xi_1, \xi_2; \xi_3)$, with $\xi_3 \neq \emptyset$ and base point $* \in F_3$ such that $p\xi_3(*)$ is the base point of $(B; B_1, B_2; B_3)$. Then

$$\Omega i\colon (\Omega F; \Omega F_1, \Omega F_2; \Omega F_3) \to (\Omega E; \Omega E_1, \Omega E_2; \Omega E_3) \to \\ \to (\Omega B; \Omega B_1, \Omega B_2; \Omega B_3)\colon \Omega p\xi$$

is a fibration, $(\Omega\xi; \Omega\xi_1, \Omega\xi_2; \Omega\xi_3)$.

HOMOTOPY THEORY. Let $\square F$ and $\square E$ denote the squares $(F; F_1, F_2; F_3)$ and $(E; E_1, E_2; E_3)$, respectively. If there is an inclusion, $i\colon \square F \to \square E$, we say that the pair $(\square E, \square F)$ is a *cube*, denoted by $\boxplus$. In this section we shall be concerned with various homotopy sequences arising from a cube.

The homotopy of a triad was defined by Blakers and Massey [1], while that of the square has been employed by Haefliger [2] and, also, Rourke and Sanderson [11].

The homotopy set, $\pi_n(\square E)$, of the square, $\square E$, is the set of arc components of the space of maps taking $(D^{n-1} \times I, S_+^{n-2} \times I, S_-^{n-2} \times I, S_+^{n-3} \times I, D^{n-1} \times \{o, 1\} \cup S_-^{n-3} \times I)$ to $(E, E_1, E_2, E_3, *)$, where $* \in E_3$ denotes the base point of the square. If $n \geqslant 3$ this set has a group structure which is commutative for $n \geqslant 4$.

It is possible to view the homotopy group of a square as the relative homotopy group of a pair of pairs. Similarly, the homotopy group of a cube may be viewed as the relative homotopy group of a pair of squares. Thus, consider the cube $(\square E, \square F)$. The homotopy set of the cube $\pi_n(\boxplus)$ is the set of arc components of the space of maps taking $(D^{n-1} \times I, S_+^{n-2} \times I, S_-^{n-2} \times I, S_+^{n-3} \times I, D^{n-1} \times o, S_+^{n-2} \times o, S_-^{n-2} \times o, S_+^{n-3} \times o, D^{n-1} \times 1 \cup S_-^{n-3} \times I)$ to $(E, E_1, E_2, E_3, F, F_1, F_2, F_3, *)$, where $* \in F_3$ is the base point of the cube. This set has a group structure for $n \geqslant 4$ which is commutative for $n \geqslant 5$.

The homotopy groups associated with a cube fit together to form a lattice of exact sequences, figure 1, involving the homotopy groups of several squares, in addition to $\square E$ and $\square F$. These squares are

$$\square_1 = (E; F, E_1; F_1),$$
$$\square_2 = (E; F, E_2; F_2),$$
$$\square_3 = (E_1; F_1, E_3; F_3) \quad \text{and}$$
$$\square_4 = (E_2; F_2, E_3; F_3).$$

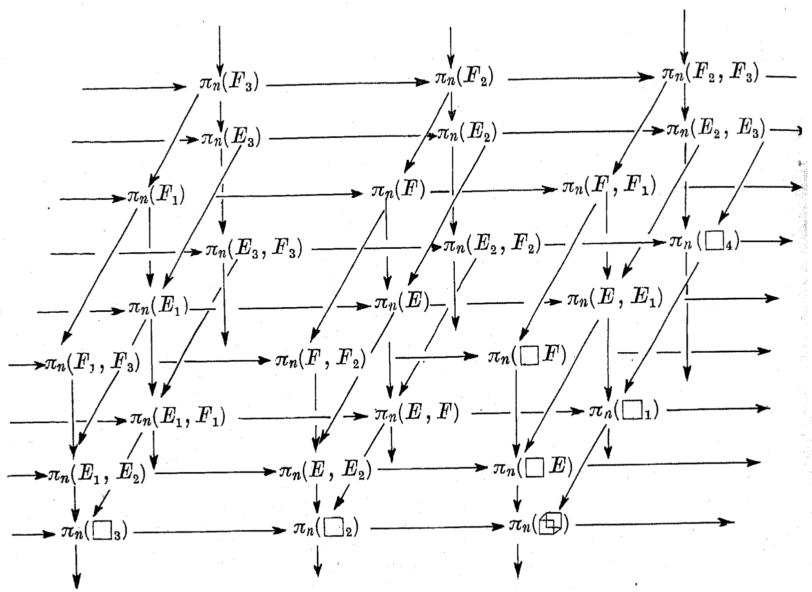


Figure 1. The homotopy lattice of a cube

We note that the lattice is comprised of two simpler types of lattices. The first of these is the lattice of a square, for example $\square E$,

$$\begin{array}{ccc}
\downarrow & \downarrow & \downarrow \\
\to \pi_n(E_3) & \to \pi_n(E_2) & \to \pi_n(E_2, E_3) \to \\
\downarrow & \downarrow & \downarrow \\
\to \pi_n(E_1) & \to \pi_n(E) & \to \pi_n(E, E_1) \to \\
\downarrow & \downarrow & \downarrow \\
\to \pi_n(E_1, E_3) \to & \pi_n(E, E_2) \to & \pi_n(\square E) \xrightarrow{\partial_1} \\
\downarrow & \downarrow & \partial_2 \downarrow
\end{array}$$

while the second, typically, involves two pairs of squares,

$$\begin{array}{ccc}
\downarrow & \downarrow & \downarrow \\
\to \pi_n(F_2, F_3) \to & \pi_n(F, F_1) \to & \pi_n(\square F) \to \\
\downarrow & \downarrow & \downarrow \\
\to \pi_n(E_2, E_3) \to & \pi_n(E, E_1) \to & \pi_n(\square E) \to \\
\downarrow & \downarrow & \downarrow \\
\to \pi_n(\square_4) & \to \pi_n(\square_1) & \to \pi_n(\boxplus) \xrightarrow{\partial_3} \\
\downarrow & \downarrow & \partial_4 \downarrow
\end{array}$$

The boundary homomorphisms, ∂_1, ∂_2, ∂_3 and ∂_4, in these exact sequences are defined as follows: ∂_1 is the restriction to $S_+^{n-2} \times I$, ∂_3 is the restriction to $S_-^{n-2} \times I$, ∂_3 is the restriction to $D^{n-1} \times o$, and ∂_4 is the restriction $S_-^{n-2} \times I$. Each of the squares involving two boundary homomorphisms is anti-commutative. All others are commutative.

In this general context there are several other homotopy sequences which are relevant. The first, which compares the homotopy groups of a square and its associated triad is

$$\to \pi_n(E; E_1, E_2; E_1 \cap E_2) \xrightarrow{\partial} \pi_{n-2}(E_1 \cap E_2, E_3) \xrightarrow{\gamma} \pi_{n-1}(\square E) \to$$

This sequence is derived from the lattice of the cube

$$\big((E; E_1, E_2; E_1 \cap E_2), (E; E_1, E_2; E_3)\big).$$

Consider the exact sequence

$$\to \pi_n(E; E_1, E_2; E_1 \cap E_2) \to \pi_n(\boxplus) \to \pi_{n-1}(E; E_1, E_2; E_3) \to$$

From the sequence

$$\to \pi_n(E; E, E_1; E_1) \to \pi_n(\boxplus) \to \pi_{n-1}(E_2; E_2, E_1 \cap E_2; E_3) \to$$

it follows that $\pi_n(\boxplus) \cong \pi_{n-1}(E_2; E_2, E_1 \cap E_2; E_3)$ since $\pi_n(E; E, E_1; E_1) = 0$. Furthermore, from the sequence

$$\to \pi_{n-1}(E_2, E_2) \to \pi_{n-1}(E_2; E_2, E_1 \cap E_2; E_3) \to \pi_{n-2}(E_1 \cap E_2, E_3) \to$$

we see that $\pi_{n-1}(E_2; E_2, E_1 \cap E_2; E_3) \cong \pi_{n-2}(E_1 \cap E_2, E_3)$, and hence $\pi_n(\boxplus) \cong \pi_{n-2}(E_1 \cap E_2, E_3)$. The boundary homomorphism, ∂, is merely the restriction to $S_+^{n-3} \times I$. The homomorphism, γ, is however somewhat more complicated and is best understood in terms of the construction of the sequence.

Recalling that a pair of pairs is a square and gives rise to a lattice, a triple of pairs gives a triple of squares and a new homotopy lattice. Thus consider the triple of pairs $\big((E, E_1), (E', E_1'), (E'', E_1'')\big)$ and the associated squares, $\square_1 = (E; E_1, E'; E_1')$, $\square_2 = (E; E_1, E''; E_1'')$ and $\square_3 = (E'; E_1', E''; E_1'')$. The associated lattice of homotopy groups is

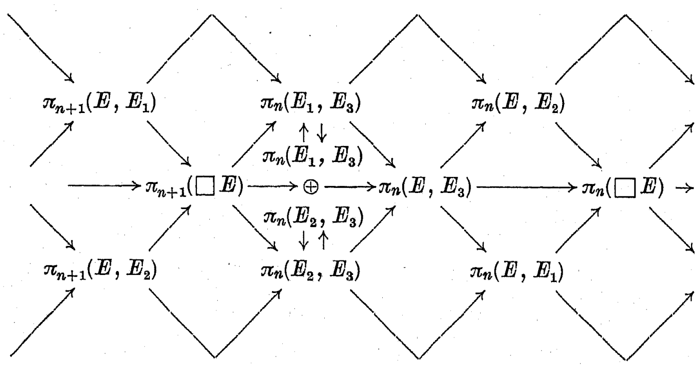
$$\begin{array}{ccccccc}
 & \downarrow & & \downarrow & & \downarrow & \\
\to \pi_n(E_1', E_1'') & \to & \pi_n(E_1, E_1'') & \to & \pi(E_1, E_1'') & \to & \\
 & \downarrow & & \downarrow & & \downarrow & \\
\to \pi_n(E', E'') & \to & \pi_n(E, E'') & \to & \pi_n(E, E') & \to & \\
 & \downarrow & & \downarrow & & \downarrow & \\
\to \pi_n(\square_3) & & \to \pi_n(\square_2) & & \to \pi_n(\square_1) & & \xrightarrow{\partial} \\
 & \downarrow & & \downarrow & & \downarrow &
\end{array}$$

where ∂ denotes, essentially, the restriction to $S_-^{n-2} \times I$.

Finally, for a square, say $\square E$, there is an associated commutative braid, [2, 11], which relates the exact sequences of the triples, (E, E_1, E_3), (E, E_2, E_3), the exact sequence of the square and a new exact sequence,

$$\to \pi_{n+1}(\square E) \to \pi_n(E_1, E_3) \oplus \pi_n(E_2, E_3) \to \pi_n(E, E_3) \to$$

which is defined via the braid.



The homotopy sequence of a fibration $i\colon F \to E \to B\colon p$ may be recovered from the homotopy sequence of the pair (E, F) by proving that $p_*\colon \pi_n(E, F) \to \pi_n(B)$ is an isomorphism. The homotopy sequences and lattices for pairs and squares of fibrations are recovered from the homotopy lattices of squares and cubes in a similar manner.

PROPOSITION 2. *If $i\colon (F, F_1) \to (E, E_1) \to (B, B_1)\colon p$ is a fibration then $p_*\colon \pi_n(E; E_1, F; F_1) \to \pi_n(B, B_1)$ is an isomorphism. Furthermore, if $i\colon \square F \to \square E \to \square B\colon p$ is a fibration then $p_*\colon \pi_n(⧉) \to \pi_n(\square B)$ is an isomorphism.*

This proposition follows easily from the result for a single fibration, the various exact sequences, and the five lemma. As a result of this proposition we have the following exact sequences of pairs and squares of fibrations, as well as the associated lattices which contain them.

$$\to \pi_n(F, F_1) \to \pi_n(E, E_1) \xrightarrow{p_*} \pi_n(B, B_1) \to \pi_{n-1}(F, F_1) \to$$

$$\to \pi_n(\square F) \to \pi_n(\square E) \xrightarrow{p_*} \pi_n(\square B) \to \pi_{n-1}(\square F) \to$$

EXAMPLES AND APPLICATIONS. Pairs of bundles, and hence fibrations, are often encountered via topological groups as described in the following proposition.

PROPOSITION 3. *Let $(G; G_1, G_2; G_3)$ be a triad of topological groups with G_1, G_2 and $G_1G_2 = \{g_1g_2 \mid g_1 \in G_1, g_2 \in G_2\}$ closed subgroups of G. If G_3 and G_1G_2 admit local cross-sections in G then, in the commutative lattice.*

$$\begin{array}{ccccc}
G_3 & \longrightarrow & G_1 & \longrightarrow & G_1/G_3 \\
\downarrow & & \downarrow & & \downarrow \\
G_2 & \longrightarrow & G & \longrightarrow & G/G_2 \\
\downarrow & & \downarrow & & \downarrow \\
G_2/G_3 & \longrightarrow & G/G_1 & \longrightarrow & G/G_1G_2
\end{array}$$

all vertical and horizontal lines are fiber bundles, as well as maps of fibrations.

Proof. Since G_3 is a closed subgroup of G which admits local cross-sections

$$G_3 \to G \to G/G_3$$

is a fiber bundle by the bundle structure theorem [12, p. 30]. This implies, by restriction, that

$$G_3 \to G_2 \to G_2/G_3$$

and

$$G_3 \to G_1 \to G_1/G_3 .$$

Observe that if G_1G_2 is a subgroup of G then $G_1G_2/G_1 = G_2/G_3$ and $G_1G_2/G_2 = G_1/G_3$. Since G_1G_2 is a closed subgroup of G admitting local cross-sections the bundle structure theorem implies that

$$G_2/G_3 = G_1G_2/G_1 \to G/G_1 \to G/G_1G_2$$

and

$$G_1/G_3 = G_1G_2/G_2 \to G/G_2 \to G/G_1G_2$$

are also fiber bundles.

To conclude the proof of the proposition it is sufficient to show that

$$G_1 \to G \to G/G_1$$

and

$$G_2 \to G \to G/G_2$$

also admit local cross-sections. Consider the first case, the second being the similar, and let $V\times W$ be a product neighborhood of $x_0 \in G/G_1$, with V and W open in G/G_1G_2 and G_2/G_3 and with local cross-sections, σ_V and σ_W, in

$$G_3 \to G_2 \to G_2/G_3$$

and $G_1G_2 \to G \to G/G_1G_2$, respectively, given by the previous bundle structure. A local cross-section at x_0, $\sigma\colon V\times W \to G$ is defined by $\sigma(x,y) = \sigma_V(x)\sigma_W(y)$.

A specific example of this is the fibrations of the Stiefel manifolds. Let $O(n)$ denote the orthogonal group with a fixed matrix representation. Corresponding to the two inclusions $i\colon R^{n-k}\to R^n$ and $j\colon R^{n-k}\to R^n$, defined by

$$i(x_1, \ldots, x_{n-k}) = (x_1, \ldots, x_{n-k}, o, \ldots, o)$$

and

$$j(x_1, \ldots, x_{n-k}) = (o, \ldots, o, x_1, \ldots, x_{n-k}),$$

there are two subgroups of $O(n)$ isomorphic to $O(n-k)$. These are denoted by $iO(n-k)$ and $jO(n-k)$. Thus, if $A\subset O(n-k)$,

$$i(A) = \begin{pmatrix} A & O \\ O & I\end{pmatrix}, \quad j(A) = \begin{pmatrix} I & O \\ O & A\end{pmatrix},$$

where I is the identity matrix. The Stiefel manifold of k-frames in n-space, $V_{n,k}$, is homeomorphic to both $O(n)/iO(n-k)$ and $O(n)/jO(n-k)$, while the Grassmann manifold of k-planes in n-space, $M_{n,k}$, is homeomorphic to both $O(n)/iO(k)jO(n-k)$ and $O(n)/iO(n-k)jO(k)$.

In the triad $\big(O(n); iO(n-k), jO(n-l); G_3\big)$ we note that $G_3 = iO(n-k)\cap jO(n-l)$ is isomorphic to $O(n-k-l)$. Thus we have a commutative diagram of fibrations

$$\begin{array}{ccccc}
O(n-k-l) & \longrightarrow & iO(n-k) & \longrightarrow & V_{n-k,l} \\
\downarrow & & \downarrow & & \downarrow \bar{i} \\
jO(n-l) & \longrightarrow & O(n) & \longrightarrow & V_{n,l} \\
\downarrow & & \downarrow & & \\
V_{n-l,k} & \xrightarrow{\bar{j}} & V_{n,k} & &
\end{array}$$

where $\bar{i}$ and $\bar{j}$ are inclusions, $O(m) = \{I\}$ if $m\leqslant 0$, and $V_{m,s} = O(m)$ if $s\geqslant m$. If $k+l\leqslant n$ $\bar{i}$ and $\bar{j}$ are the inclusions of the fiber in a fibration.

Indeed, if $k+l = n$, the common base space is $M_{n,k}$, which is homeomorphic to $M_{n,l}$.

Another simple example is the triad, for $l\geqslant k$,

$$\big(O(n); iO(n-k), iO(n-l); iO(n-l)\big),$$

giving the following lattice of fibrations.

$$\begin{array}{ccccc}
iO(n-l) & \longrightarrow & iO(n-k) & \longrightarrow & V_{n-k,l-k} \\
\downarrow & & \downarrow & & \downarrow \\
iO(n-l) & \longrightarrow & O(n) & \longrightarrow & V_{n,l} \\
\downarrow & & \downarrow & & \downarrow \\
* & \longrightarrow & V_{n,k} & \longrightarrow & V_{n,k}\,.
\end{array}$$

Thus, applying the homotopy theory of the previous sections, we see that $\pi_*\big(O(n); iO(n-k), iO(n-l); iO(n-l)\big)$ is isomorphic to $\pi_*(V_{n,k})$ and, for $k+l = n$, $\pi_*\big(O(n); iO(n-k), jO(n-l); O(n-k-l)\big)$ is isomorphic to $\pi_*(M_{n,k})$. Furthermore, the homotopy lattices may be employed to relate the various homotopy sequences of these fibrations.

An extremely elementary question one might ask concerns the notion of a section to a pair of fibrations. Whatever the definition, the existence of a section would imply the splitting of the homotopy sequence of the pair. One might ask if the existence of sections to the respective fibrations of the pair implies the splitting. The answer is, of course, no as may be seen by the triad, $\big(O(4); iO(3), jO(2); O(1)\big)$, in which both fibrations admit sections but the sequence of the pair does not split at π_2.

Another application is the recovery of the results of James in *Suspension of transgression* [4]. Briefly, let $i\colon F\to E\to B\colon p$ be a fiber space, ξ, $i\colon SF\to \Sigma E\to B\colon p'$ be its Whitney join, $\Sigma\xi$, with the trivial S^0 fibration, and $i'_\pm\colon SF_\pm \to B\colon p'_\pm$ be the restriction to the "northern and southern hemispheres" of the suspension, $\Sigma\xi_\pm$, having sections $\sigma_\pm$ determined by the north and south poles, respectively. From the homotopy lattice of the square of the fibration $(\Sigma\xi; \Sigma\xi_+, \Sigma\xi_-; \xi)$ we see that $i'_* E\partial(\alpha) = (\sigma_+)_*(\alpha) - (\sigma_-)_*(\alpha)$ where E is a suspension homomorphism and ∂ is the boundary homomorphism in the homotopy sequence of ξ. This follows by simply recognizing the suspension homomorphism in the homotopy of the square $(\Sigma F; \Sigma F_+, \Sigma F_-; F)$ and making use of the various isomorphisms resulting from the fact that all fibrations are over B. The corollaries of this result, [4], provide assistance in the following determination of the effect of the Bott map on the nonstable homotopy of $U(n)$.

Let $b'_n\colon U(n)\to \Omega^2 U(n+1)$ denote the Bott map constructed by Lundell [6]. As he remarks this map respects the inclusions of $U(n-k)$

in $U(n)$, i.e., the diagram

$$\begin{array}{ccc} b'_{n-k}\colon\ U(n-k) & \longrightarrow & \Omega^2 U(n-k+1) \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle \Omega^2 j} \\ b'_n\colon\ U(n) & \longrightarrow & \Omega^2 U(n+1) \end{array}$$

is commutative. By direct calculation, using Lundell's definition, it can shown that b'_n is actually an embedding. Thus we have a pair of fibrations

$$i\colon \big(\Omega^2 U(n-k+1),\ U(n-k)\big)\to\big(\Omega^2 U(n+1),\ U(n)\big)\to\big(\Omega^2 W_{n+1,k},\ W_{n,k}\big)\colon p$$

and hence a lattice relating the homotopy sequences of the fibrations and the Bott homomorphism induced by the Bott map.

By employing specific knowledge of the nonstable homotopy of $U(n)$, [5, 7, 8, 9, 10], Lundell's previous calculation [6], and the homotopy lattice, with $k=1$, we are able to determine the homomorphism

$$(b'_n)_*\colon\ \pi_i\big(U(n)\big)\to\pi_i\big(\Omega^2 U(n+1)\big)$$

for $i\leqslant 2n+6$. In order to present this information we include Table 1, compiled by Lundell from Matsunaga's papers. Complete responsibility for its accuracy, however, rests with the author. Note that the groups are cyclic except when $r=2$ or 6, where there is often a Z_2 summand. We let α denote the generator of this summand, when it occurs, and β the generator of the other summand.

Since $\pi_{2n+r}\big(\Omega^2 U(n+1)\big)$ is isomorphic to $\pi_{2(n+1)+r}\big(U(n+1)\big)$, $(b'_n)_*$ is a homomorphism from the group corresponding to n modulo 24 to the one corresponding to $n+1$ modulo 24, with r fixed. Table 2 gives the images of the generators under $(b'_n)_*$.

Given the Bott isomorphism, $i<2n-1$, and Lundell's calculation for $i=2n$, we use two basic facts. First, that the homomorphism induced on the homotopy of the spheres is $(n+1)E^2$, where E is the suspension homomorphism. Second, the results of Kervaire [5] and James on the homomorphism. Second, the results of Kervaire [5] and James on the transgression, i.e., boundary homomorphism, and Toda's table [13] of the generators of the stable homotopy of spheres. The technique of calculation is quite simple. We begin from Lundell's case, $r=0$, and work inductively using essentially ad hoc arguments concerning the possible image.

The case $r=1$ is trivial since, by Table 1, one of the groups is always zero.

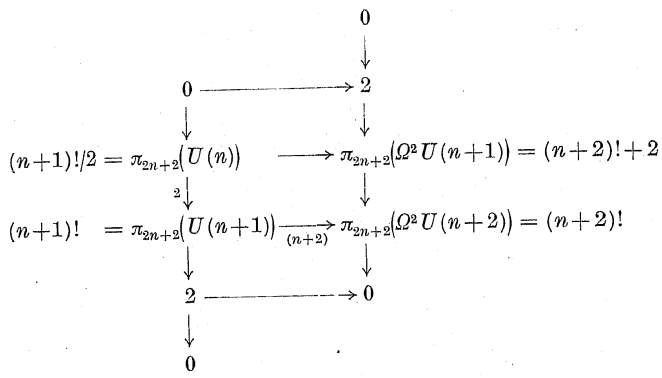
The case $r=2$ is more complicated since the groups depend upon $n \bmod 2$ and the calculation for $n=1 \bmod 2$ is related to the case $r=3$ where the groups depend upon $n \bmod 8$, by Table 1. The case $n=0 \bmod 2$ is easily determined via the following commutative diagram which occurs in the lattice of the fibration.

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow & & \\ & & 2 & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle i_2} & & \big\downarrow \\ (n+1)!+2 & = & \pi_{2n+2}\big(U(n)\big) & \longrightarrow & \pi_{2n+2}\big(\Omega^2 U(n+1)\big) = (n+2)!/2 \\ & & \big\downarrow{\scriptstyle p_2} & & \big\downarrow{\scriptstyle 2} \\ (n+1)! & = & \pi_{2n+2}\big(U(n+1)\big) & \xrightarrow[(n+2)]{} & \pi_{2n+2}\big(\Omega^2 U(n+2)\big) = (n+2)! \\ & & \downarrow & & \downarrow \\ & & 0 & \longrightarrow & 2 \\ & & & & \downarrow \\ & & & & 0 \end{array}$$

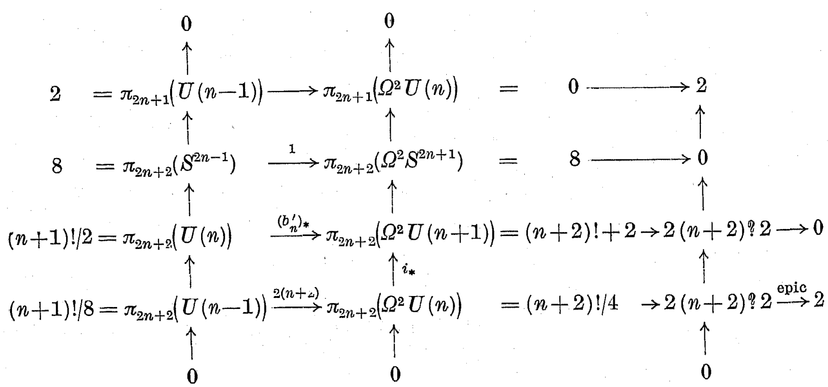
Table 1. $\pi_{2n+r}\big(U(n)\big)$

$k\geqslant 0$	$r=-2m$, $1\leqslant m\leqslant n$	$r=1-2m$, $1\leqslant m\leqslant n$	$r=0$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$
$n=1$	0	∞	0	0	0	0	0	0	0
$n=2$	0	∞	$2!$	2	12	2	2	3	15
$n=3$	0	∞	$3!$	0	$4!/2$	3	$5!/4$	4	60
$n=4$	0	∞	$4!$	2	$5!+2$	4	$6!/12$	4	$1680+2$
$n=24(k+1)$	0	∞	$n!$	2	$(n+1)!+2$	24	$(n+2)!/2$	2	$(n+3)!/6+2$
$n=24(k+1)+1$	0	∞	$n!$	0	$(n+1)!/2$	2	$(n+2)!/6$	2	$(n+3)!/24+2$
$n=24(k+1)+2$	0	∞	$n!$	2	$(n+1)!+2$	2	$(n+2)!/24$	3	$(n+3)!/4$
$n=24(k+1)+3$	0	∞	$n!$	0	$(n+1)!/2$	3	$(n+2)!/4$	4	$(n+3)!/12$
$n=24(k+1)+4$	0	∞	$n!$	2	$(n+1)!+2$	4	$(n+2)!/12$	4	$(n+3)!/3+2$
$n=24k+5$	0	∞	$n!$	0	$(n+1)!/2$	4	$(n+2)!/3$	6	$(n+3)!/8+2$
$n=24k+6$	0	∞	$n!$	2	$(n+1)!+2$	6	$(n+2)!/8$	0	$(n+3)!/12$
$n=24k+7$	0	∞	$n!$	0	$(n+1)!/2$	0	$(n+2)!/12$	8	$(n+3)!/6$
$n=24k+8$	0	∞	$n!$	2	$(n+1)!+2$	8	$(n+2)!/6$	6	$(n+3)!/2+2$
$n=24k+9$	0	∞	$n!$	0	$(n+1)!/2$	6	$(n+2)!/2$	2	$(n+3)!/24+2$
$n=24k+10$	0	∞	$n!$	2	$(n+1)!+2$	2	$(n+2)!/24$	0	$(n+3)!/12$
$n=24k+11$	0	∞	$n!$	0	$(n+1)!/2$	0	$(n+2)!/12$	12	$(n+3)!/4$
$n=24k+12$	0	∞	$n!$	2	$(n+1)!+2$	12	$(n+2)!/4$	4	$(n+3)!/3+2$
$n=24k+13$	0	∞	$n!$	0	$(n+1)!/2$	4	$(n+2)!/3$	2	$(n+3)!/24+2$
$n=24k+14$	0	∞	$n!$	2	$(n+1)!+2$	2	$(n+2)!/24$	3	$(n+3)!/4$
$n=24k+15$	0	∞	$n!$	0	$(n+1)!/2$	3	$(n+2)!/4$	8	$(n+3)!/6$
$n=24k+16$	0	∞	$n!$	2	$(n+1)!+2$	8	$(n+2)!/6$	2	$(n+3)!/6+2$
$n=24k+17$	0	∞	$n!$	0	$(n+1)!/2$	2	$(n+2)!/6$	6	$(n+3)!/8+2$
$n=24k+18$	0	∞	$n!$	2	$(n+1)!+2$	6	$(n+2)!/8$	0	$(n+3)!/12$
$n=24k+19$	0	∞	$n!$	0	$(n+1)!/2$	0	$(n+2)!/12$	4	$(n+3)!/12$
$n=24k+20$	0	∞	$n!$	2	$(n+1)!+2$	4	$(n+2)!/12$	12	$(n+3)!+2$
$n=24k+21$	0	∞	$n!$	0	$(n+1)!/2$	12	$(n+2)!$	2	$(n+3)!/24+2$
$n=24k+22$	0	∞	$n!$	2	$(n+1)!+2$	2	$(n+2)!/24$	0	$(n+3)!/12$
$n=24k+23$	0	∞	$n!$	0	$(n+1)!/2$	0	$(n+2)!/12$	24	$(n+3)!/2$

The case $n = 1 \bmod 2$ gives the following diagram



From this it is easy to see that the $(n+2)!$ component of the image is generated by $2(n+2)$. It is more difficult to find the two component of the image. A typical case is $n = 7 \bmod 8$ where we have the following commutative diagram relating the two primary components. For simplicity a number will indicate the two primary component of the cyclic group of that order. The symbol $a ? b$ denotes an extension of b and a.



If the two component of $(b'_n)_*$ is zero the cokernel is $2(n+2)+2$. Thus i_* must not have a two component, by exactness of the bottom horizontal sequence. This is impossible since the cokernel of i_* must have order 8. Thus $(b'_n)_*$ takes β to $\alpha \oplus 2(n+2)\beta$ as indicated in Table 2 if $n = 7 \bmod 8$.

Table 2. $(b'_n)_*\colon \pi_{2n+r}(U(n)) \to \pi_{2(n+1)+r}(U(n+1))$

$k \geqslant 0$	$r = -2m$, $1 \leqslant m \leqslant n$	$r = 1-2m$, $1 \leqslant m \leqslant n$	$r = 0$	$r = 1$	$r = 2$: α	$r = 2$: β	$r = 3$	$r = 4$	$r = 5$	$r = 6$: α	$r = 6$: β
$n = 1$	0	1	0	0	0	0	0	0	0	0	0
$n = 2$	0	1	3	0	0	2	0	0	0	0	4β
$n = 3$	0	1	4	0	0	$\alpha+10\beta$	0	3	1	0	1β
$n = 4$	0	1	5	0	0	3β	1	28	3	α	3β
$n = 24(k+1)$	0	1	$n+1$	0	0	$\frac{n+2}{2}\beta$	1	$(n+3)/3$	0	α	$\frac{n+4}{4}\beta$
$n = 24(k+1)+1$	0	1	$n+1$	0	0	$\alpha\oplus 2(n+2)\beta$	0	$(n+3)/4$	0	0	$6(n+4)\beta$
$n = 24(k+1)+2$	0	1	$n+1$	0	0	$\frac{n+2}{2}\beta$	0	$6(n+3)$	0	0	$\frac{n+4}{3}\beta$
$n = 24(k+1)+3$	0	1	$n+1$	0	0	$\alpha\oplus 2(n+2)\beta$	0	$(n+3)/3$	1	0	$4(n+4)\beta$
$n = 24(k+1)+4$	0	1	$n+1$	0	0	$\frac{n+2}{2}\beta$	1	$4(n+3)$	3	α	$\frac{3(n+4)}{8}\beta$
$n = 24k+5$	0	1	$n+1$	0	0	$\alpha\oplus 2(n+2)\beta$	3	$3(n+3)/8$	0	0	$\frac{2(n+4)}{3}\beta$
$n = 24k+6$	0	1	$n+1$	0	0	$\frac{n+2}{2}\beta$	0	$2(n+3)/3$	0	0	$2(n+4)\beta$
$n = 24k+7$	0	1	$n+1$	0	0	$\alpha\oplus 2(n+2)\beta$	0	$2(n+3)$	3	0	$\alpha\oplus 3(n+4)\beta$
$n = 24k+8$	0	1	$n+1$	0	0	$\frac{n+2}{2}\beta$	3	$3(n+3)$	0	α	$\frac{n+4}{12}\beta$
$n = 24k+9$	0	1	$n+1$	0	0	$\alpha\oplus 2(n+2)\beta$	0	$(n+3)/12$	0	0	$2(n+4)\beta$
$n = 24k+10$	0	1	$n+1$	0	0	$\frac{n+2}{2}\beta$	0	$2(n+3)$	0	0	$3(n+4)\beta$
$n = 24k+11$	0	1	$n+1$	0	0	$\alpha\oplus 2(n+2)\beta$	0	$3(n+3)$	1	0	$\frac{4(n+4)}{3}\beta$
$n = 24k+12$	0	1	$n+1$	0	0	$\frac{n+2}{2}\beta$	1	$4(n+3)/3$	1	α	$\frac{n+4}{8}\beta$
$n = 24k+13$	0	1	$n+1$	0	0	$\alpha\oplus 2(n+2)\beta$	1	$(n+3)/8$	0	0	$6(n+4)\beta$
$n = 24k+14$	0	1	$n+1$	0	0	$\frac{n+2}{2}\beta$	0	$6(n+3)$	0	0	$\frac{2(n+4)}{3}\beta$
$n = 24k+15$	0	1	$n+1$	0	0	$\alpha\oplus 2(n+2)\beta$	0	$2(n+3)/3$	1	0	$\alpha\oplus(n+4)\beta$
$n = 24k+16$	0	1	$n+1$	0	0	$\frac{n+2}{2}\beta$	1	$n+3$	0	α	$\frac{3(n+4)}{4}\beta$
$n = 24k+17$	0	1	$n+1$	0	0	$\alpha\oplus 2(n+2)\beta$	0	$3(n+3)/4$	0	0	$\frac{2(n+4)}{3}\beta$
$n = 24k+18$	0	1	$n+1$	0	0	$\frac{n+2}{2}\beta$	0	$2(n+3)/3$	0	0	$(n+4)\beta$
$n = 24k+19$	0	1	$n+1$	0	0	$\alpha\oplus 2(n+2)\beta$	0	$n+3$	3	0	$12(n+4)\beta$
$n = 24k+20$	0	1	$n+1$	0	0	$\frac{n+2}{2}\beta$	3	$12(n+3)$	1	α	$\frac{n+4}{24}\beta$
$n = 24k+21$	0	1	$n+1$	0	0	$\alpha\oplus 2(n+2)\beta$	1	$(n+3)/24$	0	0	$2(n+4)\beta$
$n = 24k+22$	0	1	$n+1$	0	0	$\frac{n+2}{2}\beta$	0	$2(n+3)$	0	0	$6(n+4)\beta$
$n = 24k+23$	0	1	$n+1$	0	0	$\alpha\oplus 2(n+2)\beta$	0	$6(n+3)$	1	0	$\alpha\oplus\frac{(n+1)}{3}\beta$

The remainder of the table is computed in much the same fashion making use of the lattice to relate computations at one stage with those at another, e.g., the extension question when $n = 7 \bmod 8$ above.

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On the scarcity of tame disks in certain wild cells

by

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Abstract. If K is a k-cell topologically embedded in Euclidean n-space E^n and P is a $(k-1)$-dimensional polyhedron topologically embedded in K, does there exist a re-embedding h of P in K such that $h(P)$ is locally tame relative to E^n and h is close to the inclusion map? In case $k \leq 2$ the answer is known to be affirmative. This paper aims to provide relatively simple examples, each cell K being locally tame modulo a Cantor set, to indicate that the question has a negative answer whenever $3 \leq k < n$.

Introduction. R. H. Bing proved that each disk in Euclidean 3-space E^3 contains many tame arcs [5], and using this, Martin showed that the disk contains tame arcs that pass through certain boundary points [7]. Seebeck [10] proved a similar theorem for disks in E^n $(n \geq 5)$, after which Sher established the analogue for disks in E^4 [11]. Their results are summarized in the following statement: if D is a disk in E^n $(n \geq 3)$, A an arc in D such that $A \cap \partial D$ is contained in ∂A, and $\varepsilon > 0$, then there exists an ε-homeomorphism h of D onto itself such that $h(A)$ is a tame arc. It also follows from Bing's work that for each 3-cell C in E^3 and each disk D in C such that $D \cap \partial C = \partial D$ and D is locally tame at each point of $\operatorname{Int} D$, there exist arbitrarily small homeomorphisms h of C onto itself such that $h(D)$ is a tame disk.

Our purpose here is to indicate, by exhibiting peculiar embeddings of cells in E^n, that a generalization of these results is false. For $3 \leq k \leq n$ and $n \geq 4$, we find (see Theorems 5.2 and 5.4) a k-cell K in E^n and a disk D in K such that for any sufficiently small homeomorphism h of K to itself, $h(D)$ is wildly embedded.

The cells constructed to satisfy Theorems 5 2 and 5.4 appear somewhat simple, each being locally tame modulo a Cantor set. According to the results of Section 6, such a Cantor set, whether viewed as a subset of the Euclidean space or of the embedded cell, must be wildly embedded, and most wild Cantor sets found in the literature lack the complications required for occurring in these examples.

In a sense the essence of the work here consists of the identification of the suitable complications, which are implicitly prescribed by the definition of *special defining sequence*, found in Section 4. The significance