On the compactification of closure algebras

by

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Abstract. This paper deals with the category of closure algebras and continuous homomorphisms. The latter correspond — unlike those used by most authors — formally to continuous mappings of General Topology. We present first a rather natural notion of compactness for closure algebras (due to G. Birkhoff) and then define what compactifications are to be. The existence of such compactifications for any closure algebra is proved, the construction being based on a modification of the Stone space of the algebra under consideration. We show also that, in general, a given closure algebra has many non-equivalent compactifications. A special type of compactification is then examined in more detail, exhibiting connections to the compactification theory of General Topology, particularly to the Alexandroff one-point compactification.

INTRODUCTION

Closure algebras have been investigated by several authors, chiefly by Sikorski (see [4] and the references given there). Quite a number of topological concepts have been studied with respect to the possibility of applying them to closure algebras. It seems that not much has been done to carry over questions of compactness, although a rather obvious definition of compact closure algebras has been proposed by G. Birkhoff as early as 1948 in [1].

In an earlier unpublished paper the author studied in some detail the algebraic side of the situation. In this paper the existence problem for compactifications is solved by the description of a general construction. The first part sums up the definitions and machinery used, the second part gives the construction and then treats a special case which can be brought in connection with the compactification theory for topological spaces.

I wish to put on record my indebtness to P. Wilker who drew my attention to the subject and supported me constantly with his valuable suggestions.

FIRST PART. CLOSURE ALGEBRAS

1.1. Closure algebras, continuous morphisms

1.1.1. In the following, Roman capitals $A, B, C, \ldots$ with or without subscripts will denote Boolean algebras (abbreviated BA), the operations of join, meet and complementation will be given by $\vee$, $\wedge$, and $'$, respectively. 0 will stand for the zero and 1 for the unit element of any BA in question. For the theory of BA's see [7] and [2]. A closure algebra (abbreviated CA) is a pair (A, ϱ) consisting of a BA A and a map $\varrho: A \to A$ satisfying the four well-known Kuratowski axioms. Open and closed elements are defined as usual. An order relation on the set of all closure operators definable on a given BA A is introduced by $\varrho \leqslant \sigma \leftrightarrow \varrho x \leqslant \sigma x$ for all $x \in A$.

1.1.2. Let (A, ϱ) be a CA. A filter (filter base) is called *proper*, if it does not contain 0. A filter is called *closed*, if it is generated by a filter base consisting of closed elements. Filters and filter bases will be denoted by $F, G, \ldots$; $F(a)$ will stand for the principal filter generated by a. An order relation on the set of all filters on a given BA A is introduced by $F_1 \leqslant F_2 \leftrightarrow F_1 \supseteq F_2$.

LEMMA 1. 1° *To every proper filter on A there is a smaller minimal filter* (Remark. A minimal filter is of course the same as an ultrafilter). 2° *To every proper closed filter on (A, ϱ) there is a smaller minimal closed filter.*

In the sequel, "minimal closed filter" will be abbreviated MCF, or ϱ-MCF, if the closure operator ϱ is to be emphasized.

1.1.3. In order to make a category out of the class of closure algebras, appropriate morphisms must be chosen. This can be done in different ways: In his papers [4], [5] and [6], Sikorski uses Boolean homomorphisms formally corresponding to closed mappings of General Topology; i.e. satisfying the relation $\tau(fx) \leqslant f(\varrho x)$ for all $x \in A$, where f: $(A, \varrho) \to (B, \tau)$. Several fundamental problems, e.g. the introduction of a quotient topology, or the formation of coproducts, can be solved only imposing strong additional conditions on the CA's in question. Wilker [10] succeeded in solving these problems introducing a generalized notion of closure operator (the closure of an element is not another element, but a filter). His morphisms, too, correspond formally to closed morphisms of General Topology.

We now make the following definition: Let (A, ϱ) and (B, σ) be two CA's. A Boolean homomorphism f: $(A, \varrho) \to (B, \sigma)$ is called *continuous*, if $f(\varrho x) \leqslant \sigma(fx)$ holds for all $x \in A$. Accordingly, continuous homomorphisms correspond formally to continuous mappings of General Topology. They allow, in a natural way, the formation of quotient topologies and of products. However, they don't seem to be interpretable in Stone spaces, in contrast to Sikorski's "closed" morphisms. Naturally, a continuous homomorphism f: $(A, \varrho) \to (B, \sigma)$ is called a *homeomorphism*, if 1° f is a Boolean isomorphism from A onto B and 2° f^{-1} is continuous too. Let us look briefly at the quotient topology: If (A, ϱ) is a CA, B a BA and f: $A \to B$ is a Boolean epimorphism, then the closure operator σ on B defined by the relation $\sigma(fx) := f(\varrho x)$ for all $fx \in B$ is easily seen to be well-defined and to satisfy the Kuratowski axioms. Moreover, $\sigma \leqslant \tau$ for any closure operator τ on B making f continuous. In the case $B = A/J$, where $J \lhd A$ is an ideal and A/J the corresponding quotient algebra, the quotient topology σ is given by $\sigma[x] = [\varrho x]$, the brackets denoting equivalence classes with respect to J.

1.2. Compact closure algebras

A CA (A, ϱ) will be called *compact*, if every proper closed filter on (A, ϱ) has a nontrivial (i.e. different from 0) lower bound. Obviously this definition is a direct generalization of a compactness postulate used in General Topology ("every proper closed filter in a compact topological space is fixed"). The definition was proposed by Birkhoff 1948 in [1], but it doesn't seem to have been investigated in more detail up to now. Of course, "filter" may be replaced by "filter base"; most constructions will be carried out by means of filter bases rather than filters.

An element a of a CA (A, ϱ) will be called *compact*, if the quotient algebra $A/J(a')$ provided with the quotient topology is compact — $J(a')$ being the principal ideal generated by a'. We define 0 to be compact in any CA. These definitions can also be seen to correspond to the usual definition of a compact subset of a topological space.

For an example of a compact CA consider an arbitrary BA $A \neq \{0, 1\}$ and let a closure operator ϱ be defined by $\varrho x = x \vee a$ $(x \neq 0)$ and $\varrho 0 = 0$, a being different from zero. Of course, any compact topological space may be interpreted as a compact CA. The following lemma provides the basis for the constructions of the second part:

LEMMA 2. *Let (A, ϱ) be a compact* CA. *Then, for every closed element $x \neq 0$, there exists a closed element $y \neq 0$ with $y \leqslant x$, which is minimal relative to $\leqslant$.*

Proof. Let $0 \neq x = \varrho x$. Consequently $F(x)$ is a proper closed filter. By Lemma 1 there exists a MCF M containing $F(x)$. (A, ϱ) being compact, there is $y \neq 0$ with $y \leqslant z$ for all $z \in M$. Hence $F(\varrho y) \leqslant M$, $F(\varrho y)$ proper and closed. From the minimality of M, $M = F(\varrho y)$, and from $M \leqslant F(x)$, $\varrho y \leqslant x$. If now $0 \neq v = \varrho v \leqslant \varrho y$, $v \in A$, then $F(\varrho v)$ is proper, closed and contains $F(\varrho y)$. This yields $F(\varrho v) = F(\varrho y)$ and $\varrho v = \varrho y$. Thus ϱy has the desired property. ■

1.3. Closure spaces

1.3.1. In [8], [9], Stone discusses his fundamental representation theorem for Boolean algebras: Every Boolean algebra is isomorphic to a field of sets, e.g. the field of all open-closed subsets of a compact totally disconnected Hausdorff space. For topological terminology and general background, the reader is referred to Kelley [3]. From the many possibilities to construct explicitely the Stone space of a given BA, we choose for this paper the following one: Let A be the given BA and define $X(A)$ to be the set of all ultrafilters on A. A map h from A into the power set of $X(A)$ is given by $hx := \{U \in X(A);\ x \in U\}$ for every $x \in A$. It is easily verified that h is a homomorphism from A into the field of all subsets of $X(A)$. The set $X(A)$ is topologized by taking as a closed subbase the set $\mathcal{C} := \{hx;\ x \in A\}$, the resulting topology is T_2, compact and totally disconnected and $\mathcal{C}$ consists exactly of the open-closed subsets of $X(A)$. Restricting the range of h to $\mathcal{C}$, h becomes a isomorphism between A and $\mathcal{C}$. Stone's representation theory enables us to interpret algebraic Boolean concepts in a topological way. In the following this will be done constantly, particularly for filters. Filters on A and closed subsets of $X(A)$ are in one-to-one correspondence: If $F \subseteq A$ is a filter, then $F^* := \bigcap \{hx;\ x \in F\}$ is a closed subset of $X(A)$ and if $F^* \subseteq X(A)$ is closed, then $F := \{x \in A;\ F^* \subseteq hx\}$ is a filter on A.

1.3.2. It is now our aim to modify the structure of Stone spaces in such a way that in the case of a *CA* (A, ϱ) the closure operator ϱ also appears in $X(A)$. First, we need a definition: A CA (A, ϱ) is called a *sub-closure-algebra* (abbreviated sub-CA) of a CA (B, σ) if 1° the BA A is a subalgebra of B and 2° $\sigma x = \varrho x$ for all $x \in A$. For the following, let (A, σ^*) be a fixed CA. The BA A has the Stone space $X(A)$, denoted by X. The pair (X, τ) — with τ denoting the Stone topology — may now be interpreted as a CA itself. Let $\mathcal{C}$ be the field of all open-closed subsets of X, $\mathcal{C} = \{C_k\}$ for a suitable index set K. The BA's A and $\mathcal{C}$ are isomorphic, which defines a closure operator on $\mathcal{C}$. This latter closure operator will also be denoted by σ^*. $\mathcal{C}$ together with σ^* satisfies all requirements for a partial closure operator for the whole space X; in fact:

$\sigma^* C \supseteq C$ for every $C \in \mathcal{C}$.

$\emptyset,\ X \in \mathcal{C}$ and $\sigma^*\emptyset = \emptyset$, $\sigma^* X = X$.

Let C_1 and $C_2 \in \mathcal{C}$: then $C_1 \cup C_2 \in \mathcal{C}$ and $\sigma^*(C_1 \cup C_2) = \sigma^* C_1 \cup \sigma^* C_2$.

$\sigma^*\sigma^* C = \sigma^* C$ for every $C \in \mathcal{C}$.

Thus the formula $\sigma Z := \bigcap \{\sigma^* C \in \mathcal{C};\ Z \subseteq \sigma^* C\}$ (for $Z \subseteq X$) defines a closure operator σ on X. σ coincides with σ^* on $\mathcal{C}$, and, incidentally, is the coarsest closure operator with this property. The family $\{\sigma^* C;\ C \in \mathcal{C}\}$ is a closed base for σ, hence $\sigma \geqslant \tau$. If σ^* is the discrete operator on A, we have $\sigma = \tau$, if σ^* is the indiscrete operator, the space (X, σ) will also be indiscrete. Because of $\sigma \geqslant \tau$, σ determines a compact topology on X; in the case $\sigma > \tau$, this topology is not T_2 any longer (for any compact T_2 topology is simultaneously a finest compact and a coarsest T_2 topology).

In the sequel we shall consider exclusively the space (X, σ); it will be called the *closure space* of the CA (A, σ^*). Interpreted as a CA, the space (X, σ) contains the sub-CA $(\mathcal{C}, \sigma^*)$ which is homeomorphic to (A, σ^*). Certain concepts, such as "closed filter", may now be studied topologically in (X, σ) instead of algebraically in (A, σ^*). Let F be a closed filter in (A, σ^*) and $\mathcal{F}$ its homeomorphic image in $(\mathcal{C}, \sigma^*)$. Consider the set $Z := \bigcap \{C \subseteq X;\ C \in \mathcal{F}\} \subseteq X$. Clearly, Z is σ-closed, for it is equal to the meet of all sets belonging to the closed filter base. If F is a MCF in (A, σ^*), then Z is a minimal (with respect to $\subseteq$) σ-closed subset of X: (X, σ) being compact, Z is not empty, and if there were a σ-closed nonempty Z^* properly contained in Z, the filter base $\mathcal{F}^*$ defined by $\mathcal{F}^* := \{\sigma^* C \in \mathcal{C};\ Z^* \subseteq \sigma^* C\}$ would generate a proper closed filter on the CA $(\mathcal{C}, \sigma^*)$ containing $\mathcal{F}$ properly. Conversely, the construction just above assigns to every minimal σ-closed subset of X a MCF on $(\mathcal{C}, \sigma^*)$. In general these minimal closed sets are not singletons. This is the case e.g. if σ^* is the discrete operator (since then $\sigma = \tau$).

It is not clear if and how a continuous homomorphism $f: (A, \eta) \to (B, \vartheta)$ might be brought into connection with a particular point set mapping between the corresponding closure spaces. The question under what conditions the relation between σ-closed subsets of X and σ^*-closed filters on $\mathcal{C}$ becomes one-to-one (as it is the case for BA's in the Stone theory) remains also open. Nevertheless, we need only the relations given above.

SECOND PART. COMPACTIFICATIONS

2.1. Definition and general construction

A CA (B, σ) will be called a *compactification* of a CA (A, ϱ), if the following three conditions are satisfied:

1° (B, σ) is compact,

2° (A, ϱ) is a sub-CA of (B, σ),

3° if (C, τ) is another CA satisfying 1° and 2° and if (C, τ) is a sub-CA of (B, σ), then (C, τ) is homeomorphic to (B, σ).

The meaning of 1° and 2° is clear. 3° replaces — compared with the compactification theory in General Topology — the postulate that the space considered should be dense in its compactification.

In the following, let (A, σ^*) be the CA to be compactified and (X, σ) its closure space. As shown above, (X, σ) contains the sub-CA $(\mathcal{C}, \sigma^*)$

homeomorphic to (A, σ^*). (X, σ) satisfies postulates 1° and 2° above, but — in general — not 3°. We are now going to construct a suitable subalgebra of X which will be "small" enough to fulfill 3° too, provided with an appropriate topology.

2.1.1. The family $\mathfrak{D}_a$. By Lemma 2 the compact CA (X, σ) contains minimal (with respect to $\leqslant$) closed elements — or, equivalently, the topological space (X, σ) contains minimal σ-closed subsets. Let $\mathfrak{H}'$ denote the family of all such sets and define $\mathfrak{H}$ to be $\mathfrak{H}' - \mathfrak{C}$. We write $\mathfrak{H} = \{H_m;\ m \in M\}$, with a suitable index set M. For every $H_m \in \mathfrak{H}$, we choose an arbitrary nonempty subset $D_m \subseteq H_m$ (of course, $D_m = H_m$ is also possible). Let Φ be the set of all possible choices of this kind. From now on we shall consider a fixed choice $a \in \Phi$, denoting by $\mathfrak{D}_a$ the set of all D_m $(m \in M)$ chosen by a. We list some properties of the members of $\mathfrak{D}_a$:

— $D_m \notin \mathfrak{C}$ for all $m \in M$.

Reason. Suppose $D_m \in \mathfrak{C}$ for some $m \in M$. This implies $\sigma D_m \in \mathfrak{C}$, since σ coincides with σ^* on $\mathfrak{C}$. But we have (see 2.1.3) $\sigma D_m = H_m$, and $H_m \notin \mathfrak{C}$ by definition.

— $D_i \cap D_k = \emptyset$ for $i, k \in M$, $i \neq k$.

Reason. We have $D_i \subseteq H_i$, $D_k \subseteq H_k$ and $H_i \cap H_k = \emptyset$, for otherwise, since $H_i \cap H_k$ is closed and contained both in H_i and H_k, $H_i \cap H_k = H_i = H_k$, the H's being minimal closed sets.

— $D_i \cap D_k' = D_i$ for $i, k \in M$, $i \neq k$.

Follows from the above assertion.

— Z closed, $Z \cap D_i \neq \emptyset \rightarrow Z \supseteq D_i$.

Reason. $Z \cap D_i \neq \emptyset \rightarrow Z \cap H_i \neq \emptyset$. $Z \cap H_i$ is closed and contained in H_i, thus $Z \cap H_i = H_i$, the H's being minimal closed. Consequently $Z \supseteq H_i \supseteq D_i$.

— Equations of type $C = D_1 \cup \ldots \cup D_n$, $C \in \mathfrak{C}$, are not possible.

Reason. Assume $C = D_1 \cup \ldots \cup D_n$. The corresponding H_i are disjoint σ-closed sets (see above). So there exists $\sigma^* C_0$ satisfying, say, $H_1 \subseteq \sigma^* C_0$ and $H_i \cap \sigma^* C_0 = \emptyset$, $i = 2, \ldots, n$. Thus $C \cap \sigma^* C_0 = D_1$, a contradiction, since $C \cap \sigma^* C_0 \in \mathfrak{C}$.

— If $\sigma C_1 \cup D_{11} \cup \ldots \cup D_{1r} = \sigma C_2 \cup D_{21} \cup \ldots \cup D_{2s}$, then $\sigma C_1 = \sigma C_2$, $r = s$, and apart from order, the same D's occur on both sides.

Reason. $\sigma C_1 = \sigma C_2$ follows from the two preceding assertions, the rest from the disjointness of the D's.

2.1.2. The BA M_a. We define M_a to be the BA generated by the family $\mathfrak{C} \cup \mathfrak{D}_a$ in the power set of X. Applying the properties of the D_m derived above to the general expression of an element built up from generators, one obtains the following characterization of the elements of M_a:

Every $Z \in M_a$ is representable — not uniquely in general — as a finite join of terms of the following five basic types:

a) $C \in \mathfrak{C}$,

b) $D \in \mathfrak{D}_a$,

c) $D_1' \cap \ldots \cap D_n'$, $D_i \in \mathfrak{D}_a$, $n \in N$,

d) $C \cap D$, $C \in \mathfrak{C}$, $D \in \mathfrak{D}_a$,

e) $C \cap D_1' \cap \ldots \cap D_n'$, $C \in \mathfrak{C}$, $D_i \in \mathfrak{D}_a$, $n \in N$.

All meets occurring may be assumed to be nonempty. We shall assume further that elements of type e) are minimal with respect to the number of D's involved.

2.1.3. We now investigate how σ operates on elements of M_a. Clearly it suffices to know the σ-closures of the five basic types. This causes no difficulty for types a), b) and d): We have $\sigma C = \sigma^* C$ by definition of σ; $\sigma D_m = H_m$ (for $D_m \subseteq H_m$ implies $\sigma D_m \subseteq \sigma H_m = H_m$ and H_m is minimal closed); $\sigma(C \cap D_m) = H_m$ (for $C \cap D_m \subseteq D_m$, and then the preceding argument applies).

Type c): Consider $D_1 \in \mathfrak{D}_a$. We have $\sigma D_1' = \bigcap \{\sigma^* C;\ D_1' \subseteq \sigma^* C\}$. Let $\sigma^* C \supseteq D_1'$. Certainly $\sigma^* C \neq D_1'$, for otherwise $D_1 = (\sigma^* C)' \in \mathfrak{C}$. Consequently $\sigma^* C \cap D_1 \neq \emptyset$ and $\sigma^* C \supseteq D_1$ by 2.1.1. Together, $\sigma^* C \supseteq D_1 \cup D_1' = X$. We conclude $\sigma D_1' = X$.

We proceed by induction. Clearly, we have

$$(D_1' \cap \ldots \cap D_n') \cup D_n = D_1' \cap \ldots \cap D_{n-1}'.$$

Applying σ,

$$\sigma(D_1' \cap \ldots \cap D_n') \cup \sigma D_n = \sigma(D_1' \cap \ldots \cap D_{n-1}') = X$$

by induction, or

$$\sigma(D_1' \cap \ldots \cap D_n') \cup H_n = X.$$

This implies

$$\sigma(D_1' \cap \ldots \cap D_n') \supseteq H_n'$$

and

$$\sigma^2(D_1' \cap \ldots \cap D_n') = \sigma(D_1' \cap \ldots \cap D_n') \supseteq \sigma H_n' = X,$$

the last equality being justified by the fact that the H's are special D's. Thus $\sigma(D_1' \cap \ldots \cap D_n') = X$.

Type e): Consider $C \cap D_1'$. We have

$$\sigma(C \cap D_1') = \bigcap \{\sigma^* C;\ C \cap D_1' \subseteq \sigma^* C\}.$$

Let $\sigma^* C_1 \supseteq C \cap D_1'$. We may assume $C \cap D_1 \neq \emptyset$, for otherwise $C \cap D_1' = C$, contradicting the minimality convention. Thus $\sigma C \supseteq H_1 \supseteq D_1$. Moreover, we have $\sigma^* C_1 \supseteq D_1$: If not, $\sigma^* C_1 \cap D_1 = \emptyset$ and $\sigma^* C_1 \subseteq D_1'$, thus

$$C \cap D_1' = C \cap D_1' \cap \sigma^* C_1 = C \cap \sigma^* C_1 \in \mathfrak{C},$$

contradicting again the minimality convention. We conclude

$$\sigma^* C_1 \supseteq (C \cap D_1') \cup D_1 = C \cup D_1 .$$

Applying σ we get

$$\sigma\sigma^* C_1 = \sigma^* C_1 \supseteq \sigma(C \cup D_1) = \sigma C \cup \sigma D_1 = \sigma C \cup H_1 = \sigma C .$$

Of course $\sigma C \supseteq C \cap D_1'$. So σC is the minimal set participating in the formation of $\sigma(C \cap D_1')$, thus $\sigma(C \cap D_1') = \sigma C$.

We proceed by induction again. Clearly, we have

$$(C \cap D_1' \cap \ldots \cap D_n') \cup D_n = (C \cap D_1' \cap \ldots \cap D_{n-1}') \cup D_n .$$

Applying σ,

$$\sigma(C \cap D_1' \cap \ldots \cap D_n') \cup H_n = \sigma C \cup H_n$$

by induction, or

$$\sigma C - \sigma(C \cap D_1' \cap \ldots \cap D_n') \subseteq H_n .$$

Similarly,

$$\sigma C - \sigma(C \cap D_1' \cap \ldots \cap D_n') \subseteq H_1 ,$$

say. Thus

$$\sigma C - \sigma(C \cap D_1' \cap \ldots \cap D_n') \subseteq H_1 \cap H_n = \emptyset$$

or equivalently

$$\sigma C \subseteq \sigma(C \cap D_1' \cap \ldots \cap D_n') .$$

The reverse inclusion follows from the monotony of σ, so equality holds.

Summing up, we have for the five basic types:

a) $\sigma C = \sigma^* C$,

b) $\sigma D_m = H_m$,

c) $\sigma(D_1' \cap \ldots \cap D_n') = X$, $n \in N$,

d) $\sigma(C \cap D_m) = H_m$,

e) $\sigma(C \cap D_1' \cap \ldots \cap D_n') = \sigma C$, $n \in N$.

2.1.4. The closure operator ϱ_a for M_a. Roughly speaking, we define the ϱ_a-closure of an element $Z \in M_a$ to be its σ-closure with all the H_m replaced by the corresponding D_m. This may be formulated more exactly as follows:

For any $Z \in M_a$ we have $Z = \bigcup_1^n Z_i$ where the Z_i are basic types.

For these we define:

a) $\varrho_a C := \sigma C$,

b) $\varrho_a D := D$,

c) $\varrho_a(D_1' \cap \ldots \cap D_n') := X$, $n \in N$,

d) $\varrho_a(C \cap D) := D$,

e) $\varrho_a(C \cap D_1' \cap \ldots \cap D_n') := \sigma C$, $n \in N$

and then $\varrho_a Z := \bigcup_1^n \varrho_a Z_i$.

ϱ_a is well defined: Suppose $Z_1 = Z_2$; then $\sigma Z_1 = \sigma Z_2$. For $i = 1, 2$, $\sigma Z_i = \sigma C_i \cup H_{i1} \cup \ldots \cup H_{in(i)}$. From 2.1.1 we know that $\sigma C_1 = \sigma C_2$, $n(1) = n(2)$ and $H_{1k} = H_{2k}$ after suitable renumbering (the H's are special D's). But then $\varrho_a Z_1 = \varrho_a Z_2$ trivially. The argument shows $\varrho_a(Z_1 \cup Z_2) = \varrho_a Z_1 \cup \varrho_a Z_2$; the other Kuratowski axioms are evidently satisfied.

2.1.5. (M_a, ϱ_a) is compact. The closed elements of (M_a, ϱ_a) are exactly those of the form $\sigma C_1 \cup D_1 \cup \ldots \cup D_n$, $n \in N$. By 2.1.1, the representation of closed elements by C's and D's is unique. We may thus speak of "C-free" closed elements without ambiquity, meaning those with $\sigma C = \emptyset$. Let F be a proper closed filter base in (M_a, ϱ_a). We put $F = \{A_r;\ r \in R\}$ with a suitable index set R. All A_r are ϱ_a-closed sets.

First case. No A_r is C-free. Consider $A_1, \ldots, A_m \in F$; $A_i = \sigma C_i \cup D_{i1} \cup \ldots \cup D_{in(i)}$, $1 \leqslant i \leqslant m$. Some computation yields

$$A_1 \cap \ldots \cap A_m = (\sigma C_1 \cap \ldots \cap \sigma C_m) \cup \bigcup D_{ik}$$

where $1 \leqslant i \leqslant m$ and $1 \leqslant k \leqslant \max(n(1), \ldots, n(m))$, the D_{ik} being a selection among the D's of the A_i. Now there exists $A_0 \in F$ with $A_0 \subseteq A_1 \cap \ldots \cap A_m$. This implies, since A_0 is not C-free, $\sigma C_1 \cap \ldots \cap \sigma C_m \neq \emptyset$. Thus the family $\{\sigma C_r;\ r \in R\}$ has the finite intersection property and we may form the system F^* of all finite intersections of the σC_r. F^* is evidently a proper closed filter base in (M_a, ϱ_a), but in (X, σ) as well. The latter CA being compact, we find a minimal σ-closed set $\emptyset \neq Z \subseteq X$ satisfying $Z \subseteq \sigma C_r$ for all $r \in R$. According to the construction of M_a, either Z belongs to M_a itself or there exists $D_0 \in \mathfrak{D}_a$ with $D_0 \subseteq Z$. In any case, F^* and thus F has a nontrivial lower bound.

Second case. $A_0 \in F$ is C-free; thus $A_0 = D_{0,1} \cup \ldots \cup D_{0,s}$ with $s \in N$. Let $F^* = \{A_0 \cap A_r;\ A_r \in F\}$. Clearly, F^* is a closed proper filter base. The members of F^* are by 2.1.1 joins of some of the sets $D_{0,1}, \ldots, D_{0,s}$; consequently, F^* has only a finite number of members. Hence the meet $\bigcap F^*$ is nonempty and belongs itself to F^*. Thus we may find $D_{0,k}$ $(1 \leqslant k \leqslant s)$ satisfying $D_{0,k} \subseteq A^*$ for all $A^* \in F^*$. $D_{0,k}$ is a nontrivial lower bound for F^* and, a forteriori, for F.

2.1.6. (M_a, ϱ_a) satisfies postulate 3° for compactifications. Let (B, τ) be any compact CA, (C, σ^*) a sub-CA of (B, τ) and (B, τ) a sub-CA of (M_a, ϱ_a). For every member $D_m \in \mathfrak{D}_a$ we consider the filter base F_m

defined by $F_m := \{\sigma^* C;\ D_m \subseteq \sigma^* C\}$ in $(\mathfrak{C}, \sigma^*)$. All F_m are proper and closed. From the sub-CA properties we infer that the F_m are also proper and closed filter bases in (B, τ) and in (M_a, ϱ_a). Let $U_B(F_m)$ denote the set of all nontrivial closed lower bounds of F_m in (B, τ), and $U_M(F_m)$ the corresponding set in (M_a, ϱ_a). Both sets are nonempty, since both CA's are compact. The definition of F_m implies that $U_M(F_m) = \{D_m\}$, and the sub-CA properties entail $U_B(F_m) \subseteq U_M(F_m)$. Thus we have $U_B(F_m) = \{D_m\}$, $D_m \in (B, \tau)$ and $\tau D_m = \varrho_a D_m = \overline{D}_m$ for every D_m, $m \in M$. Consequently, every Boolean polynomial in $D_m \in \mathfrak{D}_a$ and $C_k \in \mathfrak{C}$ belongs to the BA B, and M_a being generated by $\mathfrak{C} \cup \mathfrak{D}_a$, M_a and B are isomorphic. Finally, the sub-CA properties guarantee $\varrho_a = \tau$, so (B, τ) and (M_a, ϱ) are homeomorphic.

2.2. Special types of compactifications, uniqueness

2.2.1. The full compactification (M, ϱ). A natural choice of the D_m consists in putting $D_m = H_m$ for all $m \in M$. We call the compactification obtained in this way *full*; by means of the relations proved in 2.1.3 it is easy to see that (M, ϱ) is even a sub-CA of (X, σ), which is not the case for any other choice.

2.2.2. Point compactifications. The structure of the elements of (M_a, ϱ_a) will be quite easy to describe if we pick a single point out of every H_m, i.e. if we put $D_m = \{p_m\}$ for an arbitrary $p_m \in H_m$ and for every $m \in M$. The basic types b) and d) coincide in this case and it is easily verified that the elements of the M_a in question may be represented in the following form:

$$Z = C_Z \cup \{p_1, \ldots, p_{m(Z)}\} - \{p_{m(Z)+1}, \ldots, p_{m(Z)+n(Z)}\}\,.$$

Thus the elements of M_a differ from those of $\mathfrak{C}$ by a finite number of the points p_m chosen, which may be added or omitted. We call a compactification obtained in this way a *point compactification*. To simplify the notation when dealing with such compactifications, we introduce a variable P (with or without subscripts) which runs over all finite sets of p's; for any $Z \in M_a$ we then have $Z = [C_Z \cup P_{1(Z)}] - P_{2(Z)}$ where we shall omit the brackets if there is no danger of confusion.

2.2.3. Uniqueness. All compactifications (M_a, ϱ_a) coincide — and represent a point compactification — if all sets H_m are singletons. Translated into algebraic language this means that every free (i.e. bounded only by 0) MCF in $(\mathfrak{C}, \sigma^*)$ is an ultrafilter in the BA $\mathfrak{C}$. This in the case — trivially — for discrete CA's. It is not clear how other possibly existing CA's of this type could be characterized. The conjecture that such CA's exist at all is justified by the fact that the "small" closed elements are responsible for the property under discussion. It is also not clear under what conditions different — i.e. obtained by a different choise of the points p_m — point compactifications are homeomorphic. However we have

PROPOSITION 1. *If (X, σ) contains a minimal σ-closed set with at least two points, then there exist at least two non-homeomorphic compactifications of $(\mathfrak{C}, \sigma^*)$.*

Proof. Let H_1 be the required closed set and $p_0, p_1 \in H_1$. Define $D_1 := \{p_0\}$ and $D_1^* := \{p_0, p_1\}$; let the balance of the D's and D^*'s be pairwise identical singletons, i.e. $D_m = D_m^* = \{p_m\}$ for $m \neq 1$. We denote by (M_1, ϱ_1) the compactification based on the sets D_m $(m \in M)$ and by (M_1^*, ϱ_1^*) the compactification based on the sets D_m^* $(m \in M)$. Suppose there is a homeomorphism $h\colon (M_1^*, \varrho_1^*) \to (M_1, \varrho_1)$. Now consider the filter generated in (M_1^*, ϱ_1^*) by $F(D_1^*) := \{\sigma^* C;\ D_1^* \subseteq \sigma^* C\}$, it is obviously a MCF. The image under h of this filter is a MCF in (M_1, ϱ_1). The unique nontrivial lower bound of the first filter is D_1^*, that of the second one a certain D_k which is a singleton in any case. h being a homeomorphism, $hD_1^* = D_k$ must hold. Furthermore, the well-known properties of the Stone topology τ quarantee the existence of $C_0 \in \mathfrak{C}$ with $p_0 \in C_0$ and $p_1 \notin C_0$. This implies $\emptyset \neq D_1^* \cap C_0 \subseteq D_1^*$ and $D_1^* \cap C_0 \neq D_1^*$. Apply h to get $h(D_1^* \cap C_0) \subseteq hD_1^* = D_k$ and $h(D_1^* \cap C_0) \neq hD_1^*$, since h is a monomorphism. This in turn implies $h(D_1^* \cap C_0) = \emptyset$, D_k being a singleton. This is a contradiction since h is a monomorphism. ∎

2.3. Special properties of point compactifications

In this section a simplified notation is used: A always denotes a closure algebra and $M(A)$ a compactification of A constructed as in 2.1. $a(A)$ denotes the set of all atoms of the BA A, we may have $a(A) = \emptyset$. $X(A)$ stands for the Stone space of A.

LEMMA 3. *An atom of $M(A)$ is a singleton subset of $X(A)$.*

Proof. Let $Z \in a(M(A))$ and $p, q \in Z$, $p \neq q$. There exists $C_0 \in \mathfrak{C}$ with $p \in C_0$ and $q \notin C_0$. C_0 belongs to $M(A)$ and so does $Z \cap C_0$, but $Z \cap C_0 \neq \emptyset$, $Z \cap C_0 \neq Z$ and $Z \cap C_0 \subseteq Z$, a contradiction. ∎

PROPOSITION 2. *Let $M(A)$ be a point compactification. Then the atomicity of A implies that of $M(A)$.*

Proof. For $Z \in M(A)$ there is a representation $Z = C \cup P_1 - P_2$. Suppose first $P_1 \neq \emptyset$. Then every singleton $\{p_i\}$ with $p_i \in P_1$ is an atom of $M(A)$ contained in Z. If $P_1 = \emptyset$, then $C \neq \emptyset$ (otherwise the case is trivial) so there is $c \in C$. For every pair (c, p_i) with $p_i \in P_2$ there exists $C_i \in \mathfrak{C}$ with $c \in C_i$, $p_i \notin C_i$. P_2 being finite, define $C_0 := C \cap C_1 \cap C_2 \cap \ldots \cap C_n$. Obviously $C_0 \in \mathfrak{C}$ and $C_0 \subseteq C - P_2$. A being atomic there exists an atom contained in C_0 and thus in Z. ∎

If the D_m are not singletons, $M(A)$ will not be atomic in general. Of course, elements of the form $C \cap D$ may turn out to be singletons for a suitably chosen C even if D a multipoint set; but for a given D such a meet may fail to exist. The following counterexample illustrates the situation.

COUNTEREXAMPLE 1. Let A be the power set of $\boldsymbol{N}$ together with the obvious operations. Introduce the closure operator σ^* given by $\sigma^* F := [n, \infty)$ where $F \subseteq \boldsymbol{N}$, $n = \min F$. The CA (A, σ^*) has exactly one MCF: the filter generated by the family of all σ^*-closed sets. Consequently there exists exactly one minimal σ-closed set $H_0 \subseteq X(A)$. We show that

a) $\operatorname{card} H_0 \geqslant \aleph_0$.

Proof. Following the construction exhibited in 1.3.1, H_0 consists exactly of those ultrafilters on $\boldsymbol{N}$ which contain all sets $A_n = [n, \infty)$. This holds for all free ultrafilters on $\boldsymbol{N}$ (a free ultrafilters contains the complement of any finite set), thus surely $\operatorname{card} H_0 \geqslant \aleph_0$.

b) H_0 doesn't contain any atom.

Proof. Suppose there is an atom $\{q\} \subseteq H_0$. This implies the existence of $C_0 \in \mathfrak{C}$ with $C_0 \cap H_0 = \{q\}$, for the elements of the form $C \cap H_0$ are the only ones of $M(A)$ which are contained properly in H_0. C_0 is the image of some set $Z_0 \subseteq \boldsymbol{N}$ under the Stone isomorphism, and $C_0 \cap H_0 = \{q\}$ means that there is exactly one ultrafilter on $\boldsymbol{N}$ which contains Z_0 and every A_n. Clearly, then, $\operatorname{card} Z_0 \geqslant \aleph_0$.

First case. Z_0 contains only even natural numbers. Let $n_0 = \min Z_0$ and define

$$Q_1 := \{n_0\} \cup \{\text{all odd } n \in \boldsymbol{N}\}, \quad Q_2 := \{\text{all even } n \in \boldsymbol{N},\ n > n_0\}.$$

Clearly $Q_1 \cap Q_2 = \emptyset$, $Q_i \neq \emptyset$, $A_n \cap Q_i \neq \emptyset$, $Z_0 \cap Q_i \neq \emptyset$ for $i = 1, 2$ and $n \in \boldsymbol{N}$. Thus at least two ultrafilters containing Z_0 and all A_n may be generated, a contradiction.

Second case. Z_0 contains both even and odd numbers. Define

$$Q_1^* := \{\text{all even } n \in \boldsymbol{N}\}, \quad Q_2^* := \{\text{all odd } n \in \boldsymbol{N}\}$$

and apply the above argument.

PROPOSITION 3. *Let $M(A)$ be a point compactification. Suppose the index set M (for the D's) is finite. Then if A is complete, so is $M(A)$.*

Proof. The elements of $M(A)$ are of the form $Z_s = C_s \cup P_{1s} - P_{2s}$. If suffices to show that $\inf\{Z_s;\ s \in S\}$ exists in $M(A)$ for any index set S. We shall verify that

$$Z_0 := \inf_s Z_s = \operatorname{int}(\bigcap_s C_s) \cup \{p_m;\ p_m \in \bigcap_s C_s\} - \{p_m;\ p_m \notin \bigcap_s C_s\}.$$

M being finite, the brackets contain at most finitely many p's.

$Z_0 \subseteq Z_s$ for all $s \in S$ and $Z_0 \in M(A)$ is obvious ($X(A)$ being extremally disconnected).

Let $Z^* \subseteq Z_s$ for all $s \in S$, $Z^* = C^* \cup \{p_1^*, \ldots, p_k^*\} - \{p_{k+1}^*, \ldots, p_{k+n}^*\}$. This implies $C^* \subseteq C_s$ for all $s \in S$: For if $C^* \nsubseteq C_{s_0}$ for $s_0 \in S$, then $t \in C^* - C_{s_0}$ and $t = p_j$ for some $p_j \in P_{1s_0}$. P_{1s_0} being finite p_j could — by means of finitely many meets with suitably chosen $C_k \in \mathfrak{C}$ — be represented as an element of $\mathfrak{C}$, a contradiction. Thus we have $C^* \subseteq \inf_s C_s = \operatorname{int}(\bigcap_s C_s)$. For any $p_j \in Z^*$, $p_j \in Z_0$ holds anyhow, consequently $Z^* \subseteq Z_0$. ■

If $\operatorname{card} M \geqslant \aleph_0$, even a point compactification of a complete CA A my fail to be complete. This is illustrated by the following

COUNTEREXAMPLE 2. Let A be the power set of $\boldsymbol{N}$ together with the discrete closure operator. Every ultrafilter on $\boldsymbol{N}$ is then a MCF, so their number is infinite. The sets H_m — and consequently the D_m — are singletons, $D_m = \{p_m\}$. Now consider $\sup\{p_m;\ m \in M\} =: S$. If $S \in M(A)$, an equation $S = C \cup P_1 - P_2$ holds. $C \neq \emptyset$, for M is not finite and P_1 is finite. C is the image of some set $Z \subseteq \boldsymbol{N}$ under the Stone isomorphism and $Z \neq \emptyset$. Consequently C contains at least one principal ultrafilter, e.g. $q := F(n)$ for $n \in Z$. $\{q\} \in M(A)$, for $\{q\} \in \mathfrak{C}$ as the Stone image of $\{n\} \subseteq \boldsymbol{N}$. Clearly then $X(A) - \{q\} \supseteq \{p_m;\ m \in M\}$, for $\{q\} \in \mathfrak{C}$ implies $q \neq p_m$. Thus

$$C \cup P_1 - P_2 \supset (C \cup P_1 - P_2) - \{q\} \subseteq \{p_m;\ m \in M\}, \qquad S \notin M(A). \quad ■$$

A complete atomic CA is a topological space. From Propositions 2 and 3 we have

PROPOSITION 4. *Let A be a topological space with only finitely many minimal closed filters. Then an arbitrary point compactification $M(A)$ is again a topological space.*

PROPOSITION 5. *With the above assumptions $M(A)$ is even a compactification in the topological sense.*

Proof. We have to show that A is a subspace of $M(A)$ and that A is dense in $M(A)$. A and $M(A)$ are — considered as BA's — isomorphic to the power sets of their respective sets of atoms if we provide the latter with the natural operations. The isomorphisms are given by assigning to every element of A (of $M(A)$) the set of all atoms of A (of $M(A)$) contained in it. Providing the isomorphic images of A and $M(A)$ with the corresponding closure operators, we obtain homeomorphic images A^* of A and $M(A)^*$ of $M(A)$. Observing that $a(M(A)) = a(A) \cup \{p_1, \ldots, p_n\}$ and that the closed elements of A are exactly the sets σC, those of $M(A)$ exactly the sets $\sigma C \cup \{p_{k,1}, \ldots, p_{k,m}\}$, it is obvious that the closed elements of A^* are exactly the meets of A^* with the closed elements of $M(A)^*$. Moreover, the only closed set of $M(A)$ containing all atoms of A is X, so the closure of A^* in $M(A)^*$ equals $M(A)^*$. ■

EXAMPLE. Consider again the CA of Counterexample 1. Since there exists only one MCF, an arbitrary point compactification $M(A)$ is a topological space. If the added point is denoted by p, the BA $M(A)$ is isomorphic to the power set of $N \cup \{p\}$ (cf. above). The closed elements of $M(A)$ are exactly the sets of type C or $C \cup \{p\}$ with $C \in \mathfrak{C}$ closed. But $\{p\} \subseteq C$ for all closed C, for $p \in X(A)$ is an ultrafilter on N which contains all (closed) $A_n = [n, \infty) \subseteq N$. Consequently the closed elements — in the homeomorphic image $M(A)^*$ of $M(A)$ — are the sets $A_n^* \cup \{p\}$: In other words, $M(A)$ is homeomorphic to the Alexandroff 1-point-compactification of A. The next proposition generalizes this situation:

PROPOSITION 6. *If a topological space A has a unique minimal closed filter, then an arbitrary point compactification of A is homeomorphic to the Alexandroff 1-point-compactification of A.*

Proof. The closed subsets $C \subseteq X(A)$ with $\{p\} \nsubseteq C$ correspond exactly to the compact closed subsets of A (which are closed in the 1-point-compactification), and the balance of the closed C's together with the sets of type $C \cup \{p\}$ (with closed C) correspond to the closed subsets of A joined by p (which are the remaining closed sets in the 1-point-compactification). ■

PROPOSITION 7. *Let A be any* CA *and $M(A)$ a point compactification of A. Then the* BA *A is a quotient algebra of the* BA *$M(A)$ and the closure operator on A coincides with the quotient operator coinduced by $M(A)$ on A.*

Proof. Consider the family J of all finite sets $\{p_1, \ldots, p_r\}$. J is an ideal in $M(A)$. We shall prove that the BA's $M(A)/J$ and A are isomorphic. Let $h: A \to M(A)/J$ be defined by $hC := [C]$ for $C \in \mathfrak{C}$, where $[C]$ denotes the equivalence class of C in $M(A)/J$. Clearly, h is a homomorphism. Now let $Z \in M(A)$, thus $Z = C \cup P_1 - P_2$. One sees easily that $C - Z = C \cap P_2 \subseteq P_2 \in J$ and $Z - C \subseteq P_1 \in J$, consequently $[Z] = [C]$ and $[Z] = hC$ for $C \in \mathfrak{C}$. Thus h is onto. Now suppose $C_1 \neq C_2$ but $[C_1] = [C_2]$. The last equation means that $C_1 - C_2$ and $C_2 - C_1$ belong to J; since $C_1 \neq C_2$, at least one of the difference sets is nonempty. This implies $\{p_1, \ldots, p_k\} \in \mathfrak{C}$, a contradiction (see 2.1.1). Thus h is one-to-one and we are through. We now compare the closed elements: Those of $M(A)/J$ are the equivalence classes $[Z]$, where Z is closed in $M(A)$ (see 1.1.3). $Z \in M(A)$ is closed if and only if it is of the form $Z = \sigma C \cup P_0$, where $P_0 \in J$. Clearly

$$\sigma C - Z = \emptyset \in J, \quad Z - \sigma C = P_0 - \sigma C \subseteq P_0 \in J.$$

This means $[Z] = [\sigma C]$, which proves the assertion. ■

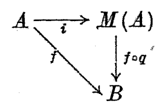
Now let $i: A \to M(A)$ be the canonical injection, $p: M(A) \to M(A)/J$ the canonical projection and $h^{-1}: M(A)/J \to A$ the inverse of the isomorphism used in the preceding proof. i is a monomorphism, $h^{-1} \circ p$ is an epimorphism and $(h^{-1} \circ p) \circ i$ is the identity on A. All mappings involved are continuous. Thus we have

PROPOSITION 8. *Every* CA *A is a retract of an arbitrary point compactification $M(A)$ of A.*

An immediate consequence is

PROPOSITION 9. *Let $M(A)$ be a point compactification of the* CA *A and $f: A \to B$ a continuous homomorphism into an arbitrary* CA *B. Then there exists a continuous homomorphism $f^*: M(A) \to B$ which is an extension of f.*

Proof. From Proposition 8, there is a continuous monomorphism $i: A \to M(A)$ and a continuous epimorphism $q: M(A) \to A$ such that $q \circ i$ is the identity on A. The diagram



commutes because of $(f \circ q \circ i)(x) = f \circ (q \circ i(x)) = f(x)$ for $x \in A$. Clearly, $f \circ q$ is continuous, so we may put $f^* = f \circ q$. ■

APPENDIX

Here we sketch some problems which remain to be solved:

Closure spaces are a suitable tool to investigate the internal algebraic and topological structure of a CA. However, continuous homomorphisms between two CA's lack a satisfying interpretation.

It is not clear whether every compactification of a given CA can be obtained by the method described in this paper.

Under what conditions are two different point compactifications homeomorphic?

The solution of several problems concerning compactifications of CA's could possibly become easier if the compactifications discussed above could be constructed in a purely algebraic way, without the use of the Stone theory.

Is there a characterization of a) the CA's in which every free MCF is an ultrafilter and b) of the topological spaces which possess only a finite number of MCF's?

Added in proof. One of the problems mentioned above has been solved by the author in the meantime: Without the use of the Stone representation theory, one may obtain the compactifications described in this paper by nonstandard methods. Given any *CA* $(A,)$, a compactification $(B,)$ in constructed as a subalgebra of $^*A$ (endowed with a suitable closure operator), where $^*A$ denotes an appropriately chosen nonstandard model of A, particularly, an enlargement.

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Homotopy sequences of fibrations

by

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Abstract. The homotopy sequence of a fibration is generalized to include pairs, triads and squares of fibrations. In accomplishing this the (three dimensional) homotopy lattice of a cube is described and is used to define an associated lattice for a fibration. The standard exact sequences are briefly described. Finally, a potpourri of examples is presented, including some calculations concerning the effect of Lundell's non-stable Bott map on the non-stable homotopy of $U(n)$, with the intention of indicating the breadth of relevance and the usefulness of this method.

Introduction. A map of pairs $i\colon (F, F') \to (E, E') \to (B, B')\colon p$ is said to be a *fibration* if both $i\colon F \to E \to B\colon p$ and $i\colon F' \to E' \to B'\colon p$ are (Serre) fibrations. Hilton [3] has described a homotopy sequence for such fibrations. We recover this sequence from the homotopy sequence of a triad [1]. This approach is then extended, via the homotopy sequences of triads, squares and cubes to provide a functorial lattice which is commutative, up to sign, and which relates the various homotopy sequences of a square of fibrations.

In the first section the basic properties of squares of fibrations are described, while, in the second, the homotopy of cubes is described and employed in defining the homotopy lattice of the fibration. We note that objects described here are special cases of a very general phenomenon. In the third section we present several examples, as well as results concerning the effect of the Bott map b'_n, [6], on the non-stable homotopy of $U(n)$.

Fibrations. By a fibration, $i\colon F \to E \to B\colon p$, we mean a fiber space in the sense of Serre, i.e., $F = p^{-1}(b_0)$ and for any CW complex, K, and commutative diagram,

$$\begin{array}{rccc} H\colon & K \times o & \to & E \\ & \downarrow & & \downarrow{\scriptstyle p} \\ F\colon & K \times I & \to & B \end{array}$$

there is an extension of H to $K \times I$ so that the diagram is commutative. The natural extension of this property to pairs, triads and squares provides