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There is no universal totally disconnected space

by

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Abstract. From the results of this note it follows that there is no universal space in the class of all separable metrizable totally disconnected spaces.

1. In this note we shall prove the following

THEOREM. *Let $\mathfrak{m}$ be an infinite cardinal and let X be a completely regular space of weight $\mathfrak{m}$ and cardinality $2^{\mathfrak{m}}$. If a completely regular space M of weight $\mathfrak{m}$ contains topologically every completely regular space of weight $\leqslant\mathfrak{m}$ which admits a one-to-one mapping onto X, then M contains topologically all completely regular spaces of weight $\leqslant\mathfrak{m}$.*

From the theorem we obtain the following.

COROLLARY. *There is no universal space in the class of all totally disconnected, completely regular spaces of weight $\leqslant\mathfrak{m}\geqslant\aleph_0$.*

The proof of our theorem is based on a construction of Hilgers that we recall in section 2. This construction gives also an interesting example of a totally disconnected, metrizable, separable space that we describe in section 5.

Our terminology and notations are as in [1] and [3]. In particular, the symbol $f\colon X\to Y$ and the word "mapping" always mean "continuous function". The symbol $X \underset{\text{top}}{\subset} Y$ means that X is topologically contained in Y. Finally $I^{\mathfrak{m}}$ and $D^{\mathfrak{m}}$ denote the Tychonoff Cube and the Cantor Cube of weight $\mathfrak{m}$, respectively. By a *totally disconnected* space we mean a space which is not connected between any pair of points, in other words such that every quasi-component consists of a single point.

2. The Hilgers construction (cf. [2]; [3], § 27, IX). Let S be a topological space, $T\subseteq S$, and let $\mathfrak{A}$ be a family of subsets of the product $S\times S$. Suppose that there exists a function φ which establishes a one-to-one correspondence between the elements of T and $\mathfrak{A}$. For every $t\in T$ let us choose an element $f(t)\in S$ such that $(t, f(t))\in\{t\}\times S\setminus\varphi(t)$ if this is possible and an arbitrary $f(t)\in S$ in the opposite case. We denote the graph $\{(t, f(t))|\ t\in T\}$ by H and call it *the Hilgers set* for the family $\mathfrak{A}$ and the set T. Let us observe that

(1) H *is mapped by projection in a one-to-one way onto* T,

(2) *if* $A \in \mathfrak{A}$ *and* $A \supseteq H$ *then* $A \underset{\text{top}}{\supset} S$.

To prove (2) let us notice that for $t \in T$ such that $A = \varphi(t)$ we have $A \in (t, f(t))$ and $A \supseteq \{t\} \times S$.

3. Proof of the theorem. Let us denote by $G_{\mathfrak{m}}$ the family of all intersections of $\leqslant \mathfrak{m}$ open subsets of $I^{\mathfrak{m}}$. We shall apply the following generalization of the classical Lavrentieff theorem:

LEMMA. *Let* $A \subseteq I^{\mathfrak{m}}$ *and let* h *be a homeomorphism of* A *onto* $B \subseteq I^{\mathfrak{m}}$. *There exist two sets* $A^*, B^* \in G_{\mathfrak{m}}$ *such that* $A^* \supseteq A$, $B^* \supseteq B$ *and a homeomorphism* h^* *of* A^* *onto* B^* *which is an extension of* h.

To prove our lemma it suffices to replace the metric argument in [3], § 35, II, analogus uniform argument.

Now let us suppose that X and M are as in the theorem. We can assume that $X \subseteq I^{\mathfrak{m}}$ and $M \subseteq I^{\mathfrak{m}}$. Let $\mathfrak{F}$ be the family of all pairs (A, f) such that $A \subseteq I^{\mathfrak{m}} \times I^{\mathfrak{m}}$, $A \in G_{\mathfrak{m}}$ and $f\colon A \to I^{\mathfrak{m}}$. Let $\mathfrak{A} = \{f^{-1}(M) |\ (A, f) \in \mathfrak{F}\}$. We have $\overline{\overline{\mathfrak{A}}} = 2^{\mathfrak{m}}$, because $\overline{\overline{(I^{\mathfrak{m}})^A}} = 2^{\mathfrak{m}}$. Thus, as in 2, we can construct, for $S = I^{\mathfrak{m}}$, $\mathfrak{A}$ and $T = X$ the Hilgers set H. By (1) and our assumption about M there exists a homeomorphism $h\colon H \to M$. By the Lemma we can extend h to a homeomorphism $h^*\colon H^* \to I^{\mathfrak{m}}$, where $H^* \supseteq H$ and $H \in G_{\mathfrak{m}}$. We have $(H^*, h^*) \in \mathfrak{F}$ and thus $h^{*-1}(M) \in \mathfrak{A}$. But $h^{*-1}(M) \supseteq h^{-1}(M) \supseteq H$ and from (2) we obtain $h^{*-1}(M) \underset{\text{top}}{\supset} I^{\mathfrak{m}}$; it follows that $I^{\mathfrak{m}} \underset{\text{top}}{\subset} M$.

4. Proof of the corollary. Let us take in our theorem $X = D^{\mathfrak{m}}$ and let M be any space of weight $\mathfrak{m}$ which contains topologically all totally disconnected spaces of weight $\leqslant \mathfrak{m}$. The assumptions are satisfied, because if a space can be mapped in a one-to-one way onto $D^{\mathfrak{m}}$, it is totally disconnected. Hence, by the Theorem, M contains the interval I and is not totally disconnected.

5. Example. *There exists a totally disconnected, metrizable, separable space* H *such that every completion of* H *contains topologically the Hilbert Cube* $I^{\aleph_0}$.

Let us take in 2: $S = I^{\aleph_0}$, $X = D^{\aleph_0} \subseteq I^{\aleph_0}$, and $\mathfrak{A}$ equal to the family of all G_δ-sets in $I^{\aleph_0} \times I^{\aleph_0}$. The Hilgers set H has in this case all the required properties. Indeed, if Y is a complete metrizable space containing H, then by the classical Lavrentieff theorem we can extend the identity $i\colon H \to H$ to a homeomorphism $i^*\colon H^* \to Y$, where H^* is a G_δ-set in $I^{\aleph_0} \times I^{\aleph_0}$. As $H^* \supseteq H$ and $H^* \in \mathfrak{A}$ we conclude by (2) that $H^* \underset{\text{top}}{\supset} I^{\aleph_0}$ and this implies $I^{\aleph_0} \underset{\text{top}}{\subset} Y$. The total disconnectedness of H follows from (1).

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Added in proof. A. D. Taĭmanov has proved independently the lemma without using uniform structures (unpublished).

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