[4] R. Duda, *On the hyperspace of subcontinua of a finite graph I*, Fund. Math. 62 (1968), pp. 265–286.
[5] C. Eberhart and S. B. Nadler, Jr., *The dimension of certain hyperspaces*, Bull. Pol. Acad. Sci. 19 (1971), pp. 1027–1034.
[6] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton 1948.
[7] J. L. Kelley, *Hyperspaces of a continuum*, Trans. Amer. Math. Soc. 52 (1942), pp. 22–36.
[8] J. Krasinkiewicz, *No 0-dimensional set disconnects the hyperspace of a continuum*, Bull. Pol. Acad. Sci. 19 (1971), pp. 755–758.
[9] K. Kuratowski, *Topology*, Vol. I, New York–London–Warszawa 1966.
[10] — *Topology*, Vol. II, New York–London–Warszawa 1968.
[11] S. Mazurkiewicz, *Sur l'existence des continus indécomposables*, Fund. Math. 25 (1935), pp. 327–328.
[12] Sam B. Nadler, Jr. and J. T. Rogers, Jr., *A note on hyperspaces and the fixed point property*, Colloq. Math. 25 (1972), pp. 255–257.
[13] Togo Nishiura and C. J. Rhee, *The hyperspace of a pseudoarc is a Cantor manifold*, Proc. Amer. Math. Soc. 31 (1972), pp. 550–556.
[14] C. J. Rhee, *On dimension of hyperspace of a metric continuum*, Bull. de la Société Royale des Sciences de Liège, 38e année, 11–12 (1969), pp. 602–604.
[15] J. T. Rogers, Jr., *The cone = hyperspace property*, Canadian J. Math. 24 (1972), pp. 279–285.
[16] — *Dimension of hyperspaces*, Bull. Pol. Acad. Sci. 20 (1972), pp. 177–179.
[17] — *Embedding the hyperspaces of circle-like plane continua*, Proc. Amer. Math. Soc. 29 (1971), pp. 165–168.
[18] J. Segal, *Hyperspaces of the inverse limit space*, Proc. Amer. Math. Soc. 10 (1959), pp. 706–709.
[19] H. Whitney, *Regular families of curves, I*, Proc. N.A.S. 18 (1932), pp. 275–278.
[20] G. T. Whyburn, *Analytic Topology*, American Math., Soc., Providence 1942.

LOYOLA UNIVERSITY
New Orleans, Louisiana
and
UNIVERSITY OF NORTH CAROLINA AT CHARLOTTE
Charlotte, North Carolina

Reçu par la Rédaction le 27. 6. 1972

Semi-confluent mappings and their invariants

by

T. Maćkowiak (Wrocław)

Abstract. A continuous mapping f of a continuum X onto Y is said to be *semi-confluent* if for every subcontinuum Q in Y and for each two components C_1 and C_2 of the inverse image $f^{-1}(Q)$ either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$. It is proved in the paper that the property of being a λ-dendroid, a dendroid, a fan, a dendrite or an arc is an invariant under a semi-confluent mapping.

§ 1. Introduction. In this paper we present a new kind of continuous mappings, called semi-confluent. The class of semi-confluent mappings comprises confluent mappings, whence also interior and monotone ones. Some theorems on confluent mappings will be generalized to semi-confluent mappings. In particular, theorems concerning the invariance of λ-dendroids proved for confluent mappings in [2] hold also for semi-confluent mappings. Moreover, dendroids, dendrites, arcs and fans (see [3], p. 32) are invariants under semi-confluent mappings.

The author is very much indebted to dr. J. J. Charatonik, who contributed to these investigations.

§ 2. Preliminaries. Recall that a continuous mapping f of a topological space X onto a topological space Y is said to be

(i) *interior* if f maps every open set in X onto an open set in Y (see [11], p. 348),

(ii) *monotone* if for any subcontinuum Q in Y the set $f^{-1}(Q)$ is a continuum in X (see [7], p. 123), or, which is equivalent provided X is a continuum, if the inverse image of each point of Y is a connected set in X (see [14], p. 127),

(iii) *quasi-monotone* if for any subcontinuum Q in Y with a non-vacuous interior the set $f^{-1}(Q)$ has a finite number of components and f maps each of them onto Q (see [12], p. 136),

(iv) *weakly monotone* if for any continuum Q in Y with a non-vacuous interior each component of the inverse image $f^{-1}(Q)$ is mapped by f onto Q (see [12], p. 136, where these mapping are called quasi-monotone and the spaces considered are locally connected continua, see also [10], p. 418),

(v) *confluent* if for every subcontinuum Q of Y each component of the inverse image $f^{-1}(Q)$ is mapped by f onto Q (see [2], p. 213).

Adopt the following definition. A continuous mapping f of a topological space X onto a topological space Y is said to be

(vi) *semi-confluent* if for every subcontinuum Q in Y and for each two components C_1 and C_2 of the inverse image $f^{-1}(Q)$ either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$.

As an easy consequence of the definitions we have

PROPOSITION 2.1. *Any confluent mapping is semi-confluent.*

It is known (see [2], p. 214) that any monotone mapping is confluent, and that any interior mapping of a compact space is confluent. Thus it follows from Proposition 2.1 that

PROPOSITION 2.2. *Any monotone mapping is semi-confluent.*

PROPOSITION 2.3. *Any interior mapping of a compact space is semi-confluent.*

It is proved (see [12], Theorem (2.1), p. 137 and Theorem (2.3), p. 138, see also [2], p. 214 and IX, p. 215) that if a mapping f is defined on a locally connected continuum, then the following conditions are equivalent: (i) f is weakly monotone, (ii) f is quasi-monotone, (iii) f is confluent. Hence we conclude by Proposition 2.1 that

PROPOSITION 2.4. *Any quasi-monotone mapping of a locally connected continuum is semi-confluent.*

PROPOSITION 2.5. *Any weakly monotone mapping of a locally connected continuum is semi-confluent.*

The class of semi-confluent mappings is essentially larger than the class of confluent mappings even for locally connected continua. This can be seen by the following

EXAMPLE 2.6. Define $f(t) = |t|$ for each $t \in [-1, 2]$. Thus f maps the segment $[-1, 2]$ onto the segment $[0, 2]$; it is semi-confluent, but not confluent; moreover, it is neither weakly monotone nor quasi-monotone.

However, the class of semi-confluent mappings fails to contain the class of quasi-monotone mappings, and hence also fails to contain the class of weakly monotone mappings. This can be shown by

EXAMPLE 2.7. Put, in the Cartesian rectangular coordinates in the plane

$$X = \left\{(x, y):\ y = \sin\frac{1}{x},\ 0 < x \leqslant 1\right\} \cup \{(0, y):\ -1 \leqslant y \leqslant 1\}.$$

Thus X is the closure of the graph of $\sin(1/x)$-function. Let $a = (0, 1)$ and $b = (0, -1)$ be the end-points of the limit segment in X. Define f to be a continuous mapping on X which identifies a and b. Therefore $f(X)$ is the union of a circle and a ray which approximates it. It is immediately seen that f is quasi-monotone and weakly monotone, but is not semi-confluent.

§ 3. **Properties of semi-confluence.** In this section we prove some properties of semi-confluent mappings, which are similar to properties of confluent ones (see [2], pp. 213–215) and which are needed to prove some theorems in Sections 4 and 5.

Henceforth the topological spaces under consideration will be assumed to be metric continua.

LEMMA 3.1. *Let a semi-confluent mapping f map a continuum X onto Y. For each subcontinuum Q of Y and for each family $\mathfrak{C}$ of components of $f^{-1}(Q)$ such that the union $\bigcup\{C:\ C \in \mathfrak{C}\}$ is closed in X there exists a component C' belonging to $\mathfrak{C}$ whose image under f is maximal in the sense that $f(C') = f(\bigcup\{C:\ C \in \mathfrak{C}\})$.*

Proof. According to Theorem 1 in [6], § 24, VII, p. 263 and to the definition of a semi-confluent mapping we can supply each component C belonging to the family $\mathfrak{C}$ with an index $t \in [0, 1]$ in such a way that $t_1 < t_2$ implies $f(C_{t_1}) \subset f(C_{t_2})$ and $f(C_{t_1}) \neq f(C_{t_2})$. Denote by T the set of indices which correspond to all members of the family $\mathfrak{C}$, and let t' be the least upper bound of T. Take a sequence of indices t_n converging to t' such that the sequence $\{C_{t_n}\}$ is convergent. Then its limit is a continuum ([7], § 47, II, Theorem 6, p. 171) which is contained in a component $C' \in \mathfrak{C}$:

$$\operatorname*{Lim}_{n\to\infty} C_{t_n} \subset C',$$

because the union of all members of the family $\mathfrak{C}$ is closed by assumption.

We shall show that for every component $C'' \in \mathfrak{C}$ of $f^{-1}(Q)$ such that $f(C'') \neq f(\bigcup\{C_t:\ t \in T\})$ there is a natural n_0 such that if $n > n_0$, then $f(C'') \subset f(C_{t_n})$, where C_{t_n} is a component from the sequence mentioned above. In fact, let $y \in f(\bigcup\{C_t:\ t \in T\}) \setminus f(C'')$. Thus there is a member C_{t_0} of $\mathfrak{C}$ such that $y \in f(C_{t_0}) \setminus f(C'')$. Since f is semi-confluent, we have $f(C'') \subset f(C_{t_0})$. Thus the index t of C'' must be less than t_0 and thus less than t'. Therefore there is a natural n such that if $n > n_0$ then $t < t_n \leqslant t'$ and consequently $f(C'') \subset f(C_{t_n})$.

This implies that if $y \in f(\bigcup\{C_t:\ t \in T\})$, i.e., if $y \in f(C_t)$ for some $t \in T$, then either C_t is mapped onto the whole union $\bigcup f(\{C_t:\ t \in T\})$ or there is for each $n > n_0$ a point $x_n \in C_{t_n}$ with $f(x_n) = y$. Since $\operatorname*{Lim}_{n\to\infty} C_{t_n} \subset C'$, the limit x of a convergent subsequence of $\{x_n\}$ is a point of C'. Therefore $f(x) = y$ by the continuity of f, and $y \in f(C')$, which shows the required equality.

Corollary 3.2. *Let a semi-confluent mapping f map a continuum X onto a continuum Y. Then for each subcontinuum Q of Y there is a component C of the set $f^{-1}(Q)$ such that $f(C) = Q$.*

Indeed, for the family $\mathfrak{C}$ of all components of the set $f^{-1}(Q)$ there is a component C' such that for each $C \in \mathfrak{C}$ we have $f(C') = f(\bigcup\{C\colon C \in \mathfrak{C}\}) = f(f^{-1}(Q)) = Q$ according to Lemma 3.1.

Theorem 3.3. *If a mappine f_1 is confluent and a mapping f_2 is semi-confluent, then $f = f_2f_1$ is semi-confluent.*

Proof. Let a confluent mapping f_1 map a continuum X onto Y, and let a semi-confluent mapping f_2 map Y onto Z. Let Q be a continuum in Z. For each two components C_1 and C_2 of the inverse image $f^{-1}(Q)$ either $f_1(C_1) = f_1(C_2)$ or $f_1(C_1) \cap f_1(C_2) = \emptyset$ holds by the confluence of f_1. Hence $f_1(C_1)$ and $f_1(C_2)$ are components of the set $f_2^{-1}(Q)$. Since f_2 is semi-confluent, we have either $f_2(f_1(C_1)) \subset f_2(f_1(C_2))$ or $f_2(f_1(C_2)) \subset f_2(f_1(C_1))$, i.e., either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$.

A superposition of two semi-confluent mappings need not be semi-confluent. This can be seen from the following

Example 3.4. We define a mapping f_1, which maps the segment $[-2, 2]$ of reals onto the segment $[0, 3]$, and a mapping f_2, which maps the segment $[0, 3]$ onto the segment $[0, 2]$, as follows: $f_1(t) = |t+1|$ and $f_2(t) = |t-2|$. Both these mappings are semi-confluent, but their superposition $f = f_2f_1$ is not.

Theorem 3.5. *If $f = f_2f_1$ is semi-confluent, then f_2 is semi-confluent.*

Proof. Let f_1 map a continuum X onto Y, and let f_2 map Y onto Z. Let Q be a continuum in Z, and let C_1 and C_2 be components of the set $f_2^{-1}(Q)$ in Y. Then, for $i = 1, 2$, the family $\mathfrak{C}_i$ of all components of the set $f_1^{-1}(C_i)$ in X is also a family of components of the set $f^{-1}(Q)$ and the union of members of $\mathfrak{C}_i$ is closed. Thus Lemma 3.1 can be applied and we conclude that for $i = 1, 2$ there is a C_i' such that

$$f(C_i') = f(\bigcup\{C\colon C \in \mathfrak{C}_i\}).$$

Observe that

$$f_2(C_i) = f_2f_1(f_1^{-1}(C_i)) = f(f_1^{-1}(C_i)) = f(\bigcup\{C\colon C \in \mathfrak{C}_i\}),$$

and thereby

$$f_2(C_i) = f(C_i') \quad \text{for} \quad i = 1, 2.$$

Since f is semi-confluent, either $f(C_1') \subset f(C_2')$ or $f(C_2') \subset f(C_1')$; thus either $f_2(C_1) \subset f_2(C_2)$ or $f_2(C_2) \subset f_2(C_1)$, which completes the proof.

If $f = f_2f_1$ is semi-confluent, then f_1 need not be semi-confluent. This can be seen by

Example 3.6. Put $f_1(x) = \sin x$ for $x \in [-\frac{5}{6}\pi, \frac{5}{6}\pi]$ and

$$f_2(x) = \begin{cases} x & \text{if} \quad |x| \leqslant \frac{1}{2}, \\ \operatorname{sgn} x & \text{if} \quad |x| \in (\frac{1}{2}, 1]. \end{cases}$$

Thus $f_1\colon [-\frac{5}{6}\pi, \frac{5}{6}\pi] \to [-1, 1]$ and $f_2\colon [-1, 1] \to [-\frac{1}{2}, \frac{1}{2}]$, both f_1 and f_2 are onto, f_2f_1 is semi-confluent (since it is monotone even) while f_1 is not.

Theorem 3.7. *If f is a semi-confluent mapping of X onto Y, B is a subset of Y, and A is the union of some components of $f^{-1}(B)$, then the partial mapping $g = f|A$ is a semi-confluent mapping of A onto $f(A)$.*

Proof. Let Q be a subcontinuum of $f(A)$ and let C_1 and C_2 be components of $g^{-1}(Q)$. Since

$$g^{-1}(Q) = A \cap f^{-1}(Q), \tag{1}$$

C_1 and C_2 lie in the components C_1' and C_2' of $f^{-1}(Q)$, respectively. It follows from $C_1 \subset A$ and $C_2 \subset A$ that

$$\emptyset \neq C_1 = A \cap C_1 \subset A \cap C_1', \tag{2}$$

and

$$\emptyset \neq C_2 = A \cap C_2 \subset A \cap C_2'. \tag{3}$$

Moreover, it follows from $Q \subset f(A) \subset B$ that

$$C_1' \subset f^{-1}(B) \quad \text{and} \quad C_2' \subset f^{-1}(B). \tag{4}$$

According to the hypothesis regarding A, conditions (2), (3) and (4) give $C_1' \subset A$ and $C_2' \subset A$, whence $C_1' \subset g^{-1}(Q)$ and $C_2' \subset g^{-1}(Q)$ by (1). Thus $C_1' = C_1$ and $C_2' = C_2$; hence $g(C_1) = f(C_1) = f(C_1')$ and $g(C_2) = f(C_2) = f(C_2')$. This implies that either $g(C_1) \subset g(C_2)$ or $g(C_2) \subset g(C_1)$ holds by the semi-confluence of f. Thus g is semi-confluent.

A set $A \subset X$ is said to be *inverse set* under a mapping $f\colon X \to Y$ if $A = f^{-1}(f(A))$ (see [14], p. 137). Theorem 3.7 implies

Corollary 3.8. *If f is a semi-confluent mapping of X and A is an inverse set under the mapping f, then the partial mapping $g = f|A$ is a semi-confluent mapping of A onto $f(A)$.*

As a direct consequence of Whyburn's factorization theorem (see [13], (2.3), p. 297) and Theorem 3.5 we obtain the following

Theorem 3.9. *If X is a continuum and if f is a semi-confluent mapping of X onto Y, then there exists a unique factorization of f into two semi-confluent mappings*

$$f(x) = f_2f_1(x) \quad \text{for each } x \in X,$$

where f_1 is monotone and f_2 is such that $\dim f_2^{-1}(y) = 0$ for each $y \in Y$.

§ 4. Mappings onto a triode and onto a circumference. This section contains some lemmas used in the next section to prove theorems which are the main results of this paper.

To begin with, recall that a *triode* is the union of three arcs which have pairwise only the end-point in common. Let the end-points of a triode T be x, y, z. Then we denote T by xyz. Further, we denote by u the common end-point of the three arcs forming the triode. The arc joining points a and b will be denoted by ab. We have the following

LEMMA 4.1. *There is no semi-confluent mapping of an arc onto a triode.*

Proof. It suffices to show that there is no semi-confluent mapping of the unit interval $I = [0,1]$ of reals onto a triode. Suppose, on the contrary, that a semi-confluent mapping f maps I onto a triode T. Put

$$a = \inf\{t \in [0,1]\colon f(t) = x\}, \qquad b = \inf\{t \in [0,1]\colon f(t) = y\}$$

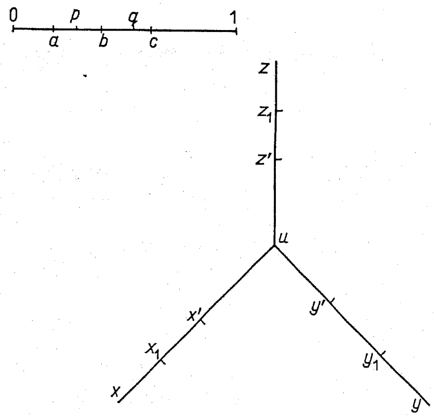
and

$$c = \inf\{t \in [0,1]\colon f(t) = z\}.$$

Thus by the continuity of f we have

(5) $$f(a) = x, \quad f(b) = y \quad \text{and} \quad f(c) = z.$$

Without loss of generality we can assume $a < b < c$. Put $p = \sup\{t \in [a,b]\colon f(t) = x\}$ and $q = \sup\{t \in [b,c]\colon f(t) = x \text{ or } f(t) = y\}$. Therefore $f(p) = x$ and $f(q) \in \{x, y\}$ by the continuity of f. Consider two cases (see the figure).



1° $f(q) = y$. According to the definitions of points p and b we have

(6) $$f([p,b]) = xyz', \quad \text{where} \quad z' \in uz\setminus\{z\},$$

and by the definitions of points q and c

(7) $$f([q,c]) = x'yz, \quad \text{where} \quad x' \in ux\setminus\{x\}.$$

Take points x_1 and z_1 such that $x_1 \in xx'\setminus\{x, x'\}$ and $z_1 \in zz'\setminus\{z, z'\}$. Thus

(8) $$x_1 \notin x'yz,$$

and

(9) $$z_1 \notin xyz'.$$

Since $x_1 \in xyz'$, there is a $t_1 \in [p,b]$ such that $f(t_1) = x_1$ by (7), and since $z_1 \in x'yz$, there is a $t_2 \in [q,c]$ such that $f(t_2) = z_1$ by (8). Consider components C_1 and C_2 of the set $f^{-1}(x_1z_1)$ such that $t_1 \in C_1$ and $t_2 \in C_2$. Since $f(p) = x$, we have by (5)

(10) $$t_1 \in C_1 \subset [p,b].$$

Similarly, $f(q) = y$ and (5) imply that

(11) $$t_2 \in C_2 \subset [q,c].$$

The mapping f being semi-confluent, either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$. If $f(C_1) \subset f(C_2)$, then $x_1 = f(t_1) \in f(C_1) \subset f(C_2) \subset f([q,c]) = x'yz$ by (11) and (7), which contradicts (8). If $f(C_2) \subset f(C_1)$, then $z_1 = f(t_2) \in f(C_2) \subset f(C_1) \subset f([p,b]) = xyz'$ by (10) and (6), which contradicts (9).

2° $f(q) = x$. Then, as above, we obtain $f([p,b]) = xyz'$ and $f([q,c]) = xy'z$. Take points y_1 and z_1 such that $y_1 \in yy'\setminus\{y, y'\}$ and $z_1 \in zz'\setminus\{z, z'\}$. Since $y_1 \in xyz'$ and $z_1 \in xy'z$, there are points t_1' and t_2' such that $t_1' \in [p,b]$, $t_2' \in [q,c]$, $f(t_1') = y_1$ and $f(t_2') = z_1$. Let C_1 and C_2 be components of $f^{-1}(y_1z_1)$ such that $t_1' \in C_1$ and $t_2' \in C_2$. As in case 1°, we have $C_1 \subset [p,b]$ and $C_2 \subset [q,c]$, and in the same way we obtain a contradiction of the semi-confluence of f, which completes the proof.

LEMMA 4.2. *Let a continuous mapping f map the unit interval $I = [0,1]$ of reals onto a circumference S and let $\dim f^{-1}(y) = 0$ for each $y \in S$. Let $[\alpha, \beta]$ be a closed interval in I such that $f([\alpha,\beta]) = pq \subset S$ and $pq \neq S$. If $t \in \operatorname{Int}([\alpha,\beta])$ is a point such that $f(t) = p$, then the one-point set $\{t\}$ is a component of $f^{-1}(px)$ for each arc $px \subset \overline{S\setminus pq}$.*

Proof. According to the assumption there exists a neighbourhood U of t in $[\alpha,\beta]$ with the property $f(U) \subset pq\setminus\{q\}$. Take an arc $px \subset \overline{S\setminus pq}$. Let A be a component of $f^{-1}(px)$ which contains t. Thus A is a closed interval and we have $t \in A \cap U$. Since $f(A \cap U) \subset f(U) \subset pq\setminus\{q\}$ and $f(A \cap U) \subset f(A) \subset px$, we have $f(A \cap U) \subset \{p\}$. But $\dim f^{-1}(p) = 0$ by assumption and therefore $A \cap U$ reduces to the point p. Since U is open and A is a closed interval, we conclude that $A = \{t\}$. In other words, the one-point set $\{t\}$ is a component of $f^{-1}(px)$.

We have also

LEMMA 4.3. *There is no semi-confluent mapping of an arc onto a circumference.*

Proof. Suppose, on the contrary, that a semi-confluent mapping f maps the unit interval $I=[0,1]$ of reals onto a circumference S. It is well known that a monotone image of an arc is an arc (see [14], (1.1), p. 165); hence we can assume that $\dim f^{-1}(y)=0$ for each $y\in S$, by Theorem 3.9. Further, let Q be a proper subcontinuum of S such that $\{f(0),f(1)\}\subset \operatorname{Int}Q$. Put $R=\overline{S\setminus Q}$. The intersection $Q\cap R$ consists of two points; denote them by a and b. Put

(12) $$t_1=\inf\{t\in[0,1]\colon f(t)=a \text{ or } f(t)=b\},$$

(13) $$t_1'=\sup\{t\in[0,1]\colon f(t)=a \text{ or } f(t)=b\}.$$

Obviously, $f(t_1)\in\{a,b\}$ and $f(t_1')\in\{a,b\}$ by the continuity of f. Consider two cases.

1° $f(t_1)\neq f(t_1')$. It follows by (12) and (13) that

$$\big(\operatorname{Int}Q\setminus f([t_1',1])\big)\cap f([0,t_1])\neq\emptyset \quad\text{and}\quad \big(\operatorname{Int}Q\setminus f([0,t_1])\big)\cap f([1,t_1'])\neq\emptyset.$$

Hence we can take an arc z_1z_2 such that

(14) $$z_1z_2\subset \operatorname{Int}Q,$$

(15) $$z_1\in f([0,t_1])\setminus f([t_1',1]),$$

(16) $$z_2\in f([t_1',1])\setminus f([0,t_1]).$$

Thus there are points $x_1\in[0,t_1]$ and $x_2\in[t_1',1]$ such that $f(x_1)=z_1$ and $f(x_2)=z_2$. Let C_1 and C_2 be components of $f^{-1}(z_1z_2)$ such that $x_1\in C_1$ and $x_2\in C_2$. By (12) and (14) we conclude that

(17) $$x_1\in C_1\subset[0,t_1],$$

and by (13) and (14) that

(18) $$x_2\in C_2\subset[t_1',1].$$

Since f is semi-confluent, either $f(C_1)\subset f(C_2)$ or $f(C_2)\subset f(C_1)$. If $f(C_1)\subset f(C_2)$, then $z_1=f(x_1)\in f(C_1)\subset f(C_2)\subset f([t_1',1])$ by (18). It is a contradiction of (15). If $f(C_2)\subset f(C_1)$, then $z_2=f(x_2)\in f(C_2)\subset f(C_1)\subset f([0,t_1])$ by (17), which contradicts (16).

2° $f(t_1)=f(t_1')$. Assume $f(t_1)=f(t_1')=a$ (if $f(t_1)=f(t_1')=b$, then the proof is the same). First we show that $[0,t_1]$ and $[t_1',1]$ are components of the set $f^{-1}(Q)$. Since $f([0,t_1])$ is a proper subcontinuum of Q (observe that $b\in Q\setminus f([0,t_1])$), it is an arc $ac=f([0,t_1])\subset Q\setminus\{b\}$. Therefore there exists a point $x_0\in[0,t_1]$ such that $f(x_0)=c$. Take the arc $bc\subset Q$ and consider $R\cup bc$. Then $R\cup bc=\overline{S\setminus ac}$, and hence by Lemma 4.2

(19) the one-point set $\{x_0\}$ is a component of $f^{-1}(R\cup bc)$.

Now let C_1 be a component of $f^{-1}(Q)$ such that $[0,t_1]\subset C_1$. Take a point t_0 with the property $C_1=[0,t_0]$. To show that $[0,t_1]$ is a component of $f^{-1}(Q)$, i.e., that $[0,t_1]=C_1$, assume, on the contrary, that $t_1<t_0$. If $f([t_1,t_0])\subset f([0,t_1])=ac$, then by Lemma 4.2 the one-point set $\{t_1\}$ is a component of $f^{-1}(R\cup bc)$. Since $f(x_0)=c\neq a=f(t_1)$, we have a contradiction of the semi-confluence of f by (19). If $f([0,t_1])\subset f([t_1,t_0])$, then we consider a point $x_1=\inf\{t\in[t_1,t_0]\colon f(t)=c\}$ and a closed interval $[0,x_1]$. Since $t_1\in\operatorname{Int}[0,x_1]$ and $R\cup bc\subset\overline{S\setminus f([0,x_1])}$, by Lemma 4.2 the one-point set $\{t_1\}$ is a component of $f^{-1}(R\cup bc)$. This contradicts the semi-confluence of f by (19). Thus

(20) the closed interval $[0,t_1]$ is a component of $f^{-1}(Q)$.

Similarly we infer that

(21) the closed interval $[t_1',1]$ is a component of $f^{-1}(Q)$.

Each component of the set $f^{-1}(R)$ and of the set $f^{-1}(Q)$ is a closed interval $[\alpha,\beta]$ which can reduce to a point if $\alpha=\beta$. Remark that if $\alpha\neq0$ and $\beta\neq1$, then

(22) $$\{f(\alpha),f(\beta)\}\subset\{a,b\}.$$

Observe that

(23) there is no component $C=[\alpha,\beta]$ of $f^{-1}(R)$ such that $f(\alpha)=f(\beta)=a$ and $\alpha\neq\beta$.

In fact, assume on the contrary that there is a component $C=[\alpha,\beta]$ of $f^{-1}(R)$ such that $f(\alpha)=f(\beta)=a$ and $\alpha\neq\beta$. Then $f([\alpha,\beta])$ is an arc $ad\subset R$ and $d\neq a$ because $\dim f^{-1}(a)=0$. Therefore there is a point $y_0\in\operatorname{Int}([\alpha,\beta])$ such that $f(y_0)=d$. Take an arc $db\subset R$ and consider an arc $db\cup bc$, where $c=f(x_0)$ as before. Since $db\cup bc\subset\overline{S\setminus ad}$, by Lemma 4.2

(24) the one-point set $\{y_0\}$ is a component of $f^{-1}(db\cup bc)$.

Since $db\cup bc\subset\overline{S\setminus ac}$, by Lemma 4.2 the one-point set $\{x_0\}$ is a component of $f^{-1}(db\cup bc)$, but $f(y_0)=d\neq c=f(x_0)$ we have a contradiction of the semi-confluence of f by (24).

Notice also that

(25) there is no component $C=[\alpha,\beta]$ of $f^{-1}(Q)$ such that $f(\alpha)=f(\beta)=b$ and $\alpha\neq\beta$.

In fact, assume on the contrary that there is a component $C=[\alpha,\beta]$ of $f^{-1}(Q)$ such that $f(\alpha)=f(\beta)=b$ and $\alpha\neq\beta$. Since $f(t_1)=a$ and $[0,t_1]$ is a component of $f^{-1}(Q)$ by (20), and $b\notin f([0,t_1])$, we have $f([0,t_1])\subset f([\alpha,\beta])$ by the semi-confluence of f. Therefore $a\in f([\alpha,\beta])$, and thus there is a point $z_0\in\operatorname{Int}([\alpha,\beta])$ such that $f(z_0)=a$. Let $z_1=\sup\{t\in[\alpha,z_0]\colon f(t)=c\}$ and $z_2=\inf\{t\in[z_0,\beta]\colon f(t)=c\}$. Then $z_0\in\operatorname{Int}([z_1,z_2])$ and $f([z_1,z_2])=f([0,t_1])=ac\subset Q$. Since $R\cup bc=\overline{S\setminus ac}$, by Lemma 4.2 the one-point set $\{z_0\}$ is a component of $f^{-1}(R \cup bc)$. This contradicts the semi-confluence of f by (19), because $f(z_0) = a \neq c = f(x_0)$.

We now show that

(26) the set $f^{-1}(R)$ has only a finite number of components with a non-empty interior.

Indeed, let $C = [a, \beta]$ be a component of $f^{-1}(R)$ such that $\operatorname{Int} C \neq \emptyset$. If $f(a) \neq f(\beta)$, then $f([a, \beta]) = R$ by (22). Since for each component $C = [a, \beta] \subset f^{-1}(R)$ such that $f(C) = R$ we have $\beta - a > \varepsilon$ for some $\varepsilon > 0$ by the continuity of f, the number of all components $C = [a, \beta]$ of $f^{-1}(R)$ with $f(a) \neq f(\beta)$ is finite. If $f(a) = f(\beta)$, then $f(a) = b$ by (23). Let $C = [a, \beta]$ be a component of $f^{-1}(R)$ such that $f(a) = f(\beta) = b$. Take components A_1 and A_2 of $f^{-1}(Q)$ such that $a \in A_1$ and $\beta \in A_2$. Since f is semi-confluent, $f([0, t_1]) \subset f(A_1)$ and $f([0, t_1]) \subset f(A_2)$ by (20). Therefore $f(A_1) = f(A_2) = Q$. But for each component $B = [\gamma, \delta]$ of $f^{-1}(Q)$ such that $f(B) = Q$ we have $\gamma - \delta > \eta$ for some $\eta > 0$ by the continuity of f. Thus $\operatorname{diam} A_1$ as well as $\operatorname{diam} A_2$ are greater than η. Thereby we have shown that for each component $C = [a, \beta]$ of $f^{-1}(R)$ with $f(a) = f(\beta) = b$ there are two components A_1 and A_2 of $f^{-1}(Q)$ having diameters greater than $\eta > 0$ and such that $A_1 \cup C \cup A_2$ is a closed interval. It follows that the number of such components C is finite. Therefore the set $f^{-1}(R)$ has only a finite number of components C such that $\operatorname{Int} C \neq \emptyset$.

Put $T = [0, 1] \setminus \bigcup \{C: C \text{ is a component of } f^{-1}(R) \text{ and } \operatorname{Int} C \neq \emptyset\}$. Since the set T has only a finite number of components by (26), also

(27) the set $\overline{T}$ has only a finite number of components.

Observe that if $e \in \operatorname{Int} R$, then for each $t \in f^{-1}(e)$ there is a connected neighbourhood G of t such that $f(G) \subset \operatorname{Int} R$. Thus G is contained in a component C of $f^{-1}(R)$ and $\operatorname{Int} C \neq \emptyset$. Therefore $e \in S \setminus f(T)$ by the definition of T. Thereby $f(T) \subset Q$, and thus $f(\overline{T}) \subset \overline{f(T)} \subset \overline{Q} = Q$. Hence it is easy to see that

(28) each component of the set $\overline{T}$ is a component of the set $f^{-1}(Q)$,

because $f(f^{-1}(Q) \cap f^{-1}(R)) = \{a, b\}$ and $\dim f^{-1}(a) = \dim f^{-1}(b) = 0$ by assumption.

Notice also that

(29) each component C of $f^{-1}(Q)$ such that $\operatorname{Int} C \neq \emptyset$ is a component of $\overline{T}$.

In fact, if C is a component of $f^{-1}(Q)$ such that $\operatorname{Int} C \neq \emptyset$, then there is a point $t \in \operatorname{Int} C$ and each component of $f^{-1}(R)$ with a non-empty interior fails to contain t. Therefore $t \in \overline{T}$, and thus t belongs to some component A of $\overline{T}$. It follows by (28) that $C = A$.

According to (26) we can supply each component C of $f^{-1}(R)$ having a non-empty interior with an index $i = 1, \ldots, n$ in such a way that $C_i = [a_i, \beta_i]$, $C_j = [a_j, \beta_j]$ and $i < j$ implies $\beta_i < a_j$.

It follows by (20), (21) and (29) that $[0, t_1]$ and $[t_1', 1]$ are components of $\overline{T}$. Therefore by the definition of T we see that

(30) $$a_1 = t_1 \quad \text{and} \quad \beta_n = t_1'.$$

We shall prove by induction that

(31) $$f(a_i) = a \quad \text{and} \quad f(\beta_i) = b \quad \text{for each } i = 1, \ldots, n.$$

Indeed, since $f(a_1) = f(t_1) = a$ by (30), we have $f(\beta_1) = b$ by (22) and (23). Assume that $f(a_k) = a$ and $f(\beta_k) = b$. We show that then $f(a_{k+1}) = a$ and $f(\beta_{k+1}) = b$. Namely, since $f(\beta_k) = b$ and a closed interval $[\beta_k, a_{k+1}]$ is a component of $f^{-1}(Q)$ by (28), we have $f(a_{k+1}) = a$ by (22) and (25). Therefore $f(\beta_{k+1}) = b$ by (22) and (23). Thus (31) is true. It follows from (31) that $f(\beta_n) = b$, but by (30) $f(\beta_n) = f(t_1')$, which contradicts the assumption that $f(t_1') = a$.

Let $\mathcal{S}$ be the circumference $|z| = 1$, and let $\mathcal{R}$ denote the real line. Recall that a continuous mapping f of a separable metric space X into $\mathcal{S}$ is said to be *inessential* if it belongs to the same component of the functional space $\mathcal{S}^X$ as the mapping $f_0(x) = 1$, where $x \in X$. It is known (see [2], XI, p. 217) that

PROPOSITION 4.4. *Every continuous mapping of a hereditarily decomposable and hereditarily unicoherent continuum into a circumference is inessential.*

A hereditarily decomposable and hereditarily unicoherent continuum is called a *λ-dendroid* (see [4], Theorem 1, p. 16). The following theorem will now be proved.

THEOREM 4.5. *There is no semi-confluent mapping of a λ-dendroid onto a circumference.*

Proof. Let X be a λ-dendroid and suppose that f is a semi-confluent mapping of X onto a circumference. Since f is inessential by Proposition 4.4, there exists by Eilenberg's Theorem 1 in [5], p. 162, a continuous mapping $\varphi\colon X \to \mathcal{R}$ such that $f(x) = e^{i\varphi(x)}$ for each $x \in X$. Thus f can be factored into two mappings ψ and φ, i.e., $f(x) = \psi(\varphi(x))$ for $x \in X$, where $\psi(t) = e^{it}$ for $t = \varphi(x) \in \mathcal{R}$. The image $\varphi(X)$ is a segment. Since f is semi-confluent by hypothesis, by Theorem 3.5 ψ is a semi-confluent mapping which maps the segment onto a circumference. But this contradicts Lemma 4.3.

§ 5. Semi-confluent images of λ-dendroids.

First we have the following

THEOREM 5.1. *The hereditary decomposability of continua is an invariant under semi-confluent mappings.*

In fact, let Q be an indecomposable subcontinuum of a semi-confluent image $f(X)$ of a continuum X. Then there is a component C of $f^{-1}(Q)$ by Corollary 3.2 such that $f(C) = Q$. Therefore C would contain an indecomposable subcontinuum by Kuratowski's theorem proved in [7], § 48, V, 4, p. 208.

Now we have the following

THEOREM 5.2. *A semi-confluent image of a λ-dendroid is a λ-dendroid.*

Proof. Let X be a λ-dendroid, and let f be a semi-confluent mapping of X. It follows by Theorem 5.1 that a continuum $f(X)$ is hereditarily decomposable. Suppose that $f(X)$ is not hereditarily unicoherent. Thus there is a subcontinuum M of $f(X)$ which is hereditarily decomposable but not unicoherent, and therefore contains (see [9], Theorem 2.6, p. 187) a subcontinuum N which has an upper semi-continuous decomposition on mutually disjoint continua N_t such that the hyperspace of this decomposition is the circumference S. This means (see [14], (3.1), p. 125) the existence of a monotone mapping ϑ of N onto S. By Corollary 3.2 there is a component C of $f^{-1}(N)$ such that $f(C) = N$. Hence by hypothesis C is a λ-dendroid. According to Theorem 3.7 the mapping $f|C$ is semi-confluent. Since the mapping ϑ is monotone, it is confluent, and thus the superposition $\vartheta(f|C)$ is semi-confluent by Theorem 3.3. A mapping $\vartheta(f|C)$ maps C onto S, which contradicts Theorem 4.5, because C is a λ-dendroid.

Recall that a *dendroid* is an arcwise connected and hereditarily unicoherent continuum (see [1], p. 239). Since every dendroid is hereditarily decomposable (see [1], (47), p. 239), and since arcwise connectedeness is an invariant under an arbitrary continuous mapping (see [14], p. 39), by Theorem 5.2 we have the following

COROLLARY 5.3. *Every semi-confluent image of a dendroid is a dendroid.*

Moreover, since a *dendrite* is a locally connected dendroid, and since the local connectedness is an invariant under arbitrary continuous mapping, we conclude from Corollary 5.3 that

COROLLARY 5.4. *Every semi-confluent image of a dendrite is a dendrite.*

As for confluent mappings (see [3], Corollary 20, p. 32), we have

THEOREM 5.5. *Every semi-confluent image of an arc is an arc.*

Proof. Let a semi-confluent mapping f map an arc X onto a continuum Y. Since X is a dendrite, Y is a dendrite. Suppose that Y contains a triode T. By Corollary 3.2 the arc X contains an arc C such that $f(C) = T$, but a mapping $f|C$ is semi-confluent by Theorem 3.7 .Therefore we have a semi-confluent mapping $f|C$ of an arc C onto a triode, which contradicts Lemma 4.1. Thus the dendrite Y contains no triode, whence it is an arc.

A point p of a dendroid X is called a *ramification point* (in the classical sense) if it is the common end-point of three (or more) arcs in X whose only common point is p. A dendroid having exactly one ramification point is called a *fan* (see [3], p. 6). The ramification point of a fan is called its *top*.

We now prove a theorem which is similar to Theorem 12 in [3], p. 32.

THEOREM 5.6. *A semi-confluent image of a fan is a fan (or an arc), and the top of the model is mapped on the top of the image.*

Proof. Let X be a fan with the top t, and let f be a semi-confluent mapping of X. Since X is a dendroid by definition, $f(X)$ is a dendroid by virtue of Corollary 5.3. Suppose that there are in $f(X)$ two different ramification points a and b. So one of them, say a, is different from $f(t)$. Thus there is a triode Q of $f(X)$ such that $a \in Q \subset f(X)\setminus\{f(t)\}$. So $f^{-1}(Q) \subset X\setminus\{t\}$, whence every component of $f^{-1}(Q)$ is an arc. Let A be the component of $f^{-1}(Q)$ such that $f(A) = Q$, which does exist by Corollary 3.2. But $f|A$ is semi-confluent by Theorem 3.7, consequently Q is an arc by Theorem 5.5, a contradiction.

A fan X with the top t is said to be *smooth* provided that if a sequence $\{a_n\}$ of points of X tends to a limit point a, then the sequence of arcs $\{ta_n\}$ is convergent and $\lim_{n\to\infty} ta_n = ta$ (see [3], p. 7). A confluent image of a smooth fan is a smooth fan (see [3], Theorem 13; p. 33). But semi-confluent mappings of fans do not preserve smoothness in general. This can be seen from the following

EXAMPLE 5.7. Put in the Cartesian coordinates in the plane $a_n = (2^{-n}, -1)$, $b_n = (2^{-n}, 0)$, $c_n = (3\cdot 2^{-(n+1)}, 0)$ and $d_n = (3\cdot 2^{-(n+1)}, 1)$. Join consecutively a_n, b_n, c_n, d_n and the point $(0, 2)$ by straight line segments and take the closure of the union of polygonal lines obtained in this way. The resulting continuum M is a smooth fan with $(0, 2)$ as the top. Define $f(x, y) = (x, |y|)$ for each point $(x, y) \in M$. Thus the mapping f is semi-confluent and the image $f(M)$ is a non-smooth fan.

A *tree* is a one-dimensional polyhedron containing no simple closed curve. Clearly, trees are dendrites having a finite set of end-points. A continuum X is said to be *tree-like* (*arc-like*) if for each $\varepsilon > 0$ there exist a tree (an arc) Y and a map f from X onto Y such that $\operatorname{diam} f^{-1}(y) < \varepsilon$ for every $y \in Y$.

McLean [8] proved that confluent images of tree-like curves are tree-like. We have the following

QUESTION 5.8. *Is a semi-confluent image of a tree-like (arc-like) curve tree-like (arc-like)?*

References

[1] J. J. Charatonik, *On ramification points in the classical sense*, Fund. Math. 51 (1962), pp. 229–252.
[2] — *Confluent mappings and unicoherence of continua*, Fund. Math. 56 (1964), pp. 213–220.
[3] — *On fans*, Dissert. Math. (Rozprawy Matem.) 54, Warszawa 1967.
[4] — *On decompositions of λ-dendroids*, Fund. Math. 67 (1970), pp. 15–30.
[5] S. Eilenberg, *Sur les transformations d'espaces métriques en circonférence*, Fund. Math. 24 (1935), pp. 160–176.
[6] K. Kuratowski, *Topology*, Vol. I, New York–London–Warszawa 1966.
[7] — *Topology*, Vol. II, New York–London–Warszawa 1968.
[8] T. B. McLean, *Confluent images of tree-like curves are tree-like*, Duke Math. J. 39 (1972), pp. 465–473.
[9] H. C. Miller, *On unicoherent continua*, Trans. Amer. Math. Soc. 69 (1950), pp. 179–194.
[10] H. Schirmer, *Coincidence producing maps onto trees*, Canad. Math. Bull. 10 (1967), pp. 417–423.
[11] S. Stoïlow, *Sur les transformations continues et le topologie des fonctions analytiques*, Ann. Sci. Ec. Norm. Sup. III, 45 (1928), pp. 347–382.
[12] A. D. Wallace, *Quasi-monotone transformations*, Duke Math. J. 7 (1940), pp. 136–145.
[13] G. T. Whyburn, *Non-alternating transformations*, Amer. J. of Math. 56 (1934), pp. 294–302.
[14] — *Analytic Topology*, New York 1942.

INSTYTYT MATEMATYCZNY UNIWERSYTETU WROCŁAWSKIEGO
INSTITUTE OF MATHEMATICS OF THE WROCŁAW UNIVERSITY

Reçu par la Rédaction le 10. 7. 1972

There is no universal totally disconnected space

by

R. Pol (Warszawa)

Abstract. From the results of this note it follows that there is no universal space in the class of all separable metrizable totally disconnected spaces.

1. In this note we shall prove the following

THEOREM. *Let $\mathfrak{m}$ be an infinite cardinal and let X be a completely regular space of weight $\mathfrak{m}$ and cardinality $2^{\mathfrak{m}}$. If a completely regular space M of weight $\mathfrak{m}$ contains topologically every completely regular space of weight $\leqslant \mathfrak{m}$ which admits a one-to-one mapping onto X, then M contains topologically all completely regular spaces of weight $\leqslant \mathfrak{m}$.*

From the theorem we obtain the following.

COROLLARY. *There is no universal space in the class of all totally disconnected, completely regular spaces of weight $\leqslant \mathfrak{m} \geqslant \aleph_0$.*

The proof of our theorem is based on a construction of Hilgers that we recall in section 2. This construction gives also an interesting example of a totally disconnected, metrizable, separable space that we describe in section 5.

Our terminology and notations are as in [1] and [3]. In particular, the symbol $f\colon X \to Y$ and the word "mapping" always mean "continuous function". The symbol $X \underset{\text{top}}{\subset} Y$ means that X is topologically contained in Y. Finally $I^{\mathfrak{m}}$ and $D^{\mathfrak{m}}$ denote the Tychonoff Cube and the Cantor Cube of weight $\mathfrak{m}$, respectively. By a *totally disconnected* space we mean a space which is not connected between any pair of points, in other words such that every quasi-component consists of a single point.

2. The Hilgers construction (cf. [2]; [3], § 27, IX). Let S be a topological space, $T \subseteq S$, and let $\mathfrak{A}$ be a family of subsets of the product $S \times S$. Suppose that there exists a function φ which establishes a one-to-one correspondence between the elements of T and $\mathfrak{A}$. For every $t \in T$ let us choose an element $f(t) \in S$ such that $(t, f(t)) \in \{t\} \times S \setminus \varphi(t)$ if this is possible and an arbitrary $f(t) \in S$ in the opposite case. We denote the graph $\{(t, f(t)) |\ t \in T\}$ by H and call it *the Hilgers set* for the family $\mathfrak{A}$ and the set T. Let us observe that