This fact is even more apparent in Corollary 3.6. Indeed, we have already seen, Lemma 3.7, that for each proper subset $J \subset I$, $E_J = \operatorname{ran} S_\tau$ for any projection τ whose essential domain is equal to J. Thus, (J_1^*)-(J_3^*) *are statements describing the properties of the sets* $\operatorname{ran} S_\tau$ *and* $\ker S_\tau$ *for all projections* $\tau \in I^I$. Consequently, if $\mathfrak{A}$ is a polyadic algebra, then the quantifier structure of $\mathfrak{A}$, as well as the connections between the quantifier structure and the transformation structure of $\mathfrak{A}$, may be described entirely in terms of the sets $\operatorname{ran} S_\tau$ and $\ker S_\tau$ for all projections $\tau \in I^I$. These, then, are the chief structural components of every polyadic algebra.

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Locating cones and Hilbert cubes in hyperspaces

by

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Abstract. Let X be a metric continuum. Let $C(X)$ denote the space of all nonempty subcontinua of X. It is shown that if X is decomposable, then $C(X)$ contains a 2-cell. This result is then used in several ways. For example, a characterization of hereditary indecomposability is obtained answering a question of B. J. Ball in a strong way. Also, for certain X, n-cells are located in $C(X)$ where they were not known to be previously, and necessary conditions are obtained in order that the cone over X be homeomorphic to $C(X)$. A general result, which locates Hilbert cubes in $C(X)$, is proved and then applied to show that certain classes of continua X have the property that $C(X)$ contains a Hilbert cube or the cone over X. Some unsolved problems are stated.

Key words and phrases. Chainable, circle-like, composant, decomposable continuum, dimension, indecomposable continuum, local dendrite, multicoherence degree, order of a point, ramification point, segment (in the sense of Kelley), upper semicontinuous decomposition.

1. Introduction. A *continuum* is a nonempty compact connected metric space. The term *nondegenerate* will be used to mean that a space has more than one point. A continuum is said to be *decomposable* if and only if it is the union of two of its proper subcontinua, *indecomposable* if and only if it is not decomposable, and *hereditarily indecomposable* if and only if each of its subcontinua is indecomposable. For definitions not given in this paper, we refer the reader to the texts listed in the references.

The *hyperspace* of a continuum X will mean, throughout this paper, the space of all (nonempty) subcontinua of X with the topology induced by the Hausdorff metric H (see [7] or [10, p. 47]); it is denoted by $C(X)$. Recognizing when and where $C(X)$ contains the cone over X or over other continua has proved to be useful information (see [12]). Much work has been done, especially recently (see [2], [15], and [17]), relating the space $C(X)$ and the cone over X. For example, J. T. Rogers, Jr. [15] investigated necessary conditions in order that $C(X)$ be homeomorphic in a "nice way" to the cone over X. We note that, in [2] and [15], the

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authors point out that if Y is an hereditarily indecomposable continuum, then the cone over Y is not homeomorphic to $C(Y)$. However, Rogers [15] showed that there is always a monotone open mapping of the cone over any hereditarily indecomposable continuum onto its hyperspace (the mapping is not arc-preserving as claimed in Theorem 9 of [15]). Recently, B. J. Ball posed the following question to the author: If Y is an hereditarily indecomposable continuum, then does the hyperspace $C(Y)$ ever contain a topological copy of the cone over Y?

In section 2 we prove what is apparently a very useful result (Theorem 1), namely that the hyperspace of a decomposable continuum contains a 2-cell. We show that the answer to B. J. Ball's question above is no; in fact, we obtain a stronger result (Theorem 2) which, by Theorem 1, has a converse. Theorem 2 and its converse provide a characterization (Theorem 3) of hereditary indecomposability. We give an example near the end of section 2 of some decomposable continua Y such that no hyperspace of any such Y contains a topological copy of the cone over Y (each such hyperspace does contain a 2-cell and, hence, cones over other continua). We note that, by results in [3] (also see [5]), there are hereditarily indecomposable continua Y such that $C(Y)$ is infinite-dimensional; yet, by Theorem 2 below, $C(Y)$ does not contain a topological copy of the cone over *any* nondegenerate continuum (recall [7] that $C(Y)$ is arcwise connected). In contrast to this, $C(X)$ may contain, or even be equal to, the cone over X when X is indecomposable. For example, Rogers [17] showed that $C(\Sigma)$ is homeomorphic to the cone over Σ when Σ is a solenoid; also, Rogers mentioned in [15] that the indecomposable chainable continuum in [10, p. 205] has the same property. In section 3 we obtain, among other results, some necessary conditions for a continuum in order that its hyperspace and cone be homeomorphic.

In section 4 we give a result (Theorem 6) which shows that, under certain very general conditions, $C(X)$ contains a Hilbert cube. Furthermore, Theorem 6 shows where such a Hilbert cube is located and gives an explicit homeomorphism between it and the standard Hilbert cube. In 5.1 of [7], Kelley determined the infinite-dimensionality of the space of all nonempty compact subsets of any nondegenerate continuum by showing that it always contains a Hilbert cube. The question — when does $C(X)$ contain a Hilbert cube? — has not, until this paper, been investigated. Some results in the literature (see, for example, 5.3 of [7] and [16]) show that $C(X)$ is infinite-dimensional for certain types of continua X by showing that $C(X)$ contains an n-cell for each n. It is important to note the fact that a hyperspace containing an n-cell for each n does not, in general, imply that the hyperspace contains a Hilbert cube (see the example in section 5). However, using our Theorem 6, we will show (see Corollary 1 of section 4 and see section 5) that the continua with infinite-dimensional hyperspace, in section 5 of [7] and in [16], all have the property that their hyperspace contains a Hilbert cube. In section 4 we also show that the hyperspace of any locally connected continuum L contains the cone over L (see Theorem 8).

We make some comments about the dimension-theoretic consequences of some of our results. In [5] we answered to some extent the question implicit in [7, p. 22] of the global dimension of the hyperspace of a non-locally connected continuum; this question was investigated further in [16]. In fact, results in [5] and [16] solve the problem of the global dimension of $C(X)$ for most continua X. The question of the dimension, at certain points, of a hyperspace is of interest (see 5.3 of [7], [8], and [13]). Locating 2-cells as in section 2 (Theorem 1 and Example 1), Hilbert cubes as in section 4, etc. is useful in determining the dimension of a hyperspace at certain points and in constructing a geometric model for a hyperspace (see Example 1 and Example 2).

The following pertinent definitions and notation are not completely standard. A *figure "T"* is a space homeomorphic to $\{(x, y) \in R^2\colon x = 0$ and $0 \leqslant y \leqslant 1\} \cup \{(x, y) \in R^2\colon -1 \leqslant x \leqslant +1$ and $y = 1\}$, where R^2 denotes the cartesian product of the reals with themselves. Let (X, d) be a continuum. The *cone over* X, denoted by $\mathrm{Cone}(X)$, is the decomposition space of the upper semicontinuous decomposition $X\times[0,1]/X\times\{1\}$. The *base of* $\mathrm{Cone}(X)$ is $X\times\{0\}$ and is denoted by $B(X)$. The *vertex of* $\mathrm{Cone}(X)$ is the point $X\times\{1\}$ in $\mathrm{Cone}(X)$. We remark that if X is nondegenerate and if X contains no arc, then the base of $\mathrm{Cone}(X)$ and the vertex of $\mathrm{Cone}(X)$ are topologically determined. If A is a nonempty compact subset of X and $\varepsilon > 0$, then $W(\varepsilon, A) = \{x \in X\colon d(x, a) < \varepsilon$ for some $a \in A\}$. The definition of *segment*, as we use it in this paper, appears in [7, p. 24). Now assume X is locally connected. A point $p \in X$ is a *point of order* n *in* X if and only if n is the smallest natural number such that p is the common noncut point of at most n arcs $A_1, A_2, \ldots, A_n$ in X such that $A_i \cap A_j = \{p\}$ for $i \neq j$; p is of *infinite order in* X if and only if p is not of order n in X for any $n = 1, 2, \ldots$ A point $p \in X$ is a *ramification point of* X if and only if p is of order greater than or equal to three in X. Throughout this paper, let I_∞ denote the countably infinite cartesian product of the interval $[0,1]$ with the product topology. A continuum homeomorphic to I_∞ is called a *Hilbert cube.*

Let X be a nondegenerate hereditarily indecomposable continuum. For each A and B in $C(X)$ such that $A \subset B$ and $A \neq B$, let $\alpha(A, B) = \{D \in C(B)\colon A \subset D \subset B\}$. Using the proof of 2.3 of [7], in particular 2.4 and 2.5, it is easy to see that $\alpha(A, B)$ is an arc in $C(B)$ with noncut points A and B.

In [19] Whitney showed that a continuous "sizing" function $\mu\colon C(X) \to [0,1]$ can be defined, for any nondegenerate continuum X. Since

Whitney's "sizing" function is used in virtually all the papers on hyperspaces referenced in this paper, we say no more about it here except to mention that, throughout this paper, μ denotes a continuous function from $C(X)$ onto $[0,1]$ satisfying 1.3 and 1.4 of [7].

Throughout this paper the symbol $\bar{S}$ means the closure of S and the symbol $\times$ will denote cartesian product.

2. Hereditarily indecomposable continua. We first prove the following lemma to be used in the proof of Theorem 2.

LEMMA 1. *Let M be an hereditarily indecomposable continuum. If A and B are two disjoint subcontinua of M which are contained in the same composant of M, then $a(A, M) \cup a(B, M)$ is a figure "T" whose point of order three is not M.*

Proof. It is easy to see, using just the indecomposability of M and the conditions on A and B, that there is a proper subcontinuum P of M containing both A and B. Let $K \subset P$ be a subcontinuum of M irreducibly containing $A \cup B$. Now, using the hereditary indecomposability of M, it is easy to see that

$$a(A, M) \cup a(B, M) = a(A, K) \cup a(B, K) \cup a(K, M)$$

(note that $K \neq M$ so that $a(K, M)$ is defined). Therefore, since K is a proper subcontinuum of M irreducibly containing $A \cup B$, it follows that $a(A, M) \cup a(B, M)$ is a figure "T" with $K \neq M$ as its point of order three. This proves Lemma 1.

THEOREM 1. *If D is a decomposable continuum, then $C(D)$ contains a 2-cell.*

Proof. Let D be a decomposable continuum and let A and B be proper subcontinua of D such that $D = A \cup B$. We will consider two cases.

Case I. $A \cap B$ is connected; in this case let σ_1: $[0,1] \to C(D)$ be a segment (in the sense of [7, p. 24]) from $A \cap B$ to A and let σ_2: $[0,1] \to C(D)$ be a segment from $A \cap B$ to B; note that $\sigma_1(t) \subset A$ and $\sigma_2(t) \subset B$ for all $t \in [0,1]$. Define f: $[0,1] \times [0,1] \to C(D)$ by $f(s,t) = \sigma_1(s) \cup \sigma_2(t)$, for any $(s,t) \in [0,1] \times [0,1]$ ($f(s,t) \in C(D)$ for any $(s,t) \in [0,1] \times [0,1]$ because $\sigma_1(s) \cap \sigma_2(t) = A \cap B$ and, therefore, $\sigma_1(s) \cap \sigma_2(t) \neq \emptyset$ for any $(s,t) \in [0,1] \times [0,1]$). It is easy to verify that f is continuous. It is easy to show, using that $\sigma_1(s) \cap \sigma_2(t) = A \cap B$, that f is one-to-one. Therefore, f is a homeomorphism which proves that $C(D)$ contains a 2-cell.

Case II. $A \cap B$ is not connected; let K and L be two distinct components of $A \cap B$ and let U and V be open subsets of D such that $K \subset U$, $L \subset V$, and $\bar{U} \cap \bar{V} = \emptyset$. Let E denote the component of $\overline{U \cap A}$ containing K and let F denote the component of $\overline{V \cap A}$ containing L. We remark that, by 10.1 of [20, p. 16], K is a proper subset of E and L is a proper subset of F. Now, Case I applies by simply noting that $B \cup E$ and $B \cup F$ satisfy the conditions A and B satisfied in Case I. This completes the proof of Theorem 1.

Remark. The proof of Theorem 1 shows *where* certain 2-cells in $C(X)$ are located (cf. part (2) of section 3).

THEOREM 2. *If X is a nondegenerate continuum and Y is an hereditarily indecomposable continuum, then* Cone(X) *can not be (topologically) embedded in $C(Y)$.*

Proof. We can obviously assume Y is nondegenerate. Suppose Cone(X) can be embedded in $C(Y)$ and let $Z \subset C(Y)$ be homeomorphic, by a homeomorphism h, to Cone(X). Let $M = \bigcup Z$ (clearly, by 1.2 of [7], M is a continuum). Then:

(1) $Z \subset C(M)$ and $M \in Z$. The first part is obvious, so we now prove $M \in Z$. Since M is indecomposable, there are points x and y in different composants of M. Let $A_x, A_y \in Z$ such that $x \in A_x$ and $y \in A_y$. If $A_x = A_y$, then $A_x = M = A_y$ (because x and y are in different composants of M) and $M \in Z$. Thus, we assume $A_x \neq A_y$. Since Z is arcwise connected, there is an arc $\gamma \subset Z$ with noncut points A_x and A_y. From 1.2 of [7], $\bigcup \gamma$ is a subcontinuum of M. Therefore, since $x, y \in (\bigcup \gamma)$ and x and y are in different composants of M, $\bigcup \gamma = M$. Hence, by 8.1 of [7], $M \in \gamma$. Thus, $M \in Z$. This completes the proof of (1).

(2) If $K \in Z$ and $K \subset L \in C(M)$, then $L \in Z$. Assume $K \neq M$ (the case when $K = M$ is taken care of by (1)). Note that $L \in a(K, M)$. Since M is hereditarily indecomposable, it follows from 8.4 of [5] that $a(K, M)$ is the only arc in $C(M)$ with noncut points K and M. Since Z is arcwise connected, we now have, using (1) above, that $a(K, M) \subset Z$. Hence $L \in Z$. This proves (2).

(3) The continuum X does not contain an arc. To see this, suppose X contained an arc. Then Z, hence $C(M)$, would contain a 2-cell. Thus, $C(M)$ would not be uniquely arcwise connected. This contradiction to 8.4 of [5] proves (3).

(4) All figures "T" in Z have M as their point of order three. In view of (3) and geometric properties of cones, it suffices to show that there are arbitrarily small arcwise connected open subsets of Z to which M belongs. Let $\varepsilon > 0$ and let $B(M, \varepsilon) = \{A \in C(Y)\colon H(M, A) < \varepsilon\}$. Let $t_0 = \mu(M)$, where μ denotes Whitney's function discussed in the Introduction. First we prove

(#) there exists $\eta > 0$ such that

$$[\mu^{-1}((t_0 - \eta, t_0 + \eta) \cap Z)] \subset B(M, \varepsilon).$$

Suppose not so that, for each $n=1,2,\ldots$, there exists

$$A_n \in [\mu^{-1}((t_0-1/n,\, t_0+1/n)) \cap Z]$$

such that $H(M, A_n) \geqslant \varepsilon$. By compactness of Z, the sequence $\{A_n\}_{n=1}^{\infty}$ has a subsequence $\{A_{n_i}\}_{i=1}^{\infty}$ converging to some $A_0 \in Z$. By the continuity of μ, $\mu(A_0)=t_0$. Hence, $\mu(A_0)=\mu(M)$ which, by the fact that any member of Z is contained in M and by 1.3 of [7], implies $A_0 = M$. However, since $\{A_{n_i}\}_{i=1}^{\infty}$ converges to A_0 and $H(M, A_{n_i}) \geqslant \varepsilon$ for each $i=1,2,\ldots$, we have that $H(M, A_0) \geqslant \varepsilon$. This contradiction proves (#). Next note that since any member of Z is contained in M,

(##) $\quad \mu^{-1}((t_0-\delta,\, t_0+\delta)) \cap Z = \mu^{-1}((t_0-\delta,\, t_0]) \cap Z \quad$ for any $\delta > 0$.

Now, let $\eta > 0$ be fixed such that the containment in (#) holds. Let $W = \mu^{-1}((t_0-\eta,\, t_0+\eta)) \cap Z$. Then, W is an open subset of Z, $M \in W$, and $W \subset B(M, \varepsilon)$. Furthermore, using (2) above and (##), it is easy to see that W is arcwise connected. This proves (4).

(5) If $K, L \in Z$, then K and L are in different composants of M or at least one of K or L is contained in the other. Assume $K \not\subset L$ and $L \not\subset K$; then, since Y is hereditarily indecomposable, $K \cap L = \emptyset$. Suppose K and L were in the same composant of M. Then, by (2) and Lemma 1, $\alpha(K, M) \cup \alpha(L, M)$ would be a figure "T" in Z whose point of order three would not be M. This contradicts (4). We have now proved (5).

Now, to complete the proof of Theorem 2, let

$$\Gamma = \{F \in Z\colon \text{ if } E \in Z \text{ and } E \subset F, \text{ then } E = F\}.$$

We show that $\Gamma = h(B(X))$. Let $A \in h(B(X))$. Since (by (3)) $B(X)$ does not contain an arc, A is not a cut point of any arc in Z. Suppose $A \notin \Gamma$. Then there would exist $E \in Z$ such that $E \subset A$ and $E \neq A$. This implies that A is a cut point of $\alpha(E, M)$. Since, by (2), $\alpha(E, M) \subset Z$, A is a cut point of an arc in Z, a contradiction. Thus, $A \in \Gamma$ and we have proved that $h(B(X)) \subset \Gamma$. Conversely, let $F \in \Gamma$. Then, since M is hereditarily indecomposable and since $F \neq M$, it follows that F is not a cut point of any arc in Z. Thus, $F \in h(B(X))$ which proves $\Gamma \subset h(B(X))$. This completes the proof that $\Gamma = h(B(X))$. Let $F, F' \in \Gamma$ such that $F \neq F'$. By (5) and the definition of Γ, F and F' are contained in different composants of M; also, since Γ is a continuum (because $\Gamma = h(B(X))$), $\bigcup \Gamma$ is a continuum (use 1.2 of [7]). It now follows that $\bigcup \Gamma = M$. Now let $F_0 \in \Gamma$ (note that $F_0 \neq M$). Let y_0 be in the composant of M determined by F_0 such that $y_0 \notin F_0$. Since $\bigcup \Gamma = M$, there is a member $F_1 \in \Gamma$ such that $y_0 \in F_1$. By (5), $F_0 \subset F_1$ or $F_1 \subset F_0$. Therefore, since $F_0, F_1 \in \Gamma$, it follows that $F_0 = F_1$. This contradicts the fact that $y_0 \in (F_1 - F_0)$ and completes the proof of Theorem 2.

Combining Theorem 1 and Theorem 2, we see that the property

(*) $\quad C(Y)$ does not contain a topological copy of the cone over any nondegenerate continuum

actually characterizes hereditary indecomposability. We state this observation as Theorem 3.

THEOREM 3. *A continuum Y is hereditarily indecomposable if and only if $C(Y)$ does not (topologically) contain the cone over any nondegenerate continuum.*

Remark. For a continuum Y, let (**) be the statement

(**) $\quad C(Y)$ does not contain a topological copy of Cone(Y).

If Y is hereditarily indecomposable, then (**) holds — in fact, this is the exact answer to B. J. Ball's question stated in the Introduction. However, (**) does not imply Y is hereditarily indecomposable, as the following example shows.

EXAMPLE 1. Let Y be (the topological sum of) two nondegenerate hereditarily indecomposable continua Y_1 and Y_2 joined together at one and only one point of each. We denote this "wedge" point by w. The continuum Y is obviously decomposable; we show Y satisfies (**). This is done in several steps. Suppose Cone(Y) can be embedded in $C(Y)$ and let $Z \subset C(Y)$ be homeomorphic to Cone(Y). Let $C_w(Y) = \{A \in C(Y)\colon w \in A\}$.

Step 1: $C_w(Y)$ is a 2-cell and $C(Y) - [C(Y_1) \cup C(Y_2)]$ is an open subset of $C(Y)$ whose closure (in $C(Y)$) is the 2-cell $C_w(Y)$. The proof of this step uses techniques similar to those used in Theorem 1 above. Let $\sigma_1\colon [0,1] \to C_w(Y)$ be a segment from $\{w\}$ to Y_1 and let $\sigma_2\colon [0,1] \to C_w(Y)$ be a segment from $\{w\}$ to Y_2. Define $f\colon [0,1]\times[0,1] \to C_w(Y)$ by

$$f(s,t) = \sigma_1(s) \cup \sigma_2(t) \quad \text{for all } (s,t) \in [0,1]\times[0,1].$$

It is easy, using that $(K \cap Y_i) \in C(Y_i)$ for each $K \in C_w(Y)$ and each $i=1$ or 2, to verify that f is a homeomorphism of $[0,1]\times[0,1]$ onto $C_w(Y)$. Hence, $C_w(Y)$ is a 2-cell. Let $U = f((0,1]\times(0,1])$. It is easy to see that $U = C(Y) - [C(Y_1) \cup C(Y_2)]$ and that $\overline{U} = C_w(Y)$. This completes the proof of Step 1.

Step 2: If $A \in Z$, then $A \in C(Y_1)$ or $A \in C(Y_2)$. To prove this we will use the following lemma. Its full statement is not really necessary here but we include it for the sake of completeness.

LEMMA 2. *If K and L are nondegenerate continua such that $K \times L$ is embeddable in the plane, then either K and L are both arcs or one of them is an arc and the other is a simple closed curve.*

Proof. Assume $K \times L$ is embeddable in the plane. Let N be a nondegenerate subcontinuum of K. Since $K \times L$ is embeddable in the plane, L is embeddable in the plane. Therefore, by Theorem IV3 of [6, p. 44] and the fact that K is nondegenerate, L is one-dimensional. Therefore, $N \times L$ is 2-dimensional (use the Remark in [6, p. 34] and the fact that $N \times L$ is embeddable in the plane). Hence, since $N \times L$ is 2-dimensional and embeddable in the plane, $N \times L$ contains a nonempty subset U such that U is homeomorphic to an open subset of the plane (use Theorem IV3 of [6, p. 44]). Clearly, since $K \times L$ is embeddable in the plane, U is an open subset of $K \times L$. Since $U \subset N \times L$, we now have that $N \times L$ is not nowhere dense in $K \times L$. Therefore, N is not nowhere dense in K. Since N was an arbitrary nondegenerate subcontinuum of K, we conclude from Theorem 2 of [10, p. 247] that (i) K is locally connected. Similarly, (ii) L is locally connected. Observing that the cartesian product of a figure "T" and an arc is not embeddable in the plane, we have that, since $K \times L$ is embeddable in the plane and L contains an arc (use (ii)) K does not contain a figure "T". Similarly, L does not contain a figure "T". It is easy to prove that a nondegenerate locally connected continuum which contains no figure "T" must be an arc or a simple closed curve (if such a continuum contained no simple closed curve, then it would be a dendrite [20, p. 88] containing no figure "T", hence an arc; if such a continuum contained a simple closed curve as a proper subcontinuum, then arcwise connectivity could be used to show it contained a figure "T"). Hence, K and L are each an arc or a simple closed curve. Since the cartesian product of two simple closed curves is a torus and, therefore, not embeddable in the plane, we have now proved Lemma 2.

Now, to complete the proof of Step 2, suppose there is an $A \in Z$ such that $A \notin C(Y_1)$ and $A \notin C(Y_2)$. Note that, as a general fact, any nonempty open subset of a cone over a continuum contains the cartesian product of a nondegenerate subcontinuum of the base of the cone with a nondegenerate subinterval of $[0,1]$ (use 10.1 of [20, p. 16]). Now, since

$$A \in \left[Z \cap \left(C(Y) - [C(Y_1) \cup C(Y_2)]\right)\right],$$

$Z \cap \left(C(Y) - [C(Y_1) \cup C(Y_2)]\right)$ is a nonempty open subset of Z and, hence, contains a cartesian product $K \times J$ of a nondegenerate subcontinuum K of Y with an arc J. Hence, by Step 1, $K \times J$ lies in a 2-cell. This is a contradiction to Lemma 2 (because K, being a subcontinuum of Y, can not even *contain* an arc!). This completes the proof of Step 2.

Step 3: Completion of the proof that Y satisfies (**). Note that $C(Y_1) \cup C(Y_2)$ is connected and that, by Step 2, $Z \subset [C(Y_1) \cup C(Y_2)]$. Also note that $[C(Y_1) \cup C(Y_2)] - \{w\}$ is not connected and, in fact, is the union of the two disjoint nonempty open sets $C(Y_1) - \{w\}$ and $C(Y_2) - \{w\}$. Now, using Theorem 2, $Z \not\subset C(Y_1)$ and $Z \not\subset C(Y_2)$. Hence, $Z \cap [C(Y_i) - \{w\}] \neq \emptyset$ for each $i = 1$ and 2. Since Z is connected, it follows that $\{w\} \in Z$. Hence, $Z - \{w\}$ is not connected. However, this contradicts the general fact that no point of the cone over a nondegenerate continuum is a net point of the cone. Thus, $\mathrm{Cone}(Y)$ can not be (topologically) embedded in $C(Y)$, i.e., Y satisfies (**).

PROBLEM 1. Can $C(Y)$ ever contain a topological copy of $X \times [0,1]$ when Y is an hereditarily indecomposable continuum and X is a nondegenerate continuum? More generally, can $C(Y)$ ever contain the cartesian product of two nondegenerate continua when Y is hereditarily indecomposable? If the answer to these questions is no, then these properties would be equivalent, by Theorem 3, to hereditary indecomposability of Y.

3. Theorem 1 — further comments and applications. We used Theorem 1 above to prove Theorem 3. Theorem 1 seems to be quite useful in other connections. For example:

(1) In [5] we answered to some extent a dimension question implicit in [7, p. 22] and we answered the question completely for certain special cases which have been of interest to mathematicians (see [14] and [18]). Among other results, we showed, in Theorem 1 of [5], that the dimension of $C(X)$ is at least two for any nondegenerate continuum X. For the class of those continua which contain a decomposable subcontinuum, Theorem 1 above is a much stronger result than Theorem 1 of [5].

(2) Techniques used to prove Theorem 1 above can be applied to prove the following result (note: these techniques show where certain n-cells in $C(X)$ are located and, hence, may be used to determine information about the dimension of $C(X)$ at certain points).

THEOREM 4. *If a continuum X contains two subcontinua A and B such that $A \cap B$ contains n components, then $C(X)$ contains an n-cell.*

(3) As observed in [15], there are several chainable continua, some indecomposable and some decomposable, such that their hyperspace is homeomorphic to their cone. Also, [15] and [17], some non-chainable continua have this property. Recognizing certain spaces of nonempty subcontinua as cones has proved fruitful in determining properties of those spaces. For example, in [12] we used this to answer a question of Knaster by showing that there is a continuum with the fixed point property whose space of nonempty subcontinua does not have the fixed point property. These comments lead us to the following problem.

PROBLEM 2. Which continua N have the property that $C(N)$ is homeomorphic to $\mathrm{Cone}(N)$? Some chainable continua have this property and some do not (obviously, the pseudo-arc). Which chainable continua have the property that their hyperspace is homeomorphic to their cone?

From Theorem 5 of [16] we can conclude immediately that if X is a finite-dimensional continuum whose cone and hyperspace are homeomorphic, then the dimension of X is less than or equal to two. Actually, we can prove a much stronger result which we state as Lemma 3.

LEMMA 3. *If X is a finite-dimensional continuum such that* Cone(X) *is homeomorphic to $C(X)$, then X does not contain a nondegenerate hereditarily indecomposable continuum. Thus, X is one-dimensional.*

We do not prove Lemma 3 here except to mention that the second part of it is a consequence of the first part by a result of Bing [3].

Using Lemma 3 and Theorems 1 and 4 above, progress can be made on Problem 2 for the class of decomposable continua. For example, we have the following result.

THEOREM 5. *If N is a decomposable continuum such that $C(N)$ is homeomorphic to* Cone(N), *then N contains an arc; furthermore,*

(a) *$r(S) \leqslant 1$ for all subcontinua S of N (where $r(S)$ denotes the multicoherence degree of S* [20, p. 83]) *and*

(b) *N is a-triodic.*

Proof. Assume N is decomposable. Then, by Theorem 1 above, $C(N)$ contains a 2-cell. Thus, since $C(N)$ and Cone(N) are homeomorphic, Cone(N) contains a 2-cell. It follows that N must contain an arc.

(a) If there were a subcontinuum S of N such that $r(S) > 1$, then, by Theorem 4, $C(N)$ would contain a 3-cell. Hence, since $C(N)$ and Cone(N) are homeomorphic, Cone(N) would contain a 3-cell. But, by Lemma 3 above, N is one-dimensional so, by a result in [6, p. 34], the dimension of Cone(N) is exactly two. This contradiction proves (a).

(b) Since $C(N)$ is homeomorphic to Cone(N) and since, as observed in the proof of (a), the dimension of Cone(N) is two, we have that the dimension of $C(N)$ is two. By Corollary 2 in [15], N is a-triodic.

Remark. The first part of Theorem 5 seems especially interesting since there are hereditarily decomposable chainable continua which do not contain an arc (see, for example, [1]). We also remark that (a) and (b) of Theorem 5 might seem to suggest that an hereditarily decomposable continuum, whose hyperspace is homeomorphic to its cone, must be chainable or circle-like. However, in [15], Rogers showed that the hyperspace of a particular "circle-with-a-spiral" is homeomorphic to its cone.

4. A general result and some applications. In this section we prove a general theorem (Theorem 6) which shows that certain subspaces of $C(X)$ form a Hilbert cube. Then we apply this result to locally connected continua. Recall [20, p. 67] that a sequence $\{A_i\}_{i=1}^{\infty}$ of subsets of a metric space (X, d) is called a *null sequence* if and only if given $\varepsilon > 0$ there exists a natural number N such that, for $i \geqslant N$, the diameter of A_i is less than ε.

THEOREM 6. *Let (X, d) be a continuum and let Y be a subcontinuum of X. For each $i = 1, 2, \ldots$, assume Y_i is a subcontinuum of X such that*

(1) *$Y_1 \cap Y \neq \emptyset$ and $Y_i \not\subset Y$ for any $i = 1, 2, \ldots$;*

(2) *$(Y_i - Y) \cap (Y_j - Y) = \emptyset$ for $i \neq j$;*

(3) *the sequence $\{Y_i\}_{i=1}^{\infty}$ is a null sequence.*

For each $i = 1, 2, \ldots$, let $\sigma_i\colon [0,1] \to C(X)$ be a segment from Y to $Y \cup Y_i$. Then, for each $(t_1, t_2, \ldots, t_i, \ldots) \in I_\infty$,

$$\Big(\bigcup_{i=1}^{\infty} \sigma_i(t_i)\Big) \in C(X) \quad \text{and} \quad \sigma\colon I_\infty \to C(X),$$

given by

$$\sigma(t_1, t_2, \ldots, t_i, \ldots) = \bigcup_{i=1}^{\infty} \sigma_i(t_i) \quad \text{for each} \quad (t_1, t_2, \ldots, t_i, \ldots) \in I_\infty,$$

is a homeomorphism.

Proof. It is easy to verify, using condition (3), that

$$\Big(\bigcup_{i=1}^{\infty} \sigma_i(t_i)\Big) \in C(X) \quad \text{for each } (t_1, t_2, \ldots, t_i, \ldots) \in I_\infty .$$

Also, it is an easy consequence of condition (2) and 2.2 of [7] that σ is one-to-one. We show that σ is continuous. Let $t = (t_1, t_2, \ldots, t_i, \ldots) \in I_\infty$ and let $\{t^n\}_{n=1}^{\infty}$ be a sequence in I_∞ such that $\{t^k\}_{k=1}^{\infty}$ converges to t, $t^k = (t_1^k, t_2^k, \ldots, t_i^k, \ldots)$. We show that $\{\sigma(t^k)\}_{k=1}^{\infty}$ converges to $\sigma(t)$. Let $\varepsilon > 0$. Using (3) choose a natural number N such that if $i > N$, then $\operatorname{diam}(Y_i) < \varepsilon$. Next, using the continuity of σ_i for $i = 1, 2, \ldots, N$, choose a natural number K such that if $k \geqslant K$, then

$$H\big(\sigma_i(t_i^k), \sigma_i(t_i)\big) < \varepsilon \quad \text{for each } i \leqslant N$$

(where H denotes the Hausdorff metric). Now let $k \geqslant K$ and let $x \in \sigma(t^k)$. If $x \in Y$, then $x \in \sigma_i(t_i)$ for each $i = 1, 2, \ldots$, i.e., $x \in \sigma(t)$. Assume $x \notin Y$. Then there exists i_0 such that $x \in Y_{i_0}$. Since $x \in (Y_{i_0} - Y)$, we have from condition (2) that $x \in \sigma_{i_0}(t_{i_0}^k)$. If $i_0 \leqslant N$, then, since

$$H\big(\sigma_{i_0}(t_{i_0}^k), \sigma_{i_0}(t_{i_0})\big) < \varepsilon, \qquad x \in W\big(\varepsilon, \sigma_{i_0}(t_{i_0})\big);$$

hence, $x \in W\big(\varepsilon, \sigma(t)\big)$. If $i_0 > N$, then, because of the first part of (1) and the choice of N, there is a point $y \in Y$ such that $d(x, y) < \varepsilon$. Therefore, since $Y \subset \sigma(t)$, $x \in W\big(\varepsilon, \sigma(t)\big)$. In any case, we have shown that $\sigma(t^k) \subset W\big(\varepsilon, \sigma(t)\big)$. The same type of argument shows that $\sigma(t) \subset W\big(\varepsilon, \sigma(t^k)\big)$. Hence, $H\big(\sigma(t^k), \sigma(t)\big) < \varepsilon$ for all $k \geqslant K$. This proves σ is continuous. We have now shown that σ is a homeomorphism of I_∞ into $C(X)$. This completes the proof of Theorem 6.

Now we give an example of how to use Theorem 6 to "compute" a specific hyperspace.

EXAMPLE 2. Let X be the continuum pictured in Figure 1 below; X is the union of countably infinitely many "shorter and shorter" arcs $Y_1, Y_2, \ldots$ emanating from a common point p and disjoint outside p.

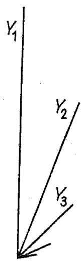

Fig. 1

For any subcontinuum Z of X such that $p \in Z$, let $C_p(Z) = \{A \in C(Z)\colon p \in A\}$. Let $Y = \{p\}$ and let σ_i, $i = 1, 2, \ldots$, and σ be as in Theorem 6. Since each Y_i is an arc, $C_p(Y_i)$ is an arc and $\sigma_i([0,1]) = C_p(Y_i)$ for each $i = 1, 2, \ldots$ Furthermore, $C_p(X)$ is a Hilbert cube and, in fact, $\sigma(I_\infty) = C_p(X)$. Clearly,

$$C(X) = C_p(X) \cup [\bigcup_{i=1}^{\infty} C(Y_i)]$$

and, for each $i = 1, 2, \ldots$, $C(Y_i)$ is a 2-cell [4] such that $C(Y_i) \cap C_p(X) = C_p(Y_i)$. Hence, $C(X)$ is a Hilbert cube together with the infinitely many "smaller and smaller free 2-cells" $C(Y_i)$ intersecting the Hilbert cube in the arc $C_p(Y_i)$. Using σ^{-1} to coordinatize the members of $C_p(X)$ as the points of I_∞, we have for each $i = 1, 2, \ldots$,

(a) $\sigma^{-1}(C_p(Y_i \cup Y_{i+1} \cup Y_{i+2})) = \{(\ldots, t_i, t_{i+1}, t_{i+2}, \ldots) \in I_\infty\colon\ t_j = 0$ for $j < i$ and for $j > i+2\}$, the 3-cell denoted by Q_i in Figure 2;

(b) $\sigma^{-1}(C_p(Y_i)) = \{(0, \ldots, 0, t_i, 0, \ldots) \in I_\infty\colon\ t_j = 0$ for $j \neq i\}$, the arc denoted by A_i in Figure 2;

(c) $\sigma^{-1}(Y_i) = \{(\ldots, 0, 1, 0, \ldots)\colon 1$ appears only in the ith coordinate and 0 appears in all other coordinates$\}$;

(d) $\sigma^{-1}(\{p\}) = \{(0, 0, \ldots, 0, \ldots)\}$.

A cross-section for this topological model of $C(X)$ is in Figure 2.

In the rest of this section we use Theorem 6 to obtain some results about locally connected continua.

A *linear graph* is a continuum which is the union of finitely many arcs and which contains only finitely many simple closed curves. In 5.4 of [7], Kelley showed that if a locally connected continuum is not a linear graph, then its hyperspace is infinite-dimensional. This was done (see the proofs of 5.3 and 5.4 of [7]) by showing that its hyperspace contains an n-cell for every n. Our next result, Theorem 7, improves this by showing that such a hyperspace actually contains a Hilbert cube.

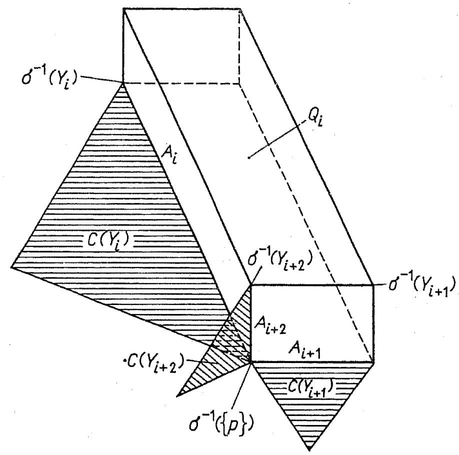

Fig. 2

THEOREM 7. *If X is a locally connected continuum which is not a linear graph, then $C(X)$ contains a Hilbert cube.*

Proof. The proof is, in spirit, motivated by the proof of 5.2 of [7]; however, more needs to be shown. We will take two main cases.

Case I: The space X contains infinitely many ramification points (see the definition at the end of section 1 above). For this case we consider two possibilities, Case I.1 and Case I.2.

Case I.1: Given any arc in X, only finitely many ramification points of X lie on that arc. Let $\{a_i\}$ be a sequence of distinct ramification points of X which, without loss of generality (by compactness of X), converges to a point $p \in X$ (also assume $p \neq a_i$ for any $i = 1, 2, \ldots$). Choose i_1 such that p and a_{i_1} lie together in a connected open subset of X of diameter less than one. Using 5.2 of [20, p. 38], let Y_1 be an arc in X with noncut points p and a_{i_1} such that the diameter of Y_1 is less than one. Assume inductively that we have defined n arcs $Y_1, Y_2, \ldots, Y_n$ in X such that $Y_i \cap Y_j = \{p\}$ for $i \neq j$ and such that, for each $i = 1, 2, \ldots, n$, the diameter of Y_i is less than $1/i$ and p is a noncut point of Y_i. Let U be a connected open subset of X such that $p \in U$, the diameter of U is less than $1/(n+1)$, and such that if $a_i \in U$, then any arc $\gamma \subset U$ with noncut points p and a_i has the property that $\gamma \cap (\bigcup_{i=1}^{n} Y_i) = \{p\}$ (if no such U existed, then consideration of smaller and smaller connected open sets containing p would produce infinitely many ramification points of X lying on one of the arcs $Y_1, Y_2, \ldots, Y_n$, a contradiction). Let $a_{i_{n+1}} \in U$. Using 5.2 of [20, p. 38], let Y_{n+1} be an arc in U with noncut points p and $a_{i_{n+1}}$. Letting $Y = \{p\}$, we see that $Y, Y_1, Y_2, \ldots$ satisfy conditions (1) through (3) of Theorem 6. Hence, by Theorem 6, $C(X)$ contains a Hilbert cube. This completes the proof for Case I.1.

Case I.2: Some arc Y in X contains infinitely many ramification points of X. Let $\{b_i\}_{i=1}^{\infty}$ be a sequence of distinct ramification points of X such that, for each $i = 1, 2, \ldots, b_i \in Y$. Then, a simple induction argument shows that there are arcs $Y_1, Y_2, \ldots, Y_i, \ldots$ such that (i) for $j \neq k$, $Y_j \cap Y_k = \emptyset$ and (ii) for $i = 1, 2, \ldots$, $Y_i \cap Y = \{b_i\}$ and the diameter of Y_i is less than $1/i$. It is clear that $Y, Y_1, Y_2, \ldots$ satisfy conditions (1) through (3) of Theorem 6. Hence, by Theorem 6, $C(X)$ contains a Hilbert cube. This completes the proof for Case I.2 which completes the proof of Case I.

Case II: The space X contains only finitely many ramification points. If each ramification point of X were of finite order in X, then X would be a linear graph. Hence, some point $p \in X$ is of infinite order in X. Let U be an open subset of X such that $p \in U$ and such that no ramification point of X, other than p, belongs to U. Let Y_1 be an arc in U such that p is a noncut point of Y_1 and the diameter of Y_1 is less than one. Assume inductively that we have defined n arcs $Y_1, Y_2, \ldots, Y_n$ in U such that $Y_i \cap Y_j = \{p\}$ for $i \neq j$ and such that, for each $i = 1, 2, \ldots, n$, the diameter of Y_i is less than $1/i$ and p is a noncut point of Y_i. Since p is of infinite order in X, there are $n+1$ arcs $Z_1, Z_2, \ldots, Z_{n+1}$ in U such that p is a noncut point of each and such that $Z_i \cap Z_j = \{p\}$ for $i \neq j$. Since U contains no ramification points of X other than p, at least one of the arcs $Z_1, Z_2, \ldots, Z_{n+1}$ must contain a subarc, call it S, with the property that $S \cap Y_i = \{p\}$ for each $i = 1, 2, \ldots, n$. Let Y_{n+1} be a subarc of S such that p is a noncut point of Y_{n+1} and the diameter of Y_{n+1} is less than $1/(n+1)$. Letting $Y = \{p\}$, we see that $Y, Y_1, Y_2, \ldots$ satisfy conditions (1) through (3) of Theorem 6. Hence, by Theorem 6, $C(X)$ contains a Hilbert cube. This completes the proof for Case II and Theorem 7 is proved.

COROLLARY 1. *Let X be a locally connected continuum. Then $C(X)$ contains a Hilbert cube if and only if X is not a linear graph.*

Proof. Use Theorem 7 above and 5.4 of [7].

We now state and prove one of our main applications of Theorem 6.

THEOREM 8. *If L is a nondegenerate locally connected continuum, then $C(L)$ contains a topological copy of* Cone(L).

Proof. First, assume that the dimension of $C(L)$ is not finite. Then, by 5.4 of [7], L is not a linear graph. Hence, by Theorem 7 above, $C(L)$ contains a Hilbert cube and, therefore, the cone over any continuum. Next assume that the dimension of $C(L)$ is finite. Then, by 5.4 of [7], L is a linear graph. Assume $C(L)$ does not contain a 3-cell; then L does not contain a figure "T". Thus, as pointed out near the end of the proof of Lemma 2, L must be an arc or a simple closed curve. In either case $C(L)$ is homeomorphic to Cone(L) (see [4]). Hence, we assume $C(L)$ contains a 3-cell. If L is embeddable in the plane, then Cone(L) is embeddable in a 3-cell, hence in $C(L)$. Assume L is not embeddable in the plane. Note that any linear graph is a local dendrite (for the definition of local dendrite see [10, p. 303]; the statement just made is an obvious consequence of Theorem 4 of [10, p. 303–304]). Hence, since we are assuming L is not embeddable in the plane, Theorem 7 of [10, p. 305–306] applies showing that L topologically contains at least one of the two skew curves in Figure 11 of [10, p. 305]. However, it is easy to show (see the proof of 5.3 of [7]) that the hyperspaces of nonempty subcontinua of these two skew curves each contains a 4-cell (actually, one contains an 8-cell and the other a 12-cell). Since any linear graph is one-dimensional, L is embeddable in a 3-cell [6, p. 56]. Hence, Cone(L) is embeddable in a 4-cell and, therefore, in $C(L)$. This proves Theorem 8.

Remark. A proof of Theorem 8 can be done which does not use results on local dendrites. It involves the two cases of whether or not $C(L)$ contains a 4-cell; but, the details are somewhat cumbersome.

5. Other applications of Theorem 6. In [16] Rogers proved several results which showed that the dimension of the hyperspaces of certain types of continua is infinite. This was done by showing that such hyperspaces contain an n-cell for each n. We give improvements of these results by showing that the hyperspaces Rogers investigated in [16] contain Hilbert cubes. First we give an example, as mentioned in the Introduction, of a continuum whose hyperspace contains an n-cell for each n but does not contain a Hilbert cube (this can not happen for locally connected continua by Theorem 7 above and 5.4 of [7]).

EXAMPLE 3. This example is due to B. J. Ball. Let X be the continuum represented in Figure 3 below. A brief description is as follows: X is composed of a sequence $\{A_n\}_{n=1}^{\infty}$ of mutually disjoint n-odds in the plane converging to a point $p \notin (\bigcup_{n=1}^{\infty} A_n)$ together with a sequence $\{L_n\}_{n=1}^{\infty}$ of mutually disjoint topological copies of the real line such that

$\{L_n\}_{n=1}^{\infty}$ converges to $p \notin (\bigcup_{n=1}^{\infty} L_n)$, $(\bigcup_{n=1}^{\infty} L_n) \cap (\bigcup_{n=1}^{\infty} A_n) = \emptyset$, and $A_n \cup L_n \cup A_{n+1}$, for each $n = 1, 2, \ldots$, is a compactification of L_n with $A_n \cup A_{n+1}$ as the remainder. Since $C(A_n)$ contains an $(n+2)$-cell for each $n = 1, 2, \ldots$, $C(X)$ contains an n-cell for every n. For each $k = 1, 2, \ldots$, let

$$M_k = (\bigcup_{n=1}^{k+1} A_n) \cup (\bigcup_{n=1}^{k} L_n).$$

Clearly, for any $k = 1, 2, \ldots$, there is an open subset W of $C(X)$ such that $C(M_k) \subset W \subset C(M_{k+1})$. From this and the finite-dimensionality of

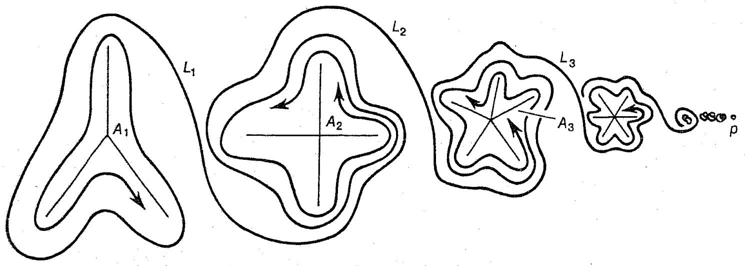

Fig. 3

$C(M_k)$ for all $k = 1, 2, \ldots$, it follows easily that no subcontinuum of $X - \{p\}$ can be a point of a Hilbert cube in $C(X)$. Now, let

$$C_p(X) = \{K \in C(X) \colon p \in K\}.$$

It is easy to see that $C_p(X)$ is an arc in $C(X)$. It now follows easily that $C(X)$ does not contain a Hilbert cube.

It is also worthwhile pointing out, in the context of this section, that there are infinite-dimensional hyperspaces which do not even contain a 2-cell (see [3] and 8.4 of [7]).

Since the techniques used in [16] are all modified in a similar way to obtain our results below, we only prove the first of them.

Our first theorem improves Corollary 2 of [16].

THEOREM 9. *If X is a continuum which contains a topological copy of the cartesian product of two nondegenerate continua, then $C(X)$ contains a Hilbert cube.*

Proof. It suffices to prove Theorem 9 for the case when X is the cartesian product of two nondegenerate continua. So, assume $X = K \times L$ where K and L are nondegenerate continua. Let $w_0 \in K$ and let $\{z_i\}_{i=1}^{\infty}$ be a sequence in L of distinct points. For each $i = 1, 2, \ldots$, let K_i be a nondegenerate subcontinuum of K containing w_0 and of diameter less than $1/i$ (such continua K_i exist by 10.1 of [20, p. 16]). Letting $Y = \{w_0\} \times L$ and $Y_i = K_i \times \{z_i\}$ for each $i = 1, 2, \ldots$, we see that Y and the continua Y_i satisfy conditions (1) through (3) of Theorem 6 above. Hence, $C(X)$ contains a Hilbert cube. This completes the proof of Theorem 9.

Our next theorem improves Theorem 2 of [16]. We say that a subcontinuum A of a continuum X is *continuumwise accessible* from a subset S of X provided there is a nondegenerate subcontinuum K of X such that $K \cap A \neq \emptyset$, $K \cap S \neq \emptyset$, and $K \subset S \cup A$.

THEOREM 10. *If X is a continuum such that some subcontinuum K of X contains infinitely many mutually disjoint subcontinua each of which is continuumwise accessible from $X - K$, then $C(X)$ contains a Hilbert cube.*

Our next theorem improves Theorem 3 of [16].

THEOREM 11. *If X is a continuum which contains a subcontinuum K such that*

(i) *K contains infinitely many arc components and*

(ii) *K is contained in an arc component of X, then $C(X)$ contains a Hilbert cube.*

The next theorem is a direct consequence of Theorem 11 above and improves Corollary 3′ of [16].

THEOREM 12. *If an arcwise connected continuum X contains a nondegenerate indecomposable subcontinuum, then $C(X)$ contains a Hilbert cube.*

Our next theorem is a consequence of Theorem 12 above and Theorem II of [11]; it improves Corollary 3″ of [16].

THEOREM 13. *If X is an arcwise connected continuum of dimension greater than or equal to two, then $C(X)$ contains a Hilbert cube.*

We remark that the following theorem can be proved using Theorem 10 above and 10.1 of [20, p. 16]. The result is related to Theorem 4 above.

THEOREM 14. *If a continuum X contains two subcontinua A and B such that $A \cap B$ contains an infinite number of components, then $C(X)$ contains a Hilbert cube.*

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Semi-confluent mappings and their invariants

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Abstract. A continuous mapping f of a continuum X onto Y is said to be *semi-confluent* if for every subcontinuum Q in Y and for each two components C_1 and C_2 of the inverse image $f^{-1}(Q)$ either $f(C_1) \subset f(C_2)$ or $f(C_2) \subset f(C_1)$. It is proved in the paper that the property of being a λ-dendroid, a dendroid, a fan, a dendrite or an arc is an invariant under a semi-confluent mapping.

§ 1. Introduction. In this paper we present a new kind of continuous mappings, called semi-confluent. The class of semi-confluent mappings comprises confluent mappings, whence also interior and monotone ones. Some theorems on confluent mappings will be generalized to semi-confluent mappings. In particular, theorems concerning the invariance of λ-dendroids proved for confluent mappings in [2] hold also for semi-confluent mappings. Moreover, dendroids, dendrites, arcs and fans (see [3], p. 32) are invariants under semi-confluent mappings.

The author is very much indebted to dr. J. J. Charatonik, who contributed to these investigations.

§ 2. Preliminaries. Recall that a continuous mapping f of a topological space X onto a topological space Y is said to be

(i) *interior* if f maps every open set in X onto an open set in Y (see [11], p. 348),

(ii) *monotone* if for any subcontinuum Q in Y the set $f^{-1}(Q)$ is a continuum in X (see [7], p. 123), or, which is equivalent provided X is a continuum, if the inverse image of each point of Y is a connected set in X (see [14], p. 127),

(iii) *quasi-monotone* if for any subcontinuum Q in Y with a non-vacuous interior the set $f^{-1}(Q)$ has a finite number of components and f maps each of them onto Q (see [12], p. 136),

(iv) *weakly monotone* if for any continuum Q in Y with a non-vacuous interior each component of the inverse image $f^{-1}(Q)$ is mapped by f onto Q (see [12], p. 136, where these mapping are called quasi-monotone and the spaces considered are locally connected continua, see also [10], p. 418),