Let $r_i = p_1 p_2 \dots p_{i-1} p_{i+1} \dots p_n$ $(i = 1, 2, \dots, n)$. Moreover, let $s_i$ be the smallest integer such that $s_i r_i \equiv 1 (\mathrm{mod}\, p_i)$. Then it is easy to check that the reduct $(G;\ x_1^{s_1 r_1} \cdot x_2^{s_2 r_2} \cdot \dots \cdot x_n^{s_n r_n})$ is an $n$-dimensional proper diagonal algebra. Q.e.d.

### References

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# A simpler set of axioms for polyadic algebras (1)

by

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**Abstract.** A new set of axioms for polyadic algebras is given. The new axioms are simple algebraic equations, having a clear algebraic content. From them are obtained some fresh insights into the structure of polyadic algebras.

**1. Introduction.** The purpose of this paper is to present a new, simpler set of axioms for polyadic algebras.

Polyadic algebras occupy a distinctive position in the scheme of algebraic logic, for they enjoy important properties which fail to hold for cylindric algebras, or even for polyadic algebras with equality. Notably, every polyadic algebra of infinite degree is representable in a very strong sense (see Daigneault and Monk [2]), and the class of all polyadic algebras has the amalgamation property (see J. Johnson [5]). Furthermore, polyadic algebras are, in a sense, richer structures than cylindric algebras, for they admit arbitrary cylindrifications as well as operations $S(\tau)$ for arbitrary transformations $\tau$.

It is unfortunate that, in one respect, polyadic algebras are less attractive to the mathematician than cylindric algebras: while the axioms for cylindric algebras are simple algebraic equations of a familiar kind, the axioms for polyadic algebras are more difficult to understand; two of them, in particular, fail to have a clear algebraic content. In our main result, we will show that these axioms may be replaced by simpler, more conventional algebraic equations. The new equations will then be used to obtain some fresh insights into the structure of polyadic agebras.

We assume the reader is acquainted with the basic papers, [3] and [4], of Halmos. In addition to the work of Halmos, we shall use an important result by P.-F. Jurie [6], which will be stated at the end of the next section.

**2. Preliminaries.** We shall use common set-theoretical notation and terminology. Small Greek letters will be used to denote *transformations*,

(1) The work reported in this paper was done while the author held an NSF Science Faculty Fellowship.

that is, functions from a set $I$ into itself. The letter $\delta$ is reserved to denote any identity mapping. The letters $\sigma$ and $\tau$ are reserved to denote *projections*, that is, transformations $\tau$ with the property that $\tau\tau = \tau$.

The value of a function $\alpha$ at an element $i$ of its domain will be denoted by $\alpha i$. The restriction of a function $\alpha$ to a subset $K$ of its domain will be denoted by $\alpha|K$. The domain and the range of a function $\alpha$ will be denoted by $\operatorname{dom}\alpha$ and $\operatorname{ran}\alpha$, respectively.

Our notation for polyadic algebras will be essentially that of Halmos, with two main exceptions: we shall write $S_\alpha$ instead of $S(\alpha)$, and $C_J$ instead of $\exists(J)$.

2.1. Definition. Let $I$ be an arbitrary set. By an *$I$-transformation algebra* we mean an algebraic system $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, S_\alpha\rangle_{\alpha\in I^I}$ where $\langle A, +, \cdot, -, 0, 1\rangle$ is a Boolean algebra, and $S_\alpha$ are unary operations which satisfy the following conditions for all $x, y \in A$ and all $\alpha, \beta \in I^I$:

($T_1$) $S_\alpha(x+y) = S_\alpha x + S_\alpha y$,

($T_2$) $S_\alpha(-x) = -S_\alpha x$,

($T_3$) $S_\alpha S_\beta = S_{\alpha\beta}$,

($T_4$) $S_\delta = \delta$.

2.2. Definition. Let $I$ be an arbitrary set. By an *$I$-polyadic algebra* we mean an algebraic system $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, S_\alpha, C_J\rangle_{\alpha\in I^I, J\subseteq I}$ where $\langle A, +, \cdot, -, 0, 1, S_\alpha\rangle_{\alpha\in I^I}$ is an $I$-transformation algebra, and $C_J$ are unary operations which satisfy the following conditions for all $x, y \in A$, all $\alpha, \beta \in I^I$ and all $J, K \subseteq I$:

($Q_1$) $C_J 0 = 0$,

($Q_2$) $x \leqslant C_J x$,

($Q_3$) $C_J(x\cdot C_J y) = C_J x\cdot C_J y$,

($P_3$) $C_\varnothing = \delta$,

($P_4$) $C_{J\cup K} = C_J C_K$,

($P_5$) $S_\alpha C_J = S_\beta C_J$ if $\alpha|I-J = \beta|I-J$,

($P_6$) $C_J S_\alpha = S_\alpha C_{\alpha^{-1}(J)}$ if $\alpha|\alpha^{-1}(J)$ is injective.

Let $A$ be a Boolean algebra. If the function $f\colon A\to A$ satisfies ($Q_1$)-($Q_3$), (more accurately, if $f0 = 0$, $x\leqslant fx$, and $f(x\cdot fy) = fx\cdot fy$ for all $x, y\in A$), then $f$ is called a *quantifier*, or *cylindrification*, of $A$. If $B$ is a Boolean subalgebra of $A$, then $B$ is called a *relatively complete* subalgebra of $A$ if it satisfies the condition

(RC) for each $x\in A$, there is a least $y\in B$ such that $y\geqslant x$.

It is known that if $f$ is a quantifier of $A$ then $\operatorname{ran} f$ is a relatively complete subalgebra of $A$, and, conversely, if $B$ is a relatively complete subalgebra of $A$ and $f$ is defined by

$$f(x) = \text{the least } y\in B \text{ such that } y\geqslant x,$$



then $f$ is a quantifier of $A$ (see Halmos [3], § 4). We shall refer to $f$ as the *quantifier associated with $B$*.

Let $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, S_\alpha\rangle_{\alpha\in I^I}$ be an $I$-transformation algebra. If $x\in A$ and $J\subseteq I$, $x$ is said to be *$J$-closed* if $S_\alpha x = x$ for every $\alpha\in I^I$ such that $\alpha|I-J = \delta$. For each $J\subseteq I$, the set of $J$-closed elements of $A$ will be denoted by $E_J$; one verifies immediately that $E_J$ is a Boolean subalgebra of $A$.

We shall make use of the following result, which is implicit in Daigneault and Monk ([2], Lemma 3.7):

2.3. Lemma. *Let $\mathfrak{A}$ be an $I$-transformation algebra, and let $J, K\subseteq I$. Then*

(i) *$J\subseteq K$ implies $E_K\subseteq E_J$, and*

(ii) *$E_{J\cup K} = E_J\cap E_K$.*

Finally, we shall need the result which follows, due to P.-F. Jurie ([6], Theorems 1 and 2):

2.4. Theorem. (i) *If $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, S_\alpha, C_J\rangle_{\alpha\in I^I, J\subseteq I}$ is an $I$-polyadic algebra, then the underlying transformation algebra $\langle A, +, \cdot, -, 0, 1, S_\alpha\rangle_{\alpha\in I^I}$ satisfies the following conditions* (²), (³):

($J_1$) *For each $J\subseteq I$, $E_J$ is a relatively complete subalgebra of $A$.*

($J_2$) *For all $J, K\subseteq I$, if $x\in E_J$ and $y\in E_K$ and $x\leqslant y$, then there is some $z\in E_J\cap E_K$ such that $x\leqslant z\leqslant y$.*

($J_3$) *For all $\alpha\in I^I$, if $\alpha|\alpha^{-1}(J)$ is injective, then every element of $\ker S_\alpha$ is dominated by an element of $\ker S_\alpha\cap E_{\alpha^{-1}(J)}$.*

(ii) *Conversely, if $\mathfrak{A}' = \langle A, +, \cdot, -, 0, 1, S_\alpha\rangle_{\alpha\in I^I}$ is an $I$-transformation algebra which satisfies ($J_1$)-($J_3$), and if, for each $J\subseteq I$, $C_J$ is the quantifier associated with $E_J$, then $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, S_\alpha, C_J\rangle_{\alpha\in I^I, J\subseteq I}$ is a polyadic algebra.*

**3. Some properties of transformation algebras.** The main purpose of this section is to show that Conditions ($J_1$)-($J_3$) of Theorem 2.4 may be replaced by simpler conditions. This result will be needed in the sequel.

3.1. Definition. If $\alpha\in I^I$ and $K\subseteq I$, we will say that $\alpha$ is *properly injective* on $K$ if

$$\alpha i = \alpha j \text{ implies } i = j, \quad \text{for all } i\in K \text{ and all } j\in I.$$

If $\alpha$ is properly injective on $K$, it is easy to see that $K = \alpha^{-1}(\alpha(K))$; in other words,

(²) Conditions ($J_1$)-($J_3$) are easily seen to be equivalent to Jurie's Conditions (R.C.), (Ind), and ($\bar{P}_6$), respectively. We have merely used the fact that $\alpha^{-1}(I-J) = I-\alpha^{-1}(J)$, and replaced $J$ by $I-J$ and $A_J$ (that is, $E_{I-J}$) by $E_J$, for every $J\subseteq I$.

(³) In Condition ($J_3$), "$x$ is dominated by $y$" means simply $y\geqslant x$.

(1) $$J = \alpha(K) \text{ implies } K = \alpha^{-1}(J)\,.$$

Furthermore, if $\alpha \in I^I$ and $J \subseteq I$, it is immediate that

(2) if $\alpha | \alpha^{-1}(J)$ is injective, then $\alpha$ is properly injective on $\alpha^{-1}(J)$.

(1) and (2) make it clear that $(J_3)$ may be re-stated as follows:

$(J_3')$ For all $\alpha \in I^I$, if $\alpha$ is properly injective on $K$, then every element of $\ker S_\alpha$ is dominated by an element of $\ker S_\alpha \cap E_K$.

The following simple lemma has important consequences:

3.2. Lemma. *For every $\alpha \in I^I$, there is a projection $\tau \in I^I$ such that $\ker S_\alpha = \ker S_\tau$. Furthermore, for any $K \subseteq I$, $\alpha$ is properly injective on $K$ iff $\tau$ is properly injective on $K$.*

Proof. It is well known that the semi-group of transformations $I^I$ has the following property: for each $\alpha \in I^I$, there is some $\theta \in I^I$ such that $\alpha = \alpha\theta\alpha$. (A semi-group with this property is called *regular*; see Clifford and Preston [1], pp. 26 and 33). Now, let $\tau = \theta\alpha$: then $\tau\tau = \theta\alpha\theta\alpha = \theta\alpha = \tau$, hence $\tau$ is a projection. Furthermore,

$$\begin{aligned} S_\alpha x = 0 &\Rightarrow S_\theta S_\alpha x = 0 \\ &\Rightarrow S_{\theta\alpha} x = S_\tau x = 0\,. \end{aligned}$$

Conversely,

$$\begin{aligned} S_\tau x = 0 &\Rightarrow S_{\theta\alpha} x = 0 \\ &\Rightarrow S_\alpha S_{\theta\alpha} x = 0 \\ &\Rightarrow S_{\alpha\theta\alpha} x = S_\alpha x = 0\,. \end{aligned}$$

Thus, $\ker S_\alpha = \ker S_\tau$.

Finally, suppose $\alpha$ is properly injective on $K$. If $i \in K$, then

$$\begin{aligned} \tau i = \tau j &\Rightarrow \theta\alpha i = \theta\alpha j \\ &\Rightarrow \alpha\theta\alpha i = \alpha\theta\alpha j \\ &\Rightarrow \alpha i = \alpha j \\ &\Rightarrow i = j\,, \end{aligned}$$

hence $\tau$ is properly injective on $K$.

Conversely, suppose $\tau$ is properly injective on $K$. If $i \in K$, then

$$\begin{aligned} \alpha i = \alpha j &\Rightarrow \theta\alpha i = \theta\alpha j \\ &\Rightarrow \tau i = \tau j \\ &\Rightarrow i = j\,, \end{aligned}$$

hence $\alpha$ is properly injective on $K$. ■

An important consequence of Lemma 3.2 is that if the condition $\alpha$ is properly injective on $K \Rightarrow$ every element of $\ker S_\alpha$ is dominated by an element of $\ker S_\alpha \cap E_K$ holds for every $\alpha$ which is a projection, then it holds for all $\alpha \in I^I$. Thus, $(J_3')$ is equivalent to



$(J_3'')$ For all projections $\tau \in I^I$, if $\tau$ is properly injective on $K$ then every element of $\ker S_\tau$ is dominated by an element of $\ker S_\tau \cap E_K$.

Let $\tau \in I^I$ be a projection. By the *essential domain* and the *essential range* of $\tau$ we mean, respectively,

$$\operatorname{edm}\tau = \operatorname{dom}(\tau - \delta) \quad \text{and} \quad \operatorname{ern}\tau = \operatorname{ran}(\tau - \delta)\,.$$

It is easy to verify that if $\tau$ is a projection, then $\operatorname{edm}\tau \cap \operatorname{ern}\tau = \emptyset$; $\tau$ maps elements of $\operatorname{edm}\tau$ onto elements of $\operatorname{ern}\tau$, and leaves all the elements of $I - \operatorname{edm}\tau$ fixed.

3.3. Lemma. *Let $\tau \in I^I$ be a projection. Then $\tau$ is properly injective on $K$ iff $K \cap (\operatorname{edm}\tau \cup \operatorname{ern}\tau) = \emptyset$.*

Proof. (i) Suppose $\tau$ is properly injective on $K$; we will show that each of the two assumptions $i \in K \cap \operatorname{edm}\tau$, $i \in K \cap \operatorname{ern}\tau$ yields a contradiction. First, suppose $i \in K \cap \operatorname{edm}\tau$: then $\tau i \neq i$; but $i \in K$ and $\tau i = \tau(\tau i)$, so by 3.1, $i = \tau i$, which is impossible. Next, suppose $i \in K \cap \operatorname{ern}\tau$: then $i = \tau j$ where $j \neq i$. But $i \in K$ and $\tau i = \tau(\tau j) = \tau j$, so by 3.1, $i = j$; again, this is impossible.

(ii) Conversely, suppose $K$ is disjoint from $\operatorname{edm}\tau$ and from $\operatorname{ern}\tau$. Let $\tau i = \tau j$, where $i \in K$; now $i \notin \operatorname{edm}\tau$, so $\tau i = i$; thus, $i = \tau j$. But then $\tau j \in K$, so $\tau j \notin \operatorname{ern}\tau$, hence $\tau j = j$. It follows that $i = j$. ■

Let $\tau$ be a projection. We shall call $\tau$ a $(J, L)$-*projection* if $\operatorname{edm}\tau = J$ and $\operatorname{ern}\tau = L$. By Lemma 3.3, $(J_3'')$ may be written in the following form:

$(J_3^*)$ For all projections $\tau \in I^I$, if $\tau$ is a $(J, L)$-projection and $K \cap (J \cup L) = \emptyset$, then every element of $\ker S_\tau$ is dominated by an element of $\ker S_\tau \cap E_K$.

We have just shown that Condition $(J_3)$ of Theorem 2.4 may be replaced by the simpler condition $(J_3^*)$. We shall now show that Conditions $(J_1)$ and $(J_2)$ similarly admit a minor improvement. We begin by stating a simple property of quantifiers:

3.4. Lemma. *Let $f$ and $g$ be quantifiers of a Boolean algebra $A$. If $fg = gf$, then $\operatorname{ran} fg = \operatorname{ran} f \cap \operatorname{ran} g$, and $fg$ is a quantifier.*

Proof. Clearly $fgx = gfx \in \operatorname{ran} f \cap \operatorname{ran} g$. Conversely, if $x \in \operatorname{ran} f \cap \operatorname{ran} g$ then $x = fx$ and $x = gx$, hence $x = fx = fgx \in \operatorname{ran} fg$. Finally, it is trivial to verify directly that $fg$ is a quantifier. ■

Jurie [6] has proved the converse of this statement; in particular, he has shown that if $f$, $g$, and $fg$ are quantifiers on $A$, then $fg = gf$. Thus, we make the interesting observation that *two quantifiers f and g commute iff their product is a quantifier.*

3.5. Lemma. *Let* $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, S_\alpha \rangle_{\alpha \in I^I}$ *be a transformation algebra in which the following conditions hold:*

($J_1^*$) *For each proper subset $J \subset I$, $E_J$ is a relatively complete subalgebra of $A$.*

($J_2^*$) *For all proper subsets $J, K \subset I$, if $x \in E_J$ and $y \in E_K$ and $x \leqslant y$, then there is some $z \in E_J \cap E_K$ such that $x \leqslant z \leqslant y$.*

*Then* ($J_1$) *and* ($J_2$) *hold.*

Proof. Suppose ($J_1^*$) and ($J_2^*$) hold; for each $J \subset I$, let $C_J$ be the quantifier associated with $E_J$. We prove, successively:

(1) $$\text{if} \quad J, K \subset I, \quad \text{then} \quad C_J C_K x \in E_K.$$

Indeed, $C_K x \in E_K$, $C_J C_K x \in E_J$, and $C_K x \leqslant C_J C_K x$. Thus, by ($J_2^*$), there is some $z \in E_J \cap E_K$ such that $C_K x \leqslant z \leqslant C_J C_K x$. But $C_J C_K x$ is the least $y \in E_J$ such that $y \geqslant C_K x$, hence $z = C_J C_K x$. Thus, $C_J C_K x \in E_K$.

(2) $$\text{If} \quad J, K \subset I, \quad \text{then} \quad C_J C_K x = C_K C_J x.$$

Indeed, $C_K C_J x$ is the least $y \in E_K$ such that $y \geqslant C_J x$. But by (1), $C_J C_K x \in E_K$, and clearly $C_J C_K x \geqslant C_J x$; thus, $C_K C_J x \leqslant C_J C_K x$. Symmetrically, $C_J C_K x \leqslant C_K C_J x$, giving (2).

It follows by (2) that if $J \neq \emptyset$, $I$, then $C_J C_{I-J} = C_{I-J} C_J$. Thus, by Lemma 3.4, $C_J C_{I-J}$ is a quantifier, and

$$\operatorname{ran} C_J C_{I-J} = \operatorname{ran} C_J \cap \operatorname{ran} C_{I-J} = E_J \cap E_{I-J}.$$

By 2.3(ii), $E_J \cap E_{I-J} = E_I$, hence $E_I$ is a relatively complete subalgebra of $A$. Thus, ($J_1$) holds.

Now by 2.3(i), $E_I \subseteq E_J$ for every $J \subseteq I$; thus, if we assume ($J_2^*$), ($J_2$) follows trivially. ■

3.6. Corollary. *Theorem 2.4 holds when* ($J_1$)-($J_3$) *are replaced by* ($J_1^*$)-($J_3^*$).

We conclude this section by deriving one more property of projections.

3.7. Lemma. *Let $J$ be a proper subset of $I$. If $\tau$ is any projection such that* $\operatorname{edm}\tau = J$, then $\operatorname{ran} S_\tau = E_J$.

Proof. By the definition of $E_J$, if $x \in E_J$ then $S_\tau x = x$, hence $x \in \operatorname{ran} S_\tau$. Conversely, if $x \in \operatorname{ran} S_\tau$, then $x = S_\tau x$. Now if $\alpha | I-J = \delta$, then $\alpha\tau = \tau$, hence $S_\alpha x = S_\alpha S_\tau x = S_{\alpha\tau} x = S_\tau x = x$; thus, $x \in E_J$. ■

**4. New axioms for polyadic algebras.** Our main results are presented in this section. The Conditions ($T_1$)-($T_4$) and ($Q_1$)-($Q_3$) to which we refer below are those of Definitions 2.1 and 2.2, respectively.



4.1. Theorem. (i) *If* $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, S_\alpha, C_J \rangle_{\alpha \in I^I, J \subseteq I}$ *is a polyadic algebra, then the following condition holds for all $K \subset I$ and every projection $\tau$:*

(PL) *If $\tau$ is a $(J, L)$-projection, then*

(a) $S_\tau C_J = C_J$,

(b) $C_J S_\tau = S_\tau$,

(c) $C_K S_\tau = S_\tau C_K$ *if* $K \cap (J \cup L) = \emptyset$, *and*

(d) $C_K C_{I-K} = C_I$.

(ii) *Conversely, if* $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, S_\tau, C_J \rangle_{\alpha \in I^I, J \subseteq I}$ *is an algebra in which* ($T_1$)-($T_4$) *hold for all* $\alpha, \beta \in I^I$, *and* ($Q_1$)-($Q_3$) *and* (PL) *hold for all $K \subset I$ and every projection $\tau$, then $\mathfrak{A}$ is a polyadic algebra.*

Proof. (i) If $\mathfrak{A}$ is a polyadic algebra, then by Daigneault and Monk ([2], Lemma 4.1) $\operatorname{ran} C_J = E_J$ for each $J \subseteq I$; (a) and (b) of (PL) follow immediately from this. (c) is an application of Axiom ($P_6$), and (d) is an application of Axiom ($P_4$).

(ii) Conversely, let $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, S_\alpha, C_J \rangle_{\alpha \in I^I, J \subseteq I}$ be an algebra in which ($T_1$)-($T_4$), ($Q_1$)-($Q_3$) and (PL) hold as in the statement of this theorem. We will prove that ($J_1$), ($J_2$) and ($J_3^*$) are satisfied and, for each $J \subseteq I$, $C_J$ is the quantifier associated with $E_J$. It will follow by Theorem 2.4 that $\mathfrak{A}$ is a polyadic algebra.

First, we will show that

(1) $$\text{for each } J \subseteq I, \quad \operatorname{ran} C_J = E_J.$$

We consider three cases:

Case 1. $J = \emptyset$. We note that $\delta$ is a $(\emptyset, \emptyset)$-projection, hence by Lemma 3.7, $E_\emptyset = \operatorname{ran} S_\delta$. But by (a) and (b) of (PL), $\operatorname{ran} S_\delta = \operatorname{ran} C_\emptyset$, so $\operatorname{ran} C_\emptyset = E_\emptyset$.

Case 2. $J \neq \emptyset, I$. If $L = \{k\} \subseteq I-J$, then there exists a $(J, L)$-projection $\tau$, and by (a) and (b), $\operatorname{ran} S_\tau = \operatorname{ran} C_J$. By Lemma 3.7, $\operatorname{ran} S_\tau = E_J$, hence $E_J = \operatorname{ran} C_J$.

Case 3. $J = I$. For any non-empty $K \subset I$, we have, by (d),

$$C_I = C_K C_{I-K} = C_{I-K} C_K.$$

Thus, by Lemma 3.4, Case 2 above, and Lemma 2.3(ii),

$$\operatorname{ran} C_I = \operatorname{ran} C_K \cap \operatorname{ran} C_{I-K} = E_K \cap E_{I-K} = E_I,$$

and $C_I$ is a quantifier. This finishes the proof of (1).

We are able to conclude from (1) that for each $J \subseteq I$, $E_J$ is a relatively complete subalgebra of $A$, and that $C_J$ is the quantifier associated with $E_J$.

From (1), we immediately derive

(2) $$\text{for each } J \subseteq I, \quad \text{if} \quad y \in E_J \quad \text{then} \quad C_J y = y.$$

To prove ($J_2$), we again consider three cases:

Case 1. $J \cap K = \emptyset$, $J \cup K \neq I$. If $L = \{k\} \subseteq I-(J \cup K)$, then there exists a $(J, L)$-projection $\tau$. Now suppose $x \in E_J$, $y \in E_K$, and $x \leqslant y$; then by (2) and the additivity of $C_K$,

$$x \leqslant C_K x \leqslant C_K y = y .$$

We will prove that ($J_2$) holds with $z = C_K x$. Clearly, $C_K x \in E_K$; on the other hand, $x = S_\tau x$, so by (c),

$$C_K x = C_K S_\tau x = S_\tau C_K x \in E_J .$$

Thus, $C_K x \in E_J \cap E_K$, and therefore ($J_2$) is satisfied.

Case 2. $J \cap K = \emptyset$, $J \cup K = I$. In this case, $K = I-J$. Now if $x \in E_J$, $y \in E_K$, and $x \leqslant y$, then, as in the preceding case, we have

$$x \leqslant C_K x \leqslant C_K y = y .$$

But $x = C_J x$; so by (d) and (1),

$$C_K x = C_{I-J} x = C_{I-J} C_J x = C_I x \in E_I = E_J \cap E_K .$$

Case 3. $J \cap K \neq \emptyset$. Again, suppose $x \in E_J$, $y \in E_K$, and $x \leqslant y$. By 2.3(i), $y \in E_{K-J}$; consequently, we have

$$x \in E_J , \quad y \in E_{K-J} , \quad x \leqslant y , \quad \text{and} \quad J \cap (K-J) = \emptyset .$$

It follows by Cases 1 and 2, above, that $x \leqslant C_{K-J} x \leqslant y$, and

$$C_{K-J} x \in E_J \cap E_{K-J} = E_{J \cup (K-J)} = E_{J \cup K} = E_J \cap E_K .$$

Thus, ($J_2$) is satisfied.

Finally, we prove ($J_3^*$): let $\tau$ be a $(J, L)$-projection and suppose that $K \cap (J \cup L) = \emptyset$. Let $x \in \ker S_\tau$, that is, $S_\tau x = 0$. Now $x \leqslant C_K x$; furthermore, by (c), $S_\tau C_K x = C_K S_\tau x = 0$, so $C_K x \in \ker S_\tau \cap E_K$. ■

4.2. THEOREM. *Let* $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, S_\alpha, C_J \rangle_{\alpha \in I^I, J \subseteq I}$ *be an algebra in which* ($T_1$)-($T_4$) *and* (PL) *hold as in* 4.1(ii). *Furthermore, assume that*

(B1) $$C_K(x+y) = C_K x + C_K y ,$$

*and*

(B2) $$x \leqslant C_K x ,$$

*for all* $K \subset I$ *and all* $x, y \in A$. *Then, for each* $K \subset I$, $C_K$ *is a quantifier of* $A$.

Proof. By part (1) of the proof of 4.1, $C_K x \in E_K$, and by (B2), $C_K x \geqslant x$. Furthermore, if $y \in E_K$ and $y \geqslant x$, then by (B1), $C_K x \leqslant C_K y$; but by part (2) of the proof of 4.1, $C_K y = y$, so $C_K x \leqslant y$. This proves that $C_K x$ is the least element of $E_K$ which dominates $x$. Thus, $E_K$ is a relatively complete subalgebra of $A$, and $C_K$ is the quantifier associated with $E_K$. ■



We may combine Theorems 4.1 and 4.2 as follows:

4.3. THEOREM. (i) *If* $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, S_\alpha, C_J \rangle_{\alpha \in I^I, J \subseteq I}$ *is polyadic algebra, its operations satisfy* ($T_1$)-($T_4$), (B1)-(B2), *and* (PL).

(ii) *Conversely, if* $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, S_\alpha, C_J \rangle_{\alpha \in I^I, J \subseteq I}$ *is an algebra in which* ($T_1$)-($T_4$), (B1)-(B2), *and* (PL) *hold for all* $\alpha, \beta \in I^I$, *all* $K \subset I$, *and all projections* $\tau$, *then* $\mathfrak{A}$ *is a polyadic algebra.*

In conclusion, we offer the following, alternative way of defining a polyadic algebra:

An $I$-polyadic algebra is an algebraic system $\mathfrak{A} = \langle A, +, \cdot, -, 0, 1, S_\alpha, C_J \rangle_{\alpha \in I^I, J \subseteq I}$ such that $\langle A, +, \cdot, -, 0, 1, S_\alpha \rangle_{\alpha \in I^I}$ is a transformation algebra, and $C_J$ are unary operations which satisfy the following conditions for all $K \subset I$, all projections $\tau$, and all $x, y \in A$: if $\tau$ is a $(J, L)$-projection, then

($PA_1$) $C_K(x+y) = C_K x + C_K y$,

($PA_2$) $x \leqslant C_K x$,

($PA_3$) $S_\tau C_J = C_J$,

($PA_4$) $C_J S_\tau = S_\tau$,

($PA_5$) $C_K S_\tau = S_\tau C_K$ if $K \cap (J \cup L) = \emptyset$,

($PA_6$) $C_K C_{I-K} = C_I$.

Remark. It is worth noting that in the preceding set of conditions, ($PA_1$)-($PA_5$) are assumed to hold for *proper* subsets $J, K \subset I$; thus, only ($PA_6$) states any property of $C_I$. Assuming all of the preceding condition except ($PA_6$), it can be shown that for all $J, K \subset I$, $C_J C_{I-J} = C_K C_{I-K}$, and $C_J C_{I-J}$ is a quantifier. Thus, $C_I$ may be regarded as a defined, rather than a primitive, operation, and ($PA_6$) can be taken as its definition.

**5. Conclusion.** The results of the last two sections allow us to make some interesting observations regarding the structure of polyadic algebras.

First, we note that conditions ($PA_1$)-($PA_6$) say nothing about operations $S_\alpha$ where $\alpha$ is not a projection. To put it another way: Axioms ($T_1$)-($T_4$), which describe the transformation structure of a polyadic algebra, state properties of all the operations $S_\alpha$, for all $\alpha \in I^I$; by contrast, Conditions ($PA_1$)-($PA_6$), which describe the quantifier structure and its connections with the transformation structure, state properties of quantifiers and only those operations $S_\alpha$ where $\alpha$ is a projection. Thus, the relationships between quantifiers and arbitrary operations $S_\alpha$ can be deduced from the relationships between quantifiers and those $S_\alpha$ where $\alpha$ is a projection.

This fact is even more apparent in Corollary 3.6. Indeed, we have already seen, Lemma 3.7, that for each proper subset $J \subset I$, $E_J = \operatorname{ran} S_\tau$ for any projection $\tau$ whose essential domain is equal to $J$. Thus, $(J_1^*)$-$(J_3^*)$ *are statements describing the properties of the sets* $\operatorname{ran} S_\tau$ *and* $\ker S_\tau$ *for all projections* $\tau \in I^I$. Consequently, if $\mathfrak{A}$ is a polyadic algebra, then the quantifier structure of $\mathfrak{A}$, as well as the connections between the quantifier structure and the transformation structure of $\mathfrak{A}$, may be described entirely in terms of the sets $\operatorname{ran} S_\tau$ and $\ker S_\tau$ for all projections $\tau \in I^I$. These, then, are the chief structural components of every polyadic algebra.

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# Locating cones and Hilbert cubes in hyperspaces


by

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**Abstract.** Let $X$ be a metric continuum. Let $C(X)$ denote the space of all nonempty subcontinua of $X$. It is shown that if $X$ is decomposable, then $C(X)$ contains a 2-cell. This result is then used in several ways. For example, a characterization of hereditary indecomposability is obtained answering a question of B. J. Ball in a strong way. Also, for certain $X$, $n$-cells are located in $C(X)$ where they were not known to be previously, and necessary conditions are obtained in order that the cone over $X$ be homeomorphic to $C(X)$. A general result, which locates Hilbert cubes in $C(X)$, is proved and then applied to show that certain classes of continua $X$ have the property that $C(X)$ contains a Hilbert cube or the cone over $X$. Some unsolved problems are stated.


**Key words and phrases.** Chainable, circle-like, composant, decomposable continuum, dimension, indecomposable continuum, local dendrite, multicoherence degree, order of a point, ramification point, segment (in the sense of Kelley), upper semicontinuous decomposition.

**1. Introduction.** A *continuum* is a nonempty compact connected metric space. The term *nondegenerate* will be used to mean that a space has more than one point. A continuum is said to be *decomposable* if and only if it is the union of two of its proper subcontinua, *indecomposable* if and only if it is not decomposable, and *hereditarily indecomposable* if and only if each of its subcontinua is indecomposable. For definitions not given in this paper, we refer the reader to the texts listed in the references.

The *hyperspace* of a continuum $X$ will mean, throughout this paper, the space of all (nonempty) subcontinua of $X$ with the topology induced by the Hausdorff metric $H$ (see [7] or [10, p. 47]); it is denoted by $C(X)$. Recognizing when and where $C(X)$ contains the cone over $X$ or over other continua has proved to be useful information (see [12]). Much work has been done, especially recently (see [2], [15], and [17]), relating the space $C(X)$ and the cone over $X$. For example, J. T. Rogers, Jr. [15] investigated necessary conditions in order that $C(X)$ be homeomorphic in a "nice way" to the cone over $X$. We note that, in [2] and [15], the

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