Homogeneous algebras are simple

by

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Abstract. Homogeneous algebras were defined by E. Marczewski (see [1]). Here we prove that, with one exception (namely the 4-element Świerczkowski algebra), all homogeneous algebras are simple. Some additional remarks connected with this topic are added.

In his paper [1] Marczewski investigated homogeneous algebras, i.e. algebras with the full symmetric group as the group of automorphisms. In this note we proove that, with a single exception, all nontrivial homogeneous algebras are simple.

Let us first recall some definitions used here: An operation $p\colon A^n \to A$ is called

$$\textit{trivial} \text{ (or a } \textit{projection}) \Leftrightarrow (\exists i \in \{1, \ldots, n\})(\forall x_1 \ldots x_n \in A)\, p(x_1, \ldots, x_n) = x_i,$$

$$\textit{quasitrivial} \Leftrightarrow (\forall x_1 \ldots x_n)\, p(x_1, \ldots, x_n) \in \{x_1, \ldots, x_n\},$$

$$\textit{idempotent} \Leftrightarrow (\forall x \in A)\, p(x, \ldots, x) = x.$$

An algebra $\mathfrak{A} = (A, F)$ is called *trivial* (*quasitrivial*, *idempotent*) iff all polynomials (i.e. algebraic operations) on $\mathfrak{A}$ are trivial (quasitrivial, idempotent).

An algebra $\mathfrak{B} = (B, G)$ is called the *idempotent reduct* of the algebra $\mathfrak{A} = (A, F)$ iff the polynomials in $\mathfrak{B}$ are exactly the idempotent polynomials on $\mathfrak{A}$, and in particular $A = B$ holds. The algebra $\mathfrak{A} = (A, F)$ is called *simple*, iff the only congruences on $\mathfrak{A}$ are id_A and $A \times A$.

THEOREM. *For a nontrivial homogeneous algebra $\mathfrak{A} = (A, F)$ then either*

(1) *$\mathfrak{A}$ is simple, or*

(2) *$\mathfrak{A}$ is the idempotent reduct of a 2-dimensional vector space over the 2-element field (or, in other words, the four-element algebra of Świerczkowski*, see [3], p. 94).

LEMMA 1. *Let q be a quasitrivial polynomial on the homogeneous algebra $\mathfrak{A} = (A, F)$, where $|A| \neq 2$; then either q is trivial or $\mathfrak{A}$ is simple.*

Proof. Let q be nontrivial, then the algebra $\mathfrak{B} = (A, q)$ is nontrivial, hence by Świerczkowski [3] (Th. 1, p. 94) $\mathfrak{B}$ possesses nontrivial polynomials of arity less than or equal to $|A|$. Let p be a nontrivial polynomial on $\mathfrak{B}$ of minimal arity n. As the composition of quasitrivial operations is again quasitrivial, we infer that p is quasitrivial, and moreover (see [1], 1.2 (v)):

(1.0) $p(x_1, \ldots, x_n) = x_i$ if all $x_1, \ldots, x_n$ are different, for some $i \in \{1, \ldots, n\}$; say $i = 1$.

As p has minimal arity, any identification of variables leads to a projection, but as p is nontrivial, not each of those projections can be the first projection. So we have three cases:

(1.1) $p(x_1, x_1, x_3, \ldots, x_n) = x_i, \quad i \in \{3, \ldots, n\}$, say $i = n$,

(1.2) $p(x_1, \ldots, x_i, x_i, \ldots, x_n) = x_i, \quad i \in \{2, \ldots, n\}$, say $i = 2$,

(1.3) $p(x_1, \ldots, x_i, x_i, \ldots, x_n) = x_j, \quad i, j \in \{2, \ldots, n\}, \ i \neq j$, say $i = 2$ and $j = n$.

In this case we have $n \geqslant 4$.

Let $\theta \neq \mathrm{id}_A$ be a congruence on $\mathfrak{B}$ and $a\,\theta\,b$ for some $a \neq b$. Whenever $a, b, a_3, \ldots, a_n$ are different, we have:

in case (1.1)

$$a = p(a, b, a_3, \ldots, a_n)\ \theta\ p(a, a, a_3, \ldots, a_n) = a_n,$$

hence $\theta = A \times A$;

in case (1.2)

$$a_3 = p(a_3, a, b, a_4, \ldots, a_n)\ \theta\ p(a_3, a, a, a_4, \ldots, a_n) = a,$$

hence $\theta = A \times A$;

in case (1.3)

$$a_3 = p(a_3, a, b, a_4, \ldots, a_n)\ \theta\ p(a_3, a, a, a_4, \ldots, a_n) = a_n,$$

hence $A - \{a, b\}$ is contained in one θ-class.

If $4 < |A|$ we get in case (1.3) also $\theta = A \times A$, and so only the case (1.3), where $|A| = 4$, remains. Let us assume that $\mathfrak{B}$ is not simple. Then we have $|A| = n = 4$ and $p(x, x, z, u)$ cannot equal z or u, as by (1.1) we would have simplicity. So we have the equations $p(x, x, z, u) = x$ and $p(x, y, y, u) = u$. For different $a, b \in A$ we get $a = p(a, a, a, b) = b$, which is a contradiction. We have now proved that $\mathfrak{B}$ is simple. As $\mathfrak{B}$ is a reduct of $\mathfrak{A}$, $\mathfrak{A}$ also has to be simple.

LEMMA 2. *Let $\mathfrak{A} = (A, F)$ be a homogeneous algebra. If $|A| \neq 4$ and $\mathfrak{A}$ is not quasitrivial, then $\mathfrak{A}$ is simple.*

Proof. By [1] 1.2(iv) A has to be finite and there is an n-ary polynomial s on $\mathfrak{A}$ such that $n = |A|-1$ and $s(x_1, \ldots, x_n) = x_{n+1}$ whenever $\{x_1, \ldots, x_{n+1}\} = A$. The polynomial $s(x_1, x_1, x_2, \ldots, x_{n-1})$ has to be quasitrivial (see [1], 1.2 (i)), and so we have, for different $x_1, \ldots, x_{n-1}$, either

(2.1) $s(x_1, x_1, x_2, \ldots, x_{n-1}) = x_1$, or

(2.2) $s(x_1, x_1, x_2, \ldots, x_{n-1}) = x_i, \quad i \in \{2, \ldots, n-1\}$, say $i = n-1$.

In this case we have $n \geqslant 3$.

Let $\theta \neq \mathrm{id}_A$ be a congruence on $\mathfrak{A}$ and $a\,\theta\,b$ for some $a \neq b$. $A = \{a, b, a_2, \ldots, a_n\}$ and we have

in case (2.1)

$$a_n = s(a, b, a_2, \ldots, a_{n-1})\ \theta\ s(a, a, a_2, \ldots, a_{n-1}) = a,$$

hence $\theta = A \times A$;

in case (2.2)

$$a_n = s(a, b, a_2, \ldots, a_{n-1})\ \theta\ s(a, a, a_2, \ldots, a_{n-1}) = a_{n-1},$$

hence $A - \{a, b\}$ is contained in one θ-class.

Because of $|A| \neq 4$ and $n \geqslant 3$ in case (2.2) we get $\theta = A \times A$ even in that case.

LEMMA 3. *Let $\mathfrak{A} = (A, F)$ be a nontrivial homogeneous algebra. If $\mathfrak{A}$ is not simple, then $\mathfrak{A}$ is the idempotent reduct of a 2-dimensional vector space over the 2-element field.*

Proof. By Lemma 1 $\mathfrak{A}$ cannot have quasitrivial polynomials which are not trivial. By Lemma 2 we have $|A| = 4$ and there is a ternary polynomial $s(x, y, z)$ satisfying $s(x, y, z) = u$ if $A = \{x, y, z, u\}$ and $s(x, x, y) = s(x, y, x) = s(y, x, x) = y$, as otherwise by (2.1) the algebra would be simple.

Define on A an operation $+\colon A^2 \to A$ such that $\mathfrak{B} = (A, +)$ is the 2-dimensional vector space over the 2-element field. It is an easy computation that $s(x, y, z) = x + y + z$. In $\mathfrak{A}$ we have a congruence which is a partition of A into two 2-element classes. As any permutation of A is an automorphism of $\mathfrak{A}$, any such partition is a congruence on $\mathfrak{A}$. So the algebras $\mathfrak{A}$ and $\mathfrak{B}$ have the same congruences.

By [4] 6.5 the algebra $\mathfrak{B}$ is affine complete, which means that any operation which preserves the congruences of $\mathfrak{B}$ is an algebraic function on $\mathfrak{B}$.

The algebraic functions on $\mathfrak{B}$ are of the form $\sum_{i=1}^{m} x_i + a$. Every polynomial of $\mathfrak{A}$ preserves the congruences of $\mathfrak{B}$ and moreover is idempotent, and so it is of the form $\sum_{i=1}^{m} x_i$, where m is odd. All these operations are generated by the polynomial $s(x, y, z) = x+y+z$, and so $\mathfrak{A}$ is the idempotent reduct of $\mathfrak{B}$.

By these three lemmas the theorem is proved.

We wish to add another fact about quasitrivial homogeneous operations, namely

THEOREM. *Every quasitrivial homogeneous operation on a finite set is generated (by composition) by the ternary discriminator*

$$d(x, y, z) = \begin{cases} x & \text{if} \quad x \neq y, \\ z & \text{if} \quad x = y. \end{cases}$$

Proof. If the set X is finite, then by [2] the algebra $\mathfrak{A} = (X, d)$ is quasi-primal, which means that any operation preserving subalgebras and isomorphisms between subalgebras is a polynomial on $\mathfrak{A}$. As any subset of X forms a subalgebra of $\mathfrak{A}$, that means that any quasitrivial homogeneous operation on X is a polynomial on $\mathfrak{A}$.

References

[1] E. Marczewski, *Homogeneous operations and homogeneous algebras*, Fund. Math. 56 (1964), pp. 81–103.

[2] A. F. Pixley, *The ternary discriminator function in universal algebra*, Math. Ann. 191 (1971), pp. 167–180.

[3] S. Świerczkowski, *Algebras which are independently generated by every n elements*, Fund. Math. 49 (1960), pp. 93–104.

[4] H. Werner, *Produkte von Kongruenzklassengeometrien universeller Algebren*, Math. Zeitschr. 121 (171), pp. 111–140.

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Addition and correction to the paper "Diagonal algebras", Fund. Math. 58 (1966), pp. 309-321

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In the paper quoted in the title the second part of Theorem 2 was formulated wrongly. That was observed by Dawid Kelly (Ludwigshafen). In this note we correct this mistake, namely the following is true:

Let $\mathfrak{D}_{A_1, A_2, \ldots, A_n} = (A_1 \times \ldots \times A_n;\ d^*(x_1, \ldots, x_n))$ *be an* n*-dimensional diagonal algebra. Then the minimal cardinal number of sets of generators of* $\mathfrak{D}_{A_1, A_2, \ldots, A_n}$ *is equal to* $\max(a_1, a_2, \ldots, a_n)$, *where* $a_p = |A_p|$ $(p = 1, \ldots, n)$.

Proof. If G is a set of generators, it must contain at least one element of each coset in each direction (see [1]). Hence,

$$|G| \geqslant \max(a_1, a_2, \ldots, a_n) .$$

We can assume without loss of generality that if $a_i \leqslant a_j$, then $A_i \subset A_j$. Let us fix $a_0 \in A_1 \cap A_2 \cap \ldots \cap A_n$. For any $a \in A_1 \cup A_2 \cup \ldots \cup A_n$ we define the n-tuple $[q_1, q_2, \ldots, q_n]$ as follows: $q_i = a$ if $a \in A_i$ and $q_i = a_0$ if $a \notin A_i$. Let G_0 be the set of all possible n-tuples $[q_1, q_2, \ldots, q_n]$. Then, by (i) from [1], G_0 is the set of generators of $\mathfrak{D}_{A_1, A_2, \ldots, A_n}$ and

$$|G_0| = \max(a_1, a_2, \ldots, a_n) . \qquad \text{Q.e.d.}$$

Additionally we show an interesting example of a diagonal algebra. We say that an algebra $\mathfrak{A}_1 = (A;\ F_1)$ is a *reduct* of algebra $\mathfrak{A}_2 = (A;\ F_2)$ if $F_1 \subset A(\mathfrak{A}_2)$. We have

THEOREM. *For each* $n \geqslant 2$ *there exists an* n*-dimensional proper diagonal algebra which is a reduct of some abelian group.*

Proof. Let $p_1, p_2, \ldots, p_n$ be a sequence of different prime integers. Let $\mathfrak{G} = (G;\ \cdot,\ ^{-1})$ be an abelian group with the exponent $m = p_1 p_2 \ldots p_n$, i.e. $\mathfrak{G}$ satisfies $x^m = 1$ and does not satisfy any equality $x^k = 1$, where $k < m$.