to $F$. Hence for some $n$, $s_j \cap \cap_{i=1}^n \left( h_1 \cup \cdots \cup h_i \right) \in X_n$. (Otherwise $h_1 \cup \cdots \cup h_j$ would already be in $F$.) Let $s_j' = s_j \cap \cap_{i=1}^n \left( h_1 \cup \cdots \cup h_i \right)$.

Let $d_n = \cup_{a_n}$. For each $a$, $m \in d_n$, $s_m \cap d_n \in X^n$ for infinitely many $r$'s. Hence if $F_n$ is generated by $d_n$ and $F$, $F_n$ obeys the induction hypothesis.

However, if $X^n \in X^r$, $d_n \cap \cap_{i=1}^n X_i$ is contained in the union of finitely many $d_n'$s for $j < n$. By the contrapositive of $D$, $d_n \cap \cap_{i=1}^n X_i$. Hence if $q$ contains $d_n$, $q \in X^n$. Finally, let $q$ be the unique ultrafilter containing $F_n$ for every $\alpha$, and $q \in X^n$. Then the only relative types of $q$ are the $p^n_\alpha$.

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Almost continuous functions on $I^a$

by

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Abstract. Suppose $n$ and $m$ are positive integers and let $I$ denote the closed unit interval $[0,1]$. It is proved that there exists a pair of almost continuous functions $f: I^m \to I^n$ and $g: I^n \to I^m$ such that the composed map $gf: I^m \to I^n$ has no fixed point and is not almost continuous. The function $f$ is a dense subset of $I^{2n}$.

The main purpose of this paper is to give a partial answer to a question posed by J. Stallings [2]. Unless otherwise stated, all functions considered have domain and range $I^a$ where $I$ denotes the closed unit interval, $[0,1]$, and $a$ is a positive integer. No distinction is made between a function and its graph. If each open set containing the function $f$ also contains a continuous function with the same domain as $f$, then $f$ is said to be almost continuous. Stallings introduced almost continuity in order to prove a generalization of the Brouwer fixed point theorem. He asked the following question. "Under what conditions is it true that if $f: X \to Y$ is almost continuous and $g: Y \to Z$ is almost continuous, then the composed map $gf: X \to Z$ is almost continuous?" In the present paper it is shown that there exists a pair of almost continuous functions $f: I^m \to I^n$ and $g: I^n \to I^m$ such that $gf$ has no fixed point. Since each almost continuous function on $I^n$ has a fixed point, it follows that $gf$ is not almost continuous.

Suppose $f: A \to B$. The statement that the subset $C$ of $A \times B$ is a blocking set of $f$ in $A \times B$ means that $C$ is closed relative to $A \times B$, $C$ contains no point of $f$ and $C$ intersects $g$ whenever $g$ is a continuous function with domain $A$ and range being a subset of $B$. If no proper subset of $C$ is a blocking set of $f$ in $A \times B$, $C$ is said to be a minimal blocking set of $f$ in $A \times B$. If the set $C$ is a minimal blocking set of some function $g: A \to B$, then $C$ is said to be a minimal blocking set in $A \times B$.

Suppose $D$ is a subset of $A \times B$. Then $p_A(D)$ will denote the projection of $D$ into $A$ and $p_B(D)$ will denote the projection of $D$ into $B$. If $K$ is a subset of $p_A(D)$, then $D/K$ denotes the part of $D$ with $A$-projection $K$.

**Theorem 1.** Suppose $f: I^m \to I^n$ is not almost continuous. (To simplify notation, we denote $I^m$ by $A$ and $I^n$ by $B$.) Then there exists a minimal blocking set $C$ of $f$ in $A \times B$. Further, $p_A(C)$ is a non-degenerate continuum and $p_B(C) = B$. 
Proof. The proof that there exists a minimal blocking set of \( f \) in \( A \times B \) is essentially the same as that given for a more restricted case in [1], and is omitted. Assume that \( p_A(C) = U \cap V \) where \( U \) and \( V \) are closed and \( U \cap V = \emptyset \). Then \( C \setminus U \) and \( C \setminus V \) are closed proper subsets of \( C \). By the minimality of \( C \), there exist continuous functions \( g : A \to B \) and \( g_1 : A \to B \) such that \( g \cap C \setminus U = \emptyset \) and \( g_1 \cap C \setminus V = \emptyset \). Using a Urysohn function, it is easy to construct a continuous function \( h : A \to B \) such that \( h \cap C \setminus V = g_1 \cap C \setminus U \). Then \( h \cap C \setminus U = \emptyset \), a contradiction. Thus \( p_A(C) \) is a constant. That \( p_A(C) \) is non-degenerate is obvious. That \( p_A(C) = B \) follows from the fact that \( C \) intersects each constant function from \( A \) to \( B \). This completes the proof.

Theorem 2. Suppose \( n \) and \( m \) are positive integers. There exist almost continuous functions \( f : I^n \to I^m \) and \( g : I^m \to I^n \) such that \( g \circ f \) has no fixed point.

Proof. Again, we simplify notation by letting \( A = I^n \) and \( B = I^m \). Denote by \( \theta \) the set to which the subset \( C \) of \( A \times B \) belongs if and only if \( C \) is closed and both \( p_A(C) \) and \( p_B(C) \) have cardinality \( \theta \). Then the set \( \theta \) also has cardinality \( \theta \). There exists a well-ordering \( C_1, C_2, \ldots, C_{\kappa}, \ldots, C_n, \ldots \) of \( \theta \) such that if \( C \) is in \( \theta \), the set of elements of \( C \) whose \( \theta \)-cardinality less than \( C \) for each \( C \) in \( \theta \) we define \( x_n, y_n, f(x_n), f(y_n) \), and \( g(f(x_n)), g(f(y_n)) \) such that \( x_n \neq g(f(x_n)) = g(f(y_n)) \). If \( g(f(x_n)) \) is in \( C_n \) and \( g(f(y_n)) \) is in \( C_m \). Choose a point \( (x_1, y_1) \) in \( C_1 \). Let \( f(x_1) = y_1 \) and let \( g(y_1) = x_1 \). Now, let \( (x_1, y_1) \) be a point in \( C_1 \), where \( y_1 \neq x_1 \). Let \( z_1 \) be in \( A \setminus \{x_1, y_1\} \). Let \( f(z_1) = y_1 \) and \( g(y_1) = x_1 \).

Suppose that \( C_\kappa \) is in \( \theta \) and assume that \( x_\kappa, y_\kappa, f(x_\kappa), f(y_\kappa) \), and \( g(f(x_\kappa)) \) exist and have the desired properties for each \( C_\kappa \) which precedes \( C_\kappa \). Denote by \( M \) the set to which \( x_\kappa \) belongs if and only if \( x_\kappa = f(x_\kappa) \). If \( x_\kappa \) is in \( M \), simply choose \( y_\kappa \) in \( A \) such that \( g(y_\kappa) \neq x_\kappa \). Then \( y_\kappa \neq g(f(y_\kappa)) \) is in \( C_\kappa \) and \( x_\kappa \neq g(f(y_\kappa)) \). Now, let \( (x_\kappa, y_\kappa) \) be in \( C_\kappa \), where \( y_\kappa \) is \( \neq \) in \( M \). Let \( z_\kappa \) be in \( A \setminus \{x_\kappa, y_\kappa\} \). Let \( f(z_\kappa) = y_\kappa \) and \( g(y_\kappa) = z_\kappa \).

Thus, by induction, \( x_\kappa, y_\kappa, f(x_\kappa), f(y_\kappa) \), \( g(f(x_\kappa)) \), and \( g(f(y_\kappa)) \) exist and have the desired properties for each \( C_\kappa \) in \( \theta \). Let \( N \) be the set to which \( x_\kappa \) belongs if \( \kappa \) only if \( x_\kappa \) is \( \neq \) in \( M \). In case \( x_\kappa \) is in \( A \setminus N \), let \( f(x_\kappa) \) be in \( f(N) \) where \( g(f(x_\kappa)) \neq x_\kappa \). Let \( y_\kappa \) be in \( B \), and choose \( x_\kappa \) in \( A \) such that \( f(x_\kappa) \neq x_\kappa \). Let \( D \) be a non-degenerate continuum in \( A \) containing \( x \). Denote by \( S \) the line segment with end-points \( P \) and \( (x, y) \), where \( P \) is the mid-point of the line segment joining \( (x, y) \) and \( (x, f(x)) \). Then \( S \setminus \{D \setminus (x, y)\} \) is in \( \theta \) and contains a point \( (x, f(x)) \) of \( f(N) \). Since \( (x, f(x)) \) is not in \( S \), \( f(x) = y_\kappa \), so \( f(N) = B \) and the above induction defines \( g(f(y_\kappa)) \) for each \( y_\kappa \) in \( B \).

If \( C \) is a minimal blocking set in \( A \times B \), by Theorem 1, \( C \) is in \( \theta \) and contains a point of \( f \). Thus \( f \) is almost continuous. Clearly, \( g(f) \) has no fixed point, and the proof is completed.

Note that each of the functions \( f \) and \( g \) defined in Theorem 2 is a dense subset of \( I^m \). This generalizes the result of Example 2 of [1].

We now make two additional definitions in order to pose some questions. The function \( f \) is said to be of Baire Class 1 if \( f \) is the pointwise limit of a sequence of continuous functions. The function \( f \) is said to be a connectivity function if \( f \) is connected whenever \( f \) is a connectivity function of the set of \( f \). Suppose \( f : I^m \to I^n \). If \( n = 1 \) and \( f \) is almost continuous, then \( f \) is a connectivity function. If \( n > 1 \) and \( f \) is a connectivity function, then \( f \) is almost continuous [2].

Question 1. To what extent can the results of Theorem 2 be extended to connectivity functions?

Question 2. What are the relationships of functions of Baire Class 1 to connectivity functions and almost continuous functions? Specifically, if \( f : I^m \to I^n \) is a connectivity function, under what conditions is \( f \) of Baire Class 1? Also, if \( f : I^n \to I^m \) is a connectivity function, is \( f \) almost continuous? (*)

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(*) A note should be added to the effect that the last part of Question 2 has been answered by J. B. Brown in a paper recently submitted to Fund. Math.