to F. Hence for some n, $e_j \cap a_{nj}^j - (h_1 \cup \ldots \cup h_j) \in X_{nj}^j$. (Otherwise $h_1 \cup \ldots \cup h_j$ would already be in F). Let $a_j = e_j \cap a_{nj}^j - (h_1 \cup \ldots \cup h_j)$.

Let $d_a = \bigcup_{n\in\omega} a_n$. For each $n, m \in \omega$, $e_n \cap d_a \in X_r^m$ for infinitely many r's. Hence if F_a is generated by d_a and F, F_a obeys the induction hypothesis.

However, if $X_m^n \in X^a$, $d_a \cap a_m^n$ is contained in the union of finitely many a_r^j's, for $j < n$. By the contrapositive of D, $d_a \cap a_m^n \notin X_m^n$. Hence if q contains d_a, $q \notin \bar{X}^a$.

Finally, let q be the unique ultrafilter containing F_a for every a, and $q \in \bigcap_{n\in\omega} \bar{X}^n$. Then the only relative types of $q^\sim$ are the $p_n^\sim$.

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Almost continous functions on I^n

by

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Abstract. Suppose n and m are positive integers and let I denote the closed unit interval $[0, 1]$. It is proved that there exists a pair of almost continuous functions $f\colon I^n \to I^m$ and $g\colon I^m \to I^n$ such that the composed map $gf\colon I^n \to I^n$ has no fixed point and is not almost continuous. The function f is a dense subset of I^{n+m}.

The main purpose of this paper is to give a partial answer to a question posed by J. Stallings [2]. Unless otherwise stated, all functions considered have domain and range I^n, where I denotes the closed unit interval, $[0, 1]$, and n is a positive integer. No distinction is made between a function and its graph. If each open set containing the function f also contains a continuous function with the same domain as f, then f is said to be *almost continuous*. Stallings introduced almost continuity in order to prove a generalization of the Brouwer fixed point theorem. He asked the following question. "Under what conditions is it true that if $f\colon X \to Y$ is almost continuous and $g\colon Y \to Z$ is almost continuous, then the composed map $gf\colon X \to Z$ is almost continuous?" In the present paper it is shown that there exists a pair of almost continuous functions $f\colon I^n \to I^m$ and $g\colon I^m \to I^n$ such that gf has no fixed point. Since each almost continuous function on I^n has a fixed point, it follows that gf is not almost continuous.

Suppose $f\colon A \to B$. The statement that the subset C of $A \times B$ is a *blocking set of f in $A \times B$* means that C is closed relative to $A \times B$, C contains no point of f and C intersects g whenever g is a continuous function with domain A and range being a subset of B. If no proper subset of C is a blocking set of f in $A \times B$, C is said to be a *minimal blocking set of f in $A \times B$*. If the set C is a minimal blocking set of some function $g\colon A \to B$, then C is said to be a *minimal blocking set in $A \times B$*.

Suppose D is a subset of $A \times B$. Then $p_A(D)$ will denote the projection of D into A and $p_B(D)$ will denote the projection of D into B. If K is a subset of $p_A(D)$, then $D|K$ denotes the part of D with A-projection K.

THEOREM 1. *Suppose $f\colon I^n \to I^m$ is not almost continuous.* (*To simplify notation, we denote I^n by A and I^m by B.*) *Then there exists a minimal blocking set C of f in $A \times B$. Further, $p_A(C)$ is a non-degenerate continuum and $p_B(C) = B$.*

Proof. The proof that there exists a minimal blocking set of f in $A\times B$ is essentially the same as that given for a more restricted case in [1], and is omitted. Assume that $p_A(C)=U\cup V$ where U and V are closed and $U\cap V=\emptyset$. Then $C|U$ and $C|V$ are closed proper subsets of C. By the minimality of C, there exist continuous functions $g_1\colon A\to B$ and $g_2\colon A\to B$ such that $g_1\cap C|U=\emptyset$ and $g_2\cap C|V=\emptyset$. Using a Urysohn function, it is easy to construct a continuous function $h\colon A\to B$ such that $h|V=g_2|V$ and $h|U=g_1|U$. Then $h\cap C=\emptyset$, a contradiction. Thus $p_A(C)$ is a continuum. That $p_A(C)$ is non-degenerate is obvious. That $p_B(C)=B$ follows from the fact that C intersects each constant function from A to B. This completes the proof.

THEOREM 2. *Suppose n and m are positive integers. There exist almost continuous functions $f\colon I^n\to I^m$ and $g\colon I^m\to I^n$ such that gf has no fixed point.*

Proof. Again, we simplify notation by letting $A=I^n$ and $B=I^m$. Denote by θ the set to which the subset C of $A\times B$ belongs if and only if C is closed and both $p_A(C)$ and $p_B(C)$ have cardinality c. Then the set θ also has cardinality c. There exists a well-ordering $C_1, C_2, \ldots, C_\omega, \ldots, C_\alpha, \ldots$ of θ such that if C is in θ, the set of elements of θ which preceed C has cardinality less than c. For each C_α in θ we will define x_α, z_α, $f(x_\alpha)$, $f(z_\alpha)$, $g(f(x_\alpha))$ and $g(f(z_\alpha))$ such that $x_\alpha\neq g(f(x_\alpha))$, $z_\alpha\neq g(f(z_\alpha))$, $(x_\alpha, f(x_\alpha))$ is in C_α and $(g(f(z_\alpha)), f(z_\alpha))$ is in C_α.

Choose a point (x_1, y_1) in C_1. Let $f(x_1)=y_1$ and let $g(y_1)=x$, where $x\neq x_1$. Now, let (z, y_2) be a point in C_1, where $y_2\neq y_1$. Let z_1 be in $A-\{z, x_1\}$. Let $f(z_1)=y_2$ and $g(y_2)=z$.

Suppose that C_α is in θ and assume that x_β, z_β, $f(x_\beta)$, $f(z_\beta)$, $g(f(x_\beta))$ and $g(f(z_\beta))$ exist and have the desired properties for each C_β which preceeds C_α. Denote by M the set to which x belongs if and only if $x=f(x_\beta)$ or $x=f(z_\beta)$ for some C_β which preceeds C_α. Let L denote the set to which x belongs if and only if x is x_β, z_β, $g(f(x_\beta))$, or $g(f(z_\beta))$ for some C_β which preceeds C_α.

Let x_α be in $p_A(C_\alpha)-(p_A(C_\alpha)\cap L)$ and choose y_1 such that (x_α, y_1) is in C_α. Let $f(x_\alpha)=y_1$. Since x_α is not in L, if y_1 is in M, then $g(y_1)\neq x_\alpha$. If y_1 is not in M, simply choose $g(y_1)$ in A such that $g(y_1)\neq x_\alpha$. Then $(x_\alpha, f(x_\alpha))$ is in C_α and $x_\alpha\neq g(f(x_\alpha))$. Now, let (z, y_2) be in C_α where y_2 is not in $M\cup\{y_1\}$. Let z_α be in $A-(L\cup\{z, x_\alpha\})$. Let $f(z_\alpha)=y_2$ and $g(y_2)=z$. Then $(g(f(z_\alpha)), f(z_\alpha))=(z, y_2)$ is in C_α and $z_\alpha\neq g(f(z_\alpha))$.

Thus, by induction, x_α, z_α, $f(x_\alpha)$, $f(z_\alpha)$, $g(f(x_\alpha))$, and $g(f(z_\alpha))$ exist and have the desired properties for each C_α in θ. Let N be the set to which x belongs if and only if x is x_α or z_α for some C_α in θ. In case x is in $A-N$, let $f(x)$ be in $f(N)$ where $g(f(x))\neq x$. Let y be in B, and choose x in A such that $f(x)\neq y$. Let D be a non-degenerate continuum in A containing x. Denote by S the line segment with end-points P and (x, y), where P is the mid-point of the line segment joining (x, y) and $(x, f(x))$. Then $S\cup(D\times\{y\})$ is in θ and must contain a point $(z, f(z))$ of $f|N$. Since $(x, f(x))$ is not in S, $f(z)=y$, so $f(N)=B$ and the above induction defines $g(y)$ for each y in B.

If C is a minimal blocking set in $A\times B$, by Theorem 1, C is in θ and contains a point of f. Thus f is almost continuous. Similarly, g is almost continuous. Clearly, gf has no fixed point, and the proof is completed.

Note that each of the functions f and g defined in Theorem 2 is a dense subset of I^{n+m}. This generalizes the result of Example 2 of [1].

We now make two additional definitions in order to pose some questions. The function f is said to be of *Baire Class* 1 if f is the pointwise limit of a sequence of continuous functions. The function f is said to be a *connectivity function* if $f|C$ is connected whenever C is a connected subset of the domain of f. Suppose $f\colon I^n\to I^n$. If $n=1$ and f is almost continuous, then f is a connectivity function. If $n>1$ and f is a connectivity function, then f is almost continuous [2].

Question 1. To what extent can the results of Theorem 2 be extended to connectivity functions?

Question 2. What are the relationships of functions of Baire Class 1 to connectivity functions and to almost continuous functions? Specifically, if $f\colon I^2\to I$ is a connectivity function, under what conditions is f of Baire Class 1? Also, if $f\colon I\to I$ is of Baire Class 1 and is a connectivity function, is f almost continuous (*)?

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(*) A note should be added to the effect that the last part of Question 2 has been answered by J. B. Brown in a paper recently submitted to Fund. Math.