A type of βN with $\aleph_0$ relative types

by

R. C. Solomon (London)

Abstract. βN is the space of ultrafilters on N, the integers. If $p, q \in \beta N - N$, and φ is a homeomorphism from βN into $\beta N - N$ such that $\varphi(p) = q$, then write $p < q$. Question: How many distinct (up to isomorphism) predecessors can an ultrafilter have in this ordering? It has been shewn that there ultrafilters with $2^{\aleph_0}$ predecessors and (assuming the continuum hypothesis) for every $n \in \omega$ there are ultrafilters with n predecessors. This paper gives a construction of an ultrafilter with $\aleph_0$ predecessors, assuming the continuum hypothesis.

1. Introduction. βN is the space of all ultrafilters on N, the integers. Its topology is generated by clopen sets of the form $W(E) = \{q\colon E \in q\}$ for each $E \subseteq N$.

We identify $n \in N$ with the principal ultrafilter generated by n. Let $N^* = \beta N - N$. N^* is the space of all non-principal ultrafilters on N.

If π is a permutation of N, and $p \in N^*$, write $\pi(p) = \{\pi[a]\colon a \in p\}$. This is also an ultrafilter, isomorphic to p. Put $p^\sim = \{q\colon \pi(p) = q$ for some permutation $\pi\}$. $p^\sim$ is called the *type* of p.

If φ is a homeomorphism of βN into N^*, and $p \notin N$, $\varphi(p) = q$, then $p^\sim$ is called a *relative type* of $q^\sim$.

In [2], Z. Frolík shewed that every type of N^* has at most $2^{\aleph_0}$ relative types.

The continuum hypothesis implies that for every finite n there are types with precisely n relative types. If a type has no relative types, it is called *minimal*.

In [4], A. K. and E. S. Steiner shewed that there is a type with exactly $2^{\aleph_0}$ relative types. They stated at the end of the paper that they did not know if a type could have precisely $\aleph_0$ relative types. This paper gives a construction of one such, assuming the continuum hypothesis.

2. Preliminaries. We will use X, Y, Z etc., with or without superscripts such as X^n etc., to denote countable subsets of N^*. The nth member of X is written X_n.

If X is a countable subset of N^*, we say X is *discrete* if there are sets $\{c_n\}_{n \in \omega}$ such that $c_n \in X_n$ and $n \neq m$ implies $c_n \cap c_m = \emptyset$.

If X is a countable discrete subset of N^*, and $p \in N^*$, write

$$\Sigma[X, p] = \{a\colon \{n\colon a \in X_n\} \in p\}\,.$$

If $q \in \bar{X}$, (the closure of X), write

$$\Omega[X, q] = \{a \subseteq \omega\colon \text{for all } b \in q,\ \exists n \in a,\ b \in X_n\}\,.$$

Say $q > p$ if there is a countable discrete set X such that $q = \Sigma[X, p]$. We also put $q^{\sim} > p^{\sim}$ whenever $q > p$.

The basic facts we shall need are in the following lemma.

LEMMA. 1) *If $X \cup Y$ is discrete and countable and $p \in \bar{X} \cap \bar{Y}$, then $p \in \overline{X \cap Y}$.*

2) *$\Sigma[X, p]$ and $\Omega[X, p]$ are ultrafilters, and $\Sigma[X, \Omega[X, p]] = p$, and $\Omega[X, \Sigma[X, p]] = p$.*

3) *$q > p$ iff $p^{\sim}$ is a relative type of $q^{\sim}$.*

4) *If $X \cap Y = \emptyset$, and $p \in \bar{X} \cap \bar{Y}$, then there are subsequences $X' \subset X$, $Y' \subset Y$, $p \in \bar{X}' \cap \bar{Y}'$, and either $X' \subset \bar{Y}'$ or $Y' \subset \bar{X}'$.*

5) *$q > p$ iff there are countable discrete sequences X and Y such that $\Sigma[X, p] = \Sigma[Y, q]$ and $X \subset \bar{Y}$, $X \cap Y = \emptyset$.*

6) *$>$ is a total ordering on $\{p^{\sim}\colon p^{\sim} < q^{\sim}\}$.*

7) *A type $p^{\sim}$ is minimal iff for no countable discrete set X does $p \in \bar{X} - X$.*

The proofs are in [1] and [3].

3. THEOREM. *Assuming the continuum hypothesis, there is an ultrafilter q such that $q^{\sim}$ has precisely $\aleph_0$ relative types.*

Proof. Let a^n_m, n, $m \in \omega$, be infinite subsets of ω such that

i) $a^n_m \cap a^n_{m'} = \emptyset$ for $m \neq m'$.

ii) $\bigcup_{m \in \omega} a^n_m = \omega$ for all n.

iii) $a^{n+1}_m = \bigcup_{r \in f_n(m)} a^n_r$, where $f_n(m)$ is an infinite subset of ω.

(i.e. $\{a^n_m\}_{m \in \omega}$ is a partition of ω, and $\{a^{n+1}_m\}_{m \in \omega}$ is coarser than $\{a^n_m\}_{m \in \omega}$.)

Now let X^1_m be minimal types s.t. $a^1_m \in X^1_m$ for all m.

We will define X^n_m for all n. Suppose we have defined X^n_m for some n and all m. Let Y^n_m be minimal types such that $f_n(m) \in Y^n_m$, and let $X^{n+1}_m = \Sigma[X^n, Y^n_m]$.

Thus we can define X^n_m for all n, m. From the construction, $a^n_m \in X^n_m$, and $X^{n+1} \subset \bar{X}^n - X$.

Our aim is to construct an ultrafilter $q \in \bigcap_{n \in \omega} \bar{X}^n$; such that if $p_n = \Omega[X^n, q]$, then the only relative types of $q^{\sim}$ are the $p^{\sim}_n$.

First we state a few facts about the construction:

A) *$q = p_0 > p_1 > \ldots > p_n > \ldots$*

B) *$p_n = \Sigma[Y^n, p_{n+1}]$.*

C) *If $p_n \geqslant p \geqslant p_{n+1}$, then either $p^{\sim} = p^{\sim}_n$ or $p^{\sim} = p^{\sim}_{n+1}$.*

D) *If $a \in X^{n+1}_m$, then for infinitely many r, $a \in X^n_r$.*

E) *If $p_n > p$ for all n, then there is a p' and a countable discrete set $X' \subset \bigcup_{n \in \omega} X^n$ such that $q \in \bar{X}'$ and $p_n > p'$ for all n and $p' = \Omega[X', q]$.*

Proofs. A, B, C and D are routine applications of the Lemma. To prove E, assume $q = \Sigma[X, p]$, where we can assume that $X \subset \bar{X}^1$. Let $X = Y \cup Z$, where $Y \subset \bigcap \bar{X}^n$ and $Z \cap \bigcap \bar{X}^n = \emptyset$.

Case 1. $q \in \bar{Y}$. Let Y be made discrete by $\{c_n\}_{n \in \omega}$. Let

$$X' = \{X^n_m\colon c_n \in X^n_m\}.$$

Case 2. $q \in \bar{Z}$. Let Z be made discrete by $\{c_n\}_{n \in \omega}$. Let

$$X' = \{X^n_m\colon Z_r \in \bar{X}^n - \bar{X}^{n+1} \text{ and } c_r \in X^n_m\}\,.$$

In both cases it is routine to check that X' is a countable discrete set such that $q \in \bar{X}'$, and for each n, there is $X'' \subset X'$, s.t. $q \in \bar{X}''$ and $X'' \subset \bar{X}^n$. So if we let $p' = \Omega[X', q]$, then $p_n > p'$ for all n.

From the facts C and E above, to ensure that the only relative types of $q^{\sim}$ are the $p^{\sim}_n$, it suffices to shew that for every countable discrete subset X of $\bigcup_{n \in \omega} X^n$, either $q \notin \bar{X}$ or else $q \in \overline{X \cap X^n}$ for some n.

Enumerate (C.H.) the countable discrete subsets of $\bigcup_{n \in \omega} X^n$ as $\langle X^{\alpha}\rangle_{\alpha < \omega_1}$. At each stage α we will add a set d_{α} to q, s.t. either $d_{\alpha} \notin X^{\alpha}_m$ for any m, or else for some fixed n, $d_{\alpha} = \{a^n_m\colon X^n_m \in X^{\alpha}\}$. The first case will ensure that $q \notin \bar{X}^{\alpha}$, and the second that $q \in \overline{X^{\alpha} \cap X^n}$. In the latter case $(\Omega[X^{\alpha}, q])^{\sim} = p^{\sim}_n$.

INDUCTION HYPOTHESIS. *At each stage α we will construct a filter F_{α} s.t. for $a \in F_{\alpha}$, and any n, $\{m\colon a \in X^n_m\}$ is infinite.*

Stage 0. Let $d_0 = \omega$.

Stage α. Suppose we have constructed d_{β}, and F_{β} for $\beta < \alpha$. Let $\bigcup F_{\beta}$ generate a filter F. F is countably generated, so assume it is generated by $\{e_n\}_{n \in \omega}$, where $e_{n+1} \subseteq e_n$.

For each n, write $h_n = \bigcup \{a^n_m\colon X^n_m \in X^{\alpha}\}$.

Case 1. For some n, the filter generated by h_n and F obeys the induction hypothesis. Then let $d_{\alpha} = h_n$.

Case 2. Otherwise.

Define sets a_n as follows: h_1 cannot be added to F. So for some n, $e_1 \cap a^1_{n_1} \in X^1_{n_1}$ and $X^1_{n_1} \notin X^{\alpha}$. Let $a_1 = a^1_{n_1} \cap e_1$.

Suppose we have defined a_i for $i < j$. $h_1 \cup \ldots \cup h_j$ cannot be added to F. Hence for some n, $e_j \cap a^j_{nj} - (h_1 \cup \ldots \cup h_j) \in X^j_{nj}$. (Otherwise $h_1 \cup \ldots \cup h_j$ would already be in F). Let $a_j = e_j \cap a^j_{nj} - (h_1 \cup \ldots \cup h_j)$.

Let $d_a = \bigcup_{n \in \omega} a_n$. For each $n, m \in \omega$, $e_n \cap d_a \in X^m_r$ for infinitely many r's. Hence if F_a is generated by d_a and F, F_a obeys the induction hypothesis.

However, if $X^n_m \in X^a$, $d_a \cap a^n_m$ is contained in the union of finitely many a^{j}_r's, for $j < n$. By the contrapositive of D, $d_a \cap a^n_m \notin X^n_m$. Hence if q contains d_a, $q \notin \bar{X}^a$.

Finally, let q be the unique ultrafilter containing F_a for every a, and $q \in \bigcap_{n \in \omega} \bar{X}^n$. Then the only relative types of $q^\sim$ are the $p_n^\sim$.

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DEPARTMENT OF MATHEMATICS
BEDFORD COLLEGE, London

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Almost continous functions on I^n

by

Kenneth R. Kellum (Birmingham, Ala.)

Abstract. Suppose n and m are positive integers and let I denote the closed unit interval $[0, 1]$. It is proved that there exists a pair of almost continuous functions $f\colon I^n \to I^m$ and $g\colon I^m \to I^n$ such that the composed map $gf\colon I^n \to I^n$ has no fixed point and is not almost continuous. The function f is a dense subset of I^{n+m}.

The main purpose of this paper is to give a partial answer to a question posed by J. Stallings [2]. Unless otherwise stated, all functions considered have domain and range I^n, where I denotes the closed unit interval, $[0, 1]$, and n is a positive integer. No distinction is made between a function and its graph. If each open set containing the function f also contains a continuous function with the same domain as f, then f is said to be *almost continuous*. Stallings introduced almost continuity in order to prove a generalization of the Brouwer fixed point theorem. He asked the following question. "Under what conditions is it true that if $f\colon X \to Y$ is almost continuous and $g\colon Y \to Z$ is almost continuous, then the composed map $gf\colon X \to Z$ is almost continuous?" In the present paper it is shown that there exists a pair of almost continuous functions $f\colon I^n \to I^m$ and $g\colon I^m \to I^n$ such that gf has no fixed point. Since each almost continuous function on I^n has a fixed point, it follows that gf is not almost continuous.

Suppose $f\colon A \to B$. The statement that the subset C of $A \times B$ is a *blocking set of f in $A \times B$* means that C is closed relative to $A \times B$, C contains no point of f and C intersects g whenever g is a continuous function with domain A and range being a subset of B. If no proper subset of C is a blocking set of f in $A \times B$, C is said to be a *minimal blocking set of f in $A \times B$*. If the set C is a minimal blocking set of some function $g\colon A \to B$, then C is said to be a *minimal blocking set in $A \times B$*.

Suppose D is a subset of $A \times B$. Then $p_A(D)$ will denote the projection of D into A and $p_B(D)$ will denote the projection of D into B. If K is a subset of $p_A(D)$, then $D|K$ denotes the part of D with A-projection K.

THEOREM 1. *Suppose $f\colon I^n \to I^m$ is not almost continuous. (To simplify notation, we denote I^n by A and I^m by B.) Then there exists a minimal blocking set C of f in $A \times B$. Further, $p_A(C)$ is a non-degenerate continuum and $p_B(C) = B$.*