A type of $\beta N$ with $\aleph_0$ relative types

by

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Abstract. $\beta N$ is the space of ultrafilters on $N$, the integers. If $p, q \in \beta N - N$, and $\varphi$ is a homeomorphism from $\beta N$ into $\beta N - N$ such that $\varphi(p) = q$, then write $p < q$. Question: How many distinct (up to isomorphism) predecessors can an ultrafilter have in this ordering? It has been shown that there are ultrafilters with $2^{\aleph_0}$ predecessors and (assuming the continuum hypothesis) for every $n < \omega$ there are ultrafilters with $n$ predecessors. This paper gives a construction of an ultrafilter with $\aleph_0$ predecessors, assuming the continuum hypothesis.

1. Introduction. $\beta N$ is the space of all ultrafilters on $N$, the integers. Its topology is generated by clopen sets of the form $W(E) = \{ q : E \in q \}$ for each $E \subseteq N$.

We identify $w \in N$ with the principal ultrafilter generated by $w$.

Let $N^* = \beta N - N$. $N^*$ is the space of all non-principal ultrafilters on $N$.

If $\pi$ is a permutation of $N$, and $p \in N^*$, write $\pi(p) = \{ \pi(a) : a \in p \}$. This is also an ultrafilter, isomorphic to $p$. Put $p^\pi = \{ q : \pi(p) = q \}$ for some permutation $\pi$. $p^\pi$ is called the type of $p$.

If $\varphi$ is a homeomorphism of $\beta N$ into $N^*$, and $p \not\in N$, $\varphi(p) = q$, then $p^\pi$ is called a relative type of $\varphi(p)$.

In [2], Z. Frolik showed that every type of $N^*$ has at most $2^{\aleph_0}$ relative types.

The continuum hypothesis implies that for every finite $n$ there are types with precisely $n$ relative types. If a type has no relative types, it is called minimal.

In [4], A. K. and E. Steiner showed that there is a type with exactly $2^{\aleph_0}$ relative types. They stated at the end of the paper that they did not know if a type could have precisely $\aleph_0$ relative types. This paper gives a construction of one such, assuming the continuum hypothesis.

2. Preliminaries. We will use $X$, $Y$, $Z$ etc., with or without superscripts such as $X^*$ etc., to denote countable subsets of $N^*$. The nth member of $X$ is written $X_n$.

If $X$ is a countable subset of $N^*$, we say $X$ is discrete if there are sets $(c_n)_{n=\omega}$ such that $c_n \subseteq X_n$ and $n \neq m$ implies $c_n \cap c_m = \emptyset$.
If $X$ is a countable discrete subset of $N^*$, and $p \in N^*$, write

$$\Sigma(X, p) = \{a \in V: a \in X \in p\}.$$  

If $q \in X$, (the closure of $X$, write

$$\Omega(X, q) = \{a \in X: \exists b \in X, a \in b \in X\}.$$  

Say $q > p$ if there is a countable discrete set $X$ such that $q = \Sigma(X, p)$. We also put $q^+ > p^+$ whenever $q > p$.

The basic facts we shall need in the following lemmas.

**Lemma 1.** If $X \cup Y$ is discrete and countable and $p \in \bar{X} \cap \bar{Y}$, then $p \in \bar{X} \cap \bar{Y}$.

2. $\Sigma(X, p)$ and $\Omega(X, p)$ are ultrafilters, and $\Sigma(X, \Omega(X, p)) = p$, and $\Omega(X, \Sigma(X, p)) = p$.

3. $q > p$ iff $q^+ > p^+$ is a relative type of $q^+$.  

4. If $X \cup Y = \emptyset$, and $p \in \bar{X} \cap \bar{Y}$, then there are subsequences $X' \subset X$, $Y' \subset Y$, and either $X' \subset Y'$ or $Y' \subset X'$.

5. $q > p$ iff there are countable discrete sequences $X$ and $Y$ such that $\Sigma(X, p) = \Sigma(Y, q)$ and $X \cup Y = \emptyset$.

6. $\triangleright$ is a total ordering on $(p^+, q^+) < q^+$.  

7. A type $p^+$ is minimal iff for no countable discrete set $X$ does $p \in \bar{X} \cap X$.

The proofs are in [1] and [3].

**3. Theorem.** Assuming the continuum hypothesis, there is an ultrafilter $q^+$ such that $q^+$ has precisely $\kappa$ relative types.

**Proof.** Let $a^*_m$, $n, m \in \omega$, be infinite subsets of $\omega$ such that

i) $a^*_m \cap a^*_n = \emptyset$ for $m \neq n$.

ii) $\bigcup_{x \in a^*_m} a^*_x$ is an infinite subset of $\omega$.

iii) $a^*_m = \bigcup_{x \in a^*_m} a^*_x$, where $f_m(a)$ is an infinite subset of $\omega$.

(i.e. $(a^*_m)_{m \in \omega}$ is a partition of $\omega$, and $(a^*_m)_{m \in \omega}$ is coarser than $(a^*_m)_{m \in \omega}$.

Now let $X^*_m$ be minimal types s.t. $a^*_m \in X^*_m$ for all $m$.

We will define $X^*_m$ for all $m$. Suppose we have defined $X^*_m$ for some $m$ and all $n$. Let $X^*_m$ be minimal types such that $f_m(a) \in X^*_m$, and let $X^*_m = \Sigma(X^*_m, X^*_m)$.

Thus we can define $X^*_m$ for all $m$. From the construction, $a^*_m \in X^*_m$, and $X^*_m \subset X^*_m \subset X^*_m$.

Our aim is to construct an ultrafilter $q \in \bar{X}^*_m$; such that if $p_0 = \Omega(X^*_m, q)$, then the only relative types of $q^+$ are the $p^*_n$.

First state a few facts about the construction:

A type $p^+$ of $\beta X$ with $\kappa_1$ relative types

A) $q = p_0 \succ p_1 \succ ... \succ p_\alpha \succ ...$

B) $p_\alpha = \Sigma(X^*_\alpha, p_{\alpha+1})$.

C) If $p_\alpha \succ p_\beta \succ p_{\alpha+1}$, then either $p^\alpha = p^\alpha_\beta$ or $p^\alpha = p^\alpha_{\alpha+1}$.

D) If $p_\alpha \succ p$, then for infinitely many $r, a \in X^*_\alpha$.

E) If $p_\alpha \succ p$ for all $n$, then there is a $p'$ and a countable discrete set $X' \subset X^*_\alpha$ such that $q \in X'$ and $p_\alpha \succ p'$ for all $n$ and $p' = \Omega(X', q)$.  

Proofs. A, B, C and D are routine applications of the Lemma.  

To prove E, assume $q = \Sigma(X, p)$, where we can assume that $X \subset X^*_\alpha$.  

Let $X = Y \cup Z$, where $Y \subset \bigcap X^*_n$ and $Z \cup Y = \emptyset$.

1) $q \in Y$. Let $Y$ be made discrete by $(\epsilon_n)_{n \in \omega}$. Let $Y' = (X^*_n; \epsilon_n \in X^*_n)$.

2) $q \in Z$. Let $Z$ be made discrete by $(\epsilon_n)_{n \in \omega}$. Let $Z = (X^*_n; \epsilon \in X^*_n \cap X^*_n)$.

In both cases it is routine to check that $X'$ is a countable discrete set such that $q \in X'$, and for each $n$, there is $X'' \subset X^*_n$, s.t. $q \in X''$, and $X'' \subset X^*_n$.  

So if we let $p' = \Omega(X', q)$, then $p_\alpha \succ p'$ for all $n$.

From the facts C and D above, to ensure that the only relative types of $q^+$ are the $p^*_n$, it suffices to show that for every countable discrete subset $X \subset \bigcup X^*_n$, either $q \notin X$ or else $q \in X^* \cap X^*$ for some $n$.  

Enumerate (C.F.) the countable discrete subsets of $\bigcup X^*_n$ as $\langle X^*_n \rangle_{n \in \omega}$.

At each stage $p_0$ we will add a set $d_\alpha$ to $q$, s.t. either $d_\alpha \in X^*_n$ for any $n$, or else for some fixed $n$, $d_\alpha = (a^*_m)_{m \in \omega}$. The first case will ensure that $q \in X^*_n$, and the second that $q \in X^*_n \cap X^*_n$. In the latter case $(\Omega(X^*_n, q))^{-1} = p^*_n$.

**Induction Hypothesis.** At each stage we will construct a filter $F_\alpha$ s.t. for $a \in F_\alpha$, and any $n, (m : \epsilon_m \in X^*_n)$ is infinite.

Stage 0. Let $d_0 = \emptyset$.

Stage $a$. Suppose we have constructed $d_\alpha$, and $F_\alpha$ for $\beta < a$. Let $\bigcup F_\beta$ generate a filter $F$. $F$ is countably generated, so assume it is generated by $(\epsilon_n)_{n \in \omega}$ where $\epsilon_n \subseteq \omega$.

For each $n$, write $h_\alpha = \bigcup (a^*_m \cap X^*_n)$.

Case 1. For some $n$, the filter generated by $h_\alpha$ and $F$ obeys the induction hypothesis. Then let $d_\alpha = h_\alpha$.

Case 2. Otherwise. Define sets $a^*_n$ as follows: $h_\alpha$ cannot be added to $F$. So for some $n$, $e_1 \cap a^*_n \in X^*_n$ and $X^*_n \subseteq X^*_n$. Let $a^*_n = a^*_n \cap e_1$.

Suppose we have defined $a_i$ for $i < \beta$. $h_\alpha \subset \bigcup_{i < \beta} h_i$ cannot be added

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to $F$. Hence for some $n$, $e_j \cap a_m^*(h_1 \cup \ldots \cup h_j) \in X^m_n$. (Otherwise $h_1 \cup \ldots \cup h_j$ would already be in $F$). Let $e_j = e_j \cap a_m^*(h_1 \cup \ldots \cup h_j)$.

Let $d_n = \bigcup_{m \leq n} e_m$. For each $a_n$, $e_n \cap d_n$ is $X_n$ for infinitely many $r$'s. Hence if $F_a$ is generated by $d_n$ and $F$, $F_a$ obeys the induction hypothesis.

However, if $X_n \subseteq X$, $d_n \cap a_n$ is contained in the union of finitely many $a_j^*$'s, for $j \neq n$. By the contrapositive of $D$, $d_n \cap a_n \subseteq X_n$. Hence if $q$ contains $d_n$, $q \subseteq X$.

Finally, let $q$ be the unique ultrafilter containing $F_a$ for every $a$, and $q \subseteq X$. Then the only relative types of $q'$ are the $p_n$.

References


Almost continuous functions on $I^n$

by

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Abstract. Suppose $n$ and $m$ are positive integers and let $I$ denote the closed unit interval $[0, 1]$. It is proved that there exists a pair of almost continuous functions $f: I^n \to I^m$ and $g: I^m \to I^n$ such that the composed map $gf: I^n \to I^n$ has no fixed point and is not almost continuous. The function $f$ is a dense subset of $I^{*+m}$.

The main purpose of this paper is to give a partial answer to a question posed by J. Stallings [2]. Unless otherwise stated, all functions considered have domain and range $I^n$ where $I$ denotes the closed unit interval, $[0, 1]$, and $n$ is a positive integer. No distinction is made between a function and its graph. If each open set containing the function $f$ also contains a continuous function with the same domain as $f$, then $f$ is said to be almost continuous. Stallings introduced almost continuity in order to prove a generalization of the Brouwer fixed point theorem. He asked the following question. "Under what conditions is it true that if $f: X \to Y$ is almost continuous and $g: Y \to Z$ is almost continuous, then the composed map $gf: X \to Z$ is almost continuous?" In the present paper it is shown that there exists a pair of almost continuous functions $f: I^n \to I^m$ and $g: I^m \to I^n$ such that $gf$ has no fixed point. Since each almost continuous function on $I^n$ has a fixed point, it follows that $gf$ is not almost continuous.

Suppose $f: A \to B$. The statement that the subset $C$ of $A \times B$ is a blocking set of $f$ in $A \times B$ means that $C$ is closed relative to $A \times B$, $C$ contains no point of $f$ and $C$ intersects $g$ whenever $g$ is a continuous function with domain $A$ and range being a subset of $B$. If no proper subset of $C$ is a blocking set of $f$ in $A \times B$, $C$ is said to be a minimal blocking set of $f$ in $A \times B$. If the set $C$ is a minimal blocking set of some function $g: A \to B$, then $C$ is said to be a minimal blocking set in $A \times B$.

Suppose $D$ is a subset of $A \times B$. Then $p_a(D)$ will denote the projection of $D$ into $A$ and $p_b(D)$ will denote the projection of $D$ into $B$. If $K$ is a subset of $p_a(D)$, then $D/K$ denotes the partition of $D$ with $A$-projection $K$.

Theorem 1. Suppose $f: I^n \to I^m$ is not almost continuous. (To simplify notation, we denote $I^n$ by $A$ and $I^m$ by $B$.) Then there exists a minimal blocking set $C$ of $f$ in $A \times B$. Further, $p_a(C)$ is a non-degenerate continuum and $p_b(C) = B$. 