Grouplike Menger algebras

by

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Abstract. This paper generalizes the concept of group to include sets with \( n \)-ary operations. An \( n \)-place Menger algebra is a set \( A \) with an \( (n+1) \)-ary operation \( o \) satisfying the superassociative law: \( o(o(a_0, a_1, ..., a_n), b_1, ..., b_n) = o(a_0, o(a_1, b_1, ..., b_n), ..., o(a_n, b_1, ..., b_n)) \). \( A \) is said to be grouplike if for any sequence \( a_0, a_1, ..., a_n, b \) of elements from \( A \) there exist unique elements \( x_0, x_1, ..., x_n \) of \( A \) such that \( o(a_0, a_1, ..., a_n, x_i) = b \) for \( i = 0, 1, ..., n \). It is proved that there exist \( n \)-place grouplike Menger algebras of every finite order if \( n \) is odd; if \( n \) is even there exist grouplike Menger algebras of every order not of the form \( 2p \) where \( p \) is an odd prime. There exist no \( 2 \)-place grouplike Menger algebras of order \( 2p \). Alternate conditions for a Menger algebra to be grouplike are presented. The existence of grouplike subalgebras is also studied.

The study of functions of many variables gives rise to a natural extension of the concept of associativity for \( n \)-ary operations. An \( n \)-place function \( f \) over a set \( S \) is any mapping of \( S^n = S \times ... \times S \) (the cross product of \( S \) with itself \( n \) times) into \( S \). If \( f_1, f_2, ..., f_n \) are \( n \)-place functions over \( S \), the composite \( f = f_1 \circ f_2 \circ ... \circ f_n \) is defined in the usual way:

\[
(f_1 \circ f_2 \circ ... \circ f_n)(a_1, ..., a_n) = f_n(f_{n-1}(f_{n-2}(... (f_2(f_1(a_1, ..., a_n) = a_n))))
\]

for each \( n \)-tuple \( (a_1, ..., a_n) \) from \( S^n \). If \( g_1, ..., g_n \) are also \( n \)-place functions over \( S \) then it is easily verified that

\[
(f_1 \circ f_2 \circ ... \circ f_n)(g_1, ..., g_n) = f_n(f_{n-1}(f_{n-2}(... (f_2(f_1(g_1, ..., g_n) = g_n)) \).
\]

For the case \( n = 1 \), Equation (1) reduces to the associative law for transformations of a set into itself.

A set \( A \) of \( n \)-place functions is said to be an algebra of functions if the composite of any \( n+1 \) functions from \( A \) is also in \( A \). Such algebras have been extensively studied in [1]-[12]. Generalizing the above concepts we define an \( n \)-place Menger algebra to be a set \( A \) with an \((n+1)\)-ary operation \( o \) satisfying what has been called the superassociative law:

\[
o(o(o(a_0, a_1, ..., a_n), a'_1, ..., a'_n), a'_0) = o(o(a_0, o(a_1, a'_1, ..., a'_n), a'_0), ..., o(a_n, a'_1, ..., a'_n)),
\]

for any elements \( a_i, a'_j, i = 0, 1, ..., n, j = 1, ..., n \), from \( A \). The case \( n = 1 \) in just the ordinary associative law and hence any \( 1 \)-place Menger algebra is a semigroup.
It follows immediately from Equation (1) that any algebra of functions is a Menger algebra. Conversely, it was shown by Dikker (cf. [2]) that any Menger algebra is isomorphic to an algebra of functions over some set. Hence, in particular, any semigroup is isomorphic to a set of transformations over some set.

In this paper we study Menger algebras which satisfy certain solvability criteria, thus extending the group concept to include sets with n-ary operations.

An n-place Menger algebra is said to be grouplike if for any sequence \( a_0, a_1, \ldots, a_n \) of elements from \( A \) there exist unique elements \( x_0, x_1, \ldots, x_n \) in \( A \) such that

\[
(3) \quad o(a_0, \ldots, a_{r-1}, x_r, a_{r+1}, \ldots, a_n) = b \quad \text{for} \quad i = 0, 1, \ldots, n.
\]

Thus any 1-place grouplike Menger algebra is a group.

Some examples of grouplike Menger algebras are the following:

(i) The set of all n-place functions \( f_i \) over the reals defined by \( f_i(x_1, \ldots, x_n) = (x_1 + \ldots + x_n + i)/n \) for each real number \( r \).

(ii) The set \( I_{n+1} \) of integers modulo \( 2k+1 \) with the \((2n+1)\)-ary operation \( o(t_0, t_1, \ldots, t_{2n}) = t_{0} - t_{1} + \ldots + t_{2n-1} + 2t_{2n} \pmod{2k+1} \).

(iii) The set \( I_{2k} \) of integers modulo \( 2k \) with the 2-ary operation \( o(t_0, t_1, \ldots, t_{2n}) = t_{0} + t_{1} + \ldots + t_{2n-1} \pmod{2k} \).

**Theorem 1.** An n-place Menger algebra \( A \) is grouplike if and only if \( A \) contains an element \( e \) (called the identity of \( A \)) such that

\[
(3a) \quad o(e, x, \ldots, x) = x = o(x, e, \ldots, e) \quad \text{for each} \quad x \in A;
\]

and

\[
(3b) \quad \text{for any sequence} \quad a_0, a_1, \ldots, a_n \quad \text{of elements of} \quad A, \quad \text{there exist unique elements} \quad x_0, x_1, \ldots, x_n \quad \text{such that}
\]

\[
o(a_0, a_1, \ldots, a_{r-1}, x_r, a_{r+1}, \ldots, a_n) = e \quad \text{for} \quad i = 0, 1, \ldots, n.
\]

First suppose \( A \) is grouplike. Let \( a \) be any element of \( A \) and let \( e \) be the element of \( A \) such that \( o(e, a, a) = a \). For any element \( x \) in \( A \), there exists an element \( y \) such that \( o(x, y, a) = x \). Hence
\[
x = o(x, y, a, a, a).
\]

To prove the second equality of Condition (3a), choose elements \( y_0, \ldots, y_n \) such that \( o(x, y_0, \ldots, y_n) = e \). Then
\[
o(x, y_1, \ldots, y_n) = o(x, o(x, y_1, \ldots, y_n), y_0) = e.
\]

Therefore by the unique solvability criterion \( x \) must equal \( o(e, a, a, \ldots, a) \) and Condition (3a) follows. Clearly Condition (3b) is satisfied.

Conversely suppose \( A \) is a Menger algebra satisfying Conditions (3a) and (3b). Let \( a_0, a_1, \ldots, a_n, b \) be any sequence of elements from \( A \). We first prove there exists a unique element \( x \) such that \( o(a_0, a_1, \ldots, a_n) = b \). By Condition (3b) there exist elements \( b_1, \ldots, b_n \) such that \( o(b_1, b_2, \ldots, b_n) = e \).

Let \( x \) be the element such that \( o(x_0, o(a_0, b_1, \ldots, b_n), \ldots, o(a_n, b_1, \ldots, b_n)) = e \). The left-hand side of this equality is just \( o(o(x_0, a_1, \ldots, a_n, b_1, \ldots, b_n)) \) and it follows from the uniqueness stipulation in Condition (3b) that \( o(x_0, a_1, \ldots, a_n, b_1, \ldots, b_n)) = e \). Then also
\[
o(x_0, o(a_0, b_1, \ldots, b_n), \ldots, o(a_n, b_1, \ldots, b_n)) = e.
\]

Again, the uniqueness stipulation of Condition (3b) implies that \( x_0 = x \).

For \( i = 1, \ldots, n \), let \( y_0 \) be the element of \( A \) such that
\[
o(a_0, o(a_1, b_1, \ldots, b_n), \ldots, y_0) = o(a_0, b_1, \ldots, b_n)) = e.
\]

By the above, there exist elements \( x \) such that \( o(x_0, b_1, \ldots, b_n)) = y_0 \), whence
\[
e = o(a_0, o(a_1, b_1, \ldots, b_n), \ldots, o(x, b_1, \ldots, b_n), \ldots, o(a_n, b_1, \ldots, b_n))
\]

And by the uniqueness stipulation in Condition (3b), \( o(a_0, a_1, \ldots, a_n, a_0) = b \). If also \( o(a_0, a_1, \ldots, a_n, a_0) = b \) for another element \( x' \), then
\[
o(x', b_1, \ldots, b_n, a_0) = o(x', b_1, \ldots, b_n), \ldots, o(a_n, b_1, \ldots, b_n))
\]

We remark that the uniqueness stipulations of Conditions (3a) and (3b) cannot be dropped. For example, the set of integers is a 2-place Menger algebra if we define
\[
o(i, j, k) = \begin{cases}
i + \frac{1}{2}(j + k), & \text{if } j + k \text{ is even}, \\
i + \frac{1}{2}(j + k - 1), & \text{if } j + k \text{ is odd}.
\end{cases}
\]

Clearly 0 is the identity of this algebra and the equality \( o(i, j, k) = m \) always has a solution if \( m \) and two of the three integers \( i, j, k \) are given. But \( o(0, 0, 0) = o(0, 1, 0) = 0 \). We prove however the following

**Theorem 2.** A finite Menger algebra \( A \) is grouplike if

\[
(3') \quad \text{for any sequence} \quad a_0, a_1, \ldots, a_n \quad \text{of elements of} \quad A, \quad \text{there exist elements} \quad x_0, x_1, \ldots, x_n \quad \text{such that} \quad b = o(a_0, a_1, \ldots, a_n, x_0, a_1, \ldots, a_n); \quad \text{or if} \quad \text{Condition (3a); and the following condition holds:}
\]

\[
(3b') \quad \text{for any sequence} \quad a_0, a_1, \ldots, a_n \quad \text{of elements of} \quad A, \quad \text{there exist elements} \quad x_0, x_1, \ldots, x_n \quad \text{such that} \quad e = o(a_0, a_1, \ldots, a_n, x_0, a_1, \ldots, a_n).
\]
Let \( A = \{ a_1, ..., a_n \} \). By Condition (3a) there exist elements \( c_i, i = 1, ..., b, j = 0, 1, ..., n \) such that \( o(a_1, ..., a_j, c_i, c_{i+1}, ..., a_n) = x_j \) for \( x \neq t \), \( c_i \neq c_j \) for \( i \neq j \). Thus \( A = \{ c_1, ..., c_n \} \) for each \( j \) and it follows that the elements postulated in Condition (3a) are unique.

Now suppose \( A \) satisfies Conditions (3a) and (3b). We first show that for every \( x \in A \) \( o(x, a_1, ..., a_n) = o(x, a_2, ..., a_n) \) for \( x \neq y \). For equality holds. By Condition (3b) there exists an element \( a' \) such that \( o(a', a_1, ..., a_n) = 0 \). But then

\[
x = o(x, a_1, ..., a_n) = o(x, a', a_1, ..., a_n) = 0 = o(x, y, a_1, ..., a_n) = o(x, y, a, ..., a_n) = o(x, a, ...) = 0,
\]

whence \( x = y \).

We now prove that Condition (3b) holds. Let \( a_1, a_2, ..., a_b \) be elements from \( A \). By Condition (3b) there exists an element \( y \) such that \( o(y_1, a_1, ..., a_b) = 0 \). Hence

\[
b = o(b, a_1, ..., a_b) = o(b, y_1, a_1, ..., a_b) = 0 = o(b, y_1, 0, a_1, ..., a_b) = 0 = o(b, y_1, a_1, ..., a_b) = 0.
\]

Hence there exists an element \( x_0 \) such that \( x_0 = o(b, y_1, a_1, ..., a_b) \) such that \( o(x_0, a_1, ..., a_b) = b \).

For \( i > 0 \) let \( y_i \) be the element of \( A \) such that \( e = o(b, y_i, a_1, ..., a_b) \). By Condition (3b) there exists an element \( a \) such that

\[
e = o(b, a_1, ..., a_b) = 0 = o(b, a_1, ..., a_b, a, ..., a) = 0 = o(b, a_1, ..., a_b, a, ..., a) = 0.
\]

On the grouplike Menger algebra \( A \) we define a binary operation \( * \) by \( a * b = o(b, a_1, ..., a_b) \). It is easily verified, that since the operation \( o \) is associative, it is associative. By Condition (3a), \( a * e = e * a = a \) and hence \( a \) is an identity of \( A \) with respect to \( * \). Condition (3b) implies the existence of left inverses, whence \( A \), with the operation \( * \), is a group, which we call the diagonal group of \( A \) and denote by \( A^* \). By direct calculation one easily shows that

\[
o(a * b, c_1, ..., c_n) = a * [o(b, c_1, ..., c_n)]
\]

for any elements \( a, b, c_1, ..., c_n \) from \( A \).

**Theorem 3.** For any sequence \( a_1, ..., a_n \) of elements from a grouplike Menger algebra \( A \) there exists an element \( a \) such that \( o(a, a_1, ..., a_n) = x * a \) for every \( x \in A \).
Since
\[ o(e, e \cdot b^{-1}, a_1, \ldots, a_{n-1}) \neq o(e, e, a_1, \ldots, a_{n-1}) \text{ for } b \neq e, \]
it follows that
\[ \varphi(a_1 \cdot b, \ldots, a_{n-1} \cdot b) \neq o(e, e, a_1, \ldots, a_{n-1}) \cdot b = \varphi(a_1, \ldots, a_{n-1}) \cdot b. \]
Conversely suppose the finite group \( G \) satisfies the conditions of the theorem. We define the \((n-1)\)-ary operation as follows:
\[ o(a_0, a_1, \ldots, a_n) = o(a_0, a_1^{-1}, \ldots, a_n^{-1})a_1. \]
It is easily verified by direct calculation that the operation \( o \) is super-associative. Now let \( b \) be any element of \( A \). Then the equality
\[ b = o(a_0, a_1, \ldots, a_n) = a_0 \varphi(a_0 a_1^{-1}, \ldots, a_n a_1^{-1}) a_1 \]
is satisfied for \( x_0 = a_0 a_1^{-1}[\varphi(a_0 a_1^{-1}, \ldots, a_n a_1^{-1})]^{-1} \). Because \( G \) is finite, in order to prove there exist unique elements \( x_1, \ldots, x_n \) from \( G \) such that
\[ o(a_0, a_1, \ldots, a_n) = b \text{ it suffices to show that} \]
\[ \varphi(a_0 x_1, \ldots, a_0 x_n) \neq \varphi(a_0 y_1, \ldots, a_0 y_n) \]
and
\[ \varphi(a_0 x_1^{-1}, \ldots, a_0 x_n^{-1}) \neq \varphi(a_0 y_1^{-1}, \ldots, a_0 y_n^{-1}) \]
for \( x \neq y \). The second inequality follows immediately from Condition (5b). In order to verify the first inequality we suppose
\[ \varphi(a_0 x_1, \ldots, a_0 x_n) = \varphi(a_0 y_1, \ldots, a_0 y_n) \].
Setting \( a_i = a_i x_i \) we obtain
\[ \varphi(a_0, \ldots, a_n) z y = \varphi(a_0, \ldots, a_n) z y \]
and it follows from Condition (5c) that \( xy^{-1} = e \), that is \( x = y \). Hence \( G \) with the operation defined above is a grouplike Menger algebra. Moreover, since \( a \cdot b = o(a, b, \ldots, b) = o(a \cdot e, e, \ldots, e) b = ab \) by Condition (5a), it follows that \( G \) is isomorphic to the diagonal group of the above defined Menger algebra.

**Theorem 5.** If \( G \) is a finite group and \( n \) an odd integer, there exists an \( n \)-place grouplike Menger algebra whose diagonal group is isomorphic to \( G \); if \( n \) is even and the order of \( G \) odd, there exists an \( n \)-place grouplike Menger algebra whose diagonal group is isomorphic to \( G \).

For \( n = 2m+1 \), we define \( \varphi(a_1, \ldots, a_n) = a_1 a_2^{-1} \cdots a_{n-1} a_n \). For \( n = 2m \) and the order of \( G \) odd, we define \( \varphi(a_1, \ldots, a_{2m}) = a_1 a_2^{-1} \cdots a_{2m-1} a_{2m} \). The maps \( \varphi \) are easily seen to satisfy the conditions of Theorem 4.

If both \( n \) and the order of \( G \) is even there need not exist \( n \)-place grouplike Menger algebras whose diagonal group is isomorphic to \( G \). For example the cyclic groups of even order and the dihedral groups of order twice an odd integer cannot be the diagonal group of any \( 2 \)-place grouplike Menger algebra. In order to show this let \( G \) be one of the above groups and assume there is a map \( \varphi \) satisfying the conditions of Theorem 4. Hence \( \varphi(a) \neq \varphi(b) \) for \( a \neq b \) and \( \varphi(ab) \neq \varphi(a)b \neq e \).

Setting \( c = ab \), we obtain from the second condition that \( \varphi(c) = \varphi(ab) = \varphi(a) \varphi(b) = \varphi(c) \). Hence the set \( \{ \varphi(x) : x \in G \} \) and \( \{ \varphi(x) \varphi^{-1} : x \in G \} \) are both equal to \( G \) itself. Hence if \( G = \{ a_1, \ldots, a_{2k} \} \), the product \( a_1 a_{2k} \cdot a_1 a_{2k} \neq a_1 \) in some order. But since \( G = \{ \varphi(x) : x \in G \} = \{ x^2 : x \in G \} \) must therefore equal the product in some order of all the elements of \( G \) taken twice. If \( G \) is the cyclic group of order \( 2k \) it is generated by, say, \( a_1 \). The product of all the elements of \( G \) is therefore \( a_1^k \), while the product of all the elements of \( G \) taken twice is \( a_1^{2k} = e \). If \( G \) is the dihedral group of order \( 2k \), where \( k \) is odd, let \( a_1 \) and \( a_2 \) be its generators such that \( a_1^2 = e = a_2^2 \). We write every element of \( G \) in the form \( a_1^i a_2^j \) where \( 0 \leq i < k \) and \( 0 \leq j < k/2 \). Since \( k \) is odd the product of all the elements of \( G \) in any order is of the form \( a_1^i a_2^j \) for some integer \( i \); but the product of all elements of \( G \) taken twice and in any order is of the form \( a_1^i a_2^j \) for some integer \( j \). Hence neither of the above groups can be the diagonal group of a grouplike Menger algebra.

Since the only groups of order \( 2p \) where \( p \) is a prime are the cyclic and the dihedral group, it follows that there exist no \( 2 \)-place Menger algebras of order \( 2p \), where \( p \) is an odd prime.

There do exist \( 2 \)-place grouplike Menger algebras of every other order, however. First, let \( H_k \) denote the group of order \( 2^n \) in which every element has order \( 2 \). Hence \( H_k \) is generated by \( k \) elements, say, \( a_1, \ldots, a_k \). We define the sequence \( x_1, x_2, \ldots, x_{k-1} \) of elements from \( H_k \) in the following manner: \( x_i = a_i \) for \( i = 1, \ldots, k \) and \( x_i = x_{i-k} x_{i+1} \) for \( i > k \). The elements \( x_i \) are all distinct and unequal the identity. Moreover, for \( k > 1 \), the map \( \varphi(e) = e, \varphi(a_i) = x_{i+1}, \varphi(a_{i-1}) = x_i \) satisfies the conditions of Theorem 4. Hence there exists a \( 2 \)-place grouplike Menger algebra of order \( 2^k \) for every integer \( k > 1 \).

If \( A \) and \( B \) are \( n \)-place grouplike Menger algebras then the direct product \( A \cdot B \) is also an \( n \)-place grouplike Menger algebra with the operation
\[ o(a_1, b_1, \ldots, a_n, b_n) = o(a_1, \ldots, a_n, b_1, \ldots, b_n). \]
Hence, since there exists \( 2 \)-place grouplike Menger algebras of every odd order and of order \( 2^k \) for \( k > 1 \), there exist \( 2 \)-place grouplike Menger algebras of order \( 2^{2m} \) where \( m \) is odd and \( k > 1 \).

Let \( A \) be an \( n \)-place grouplike Menger algebra of order \( m \). We show there exists an \((n+2)\)-place grouplike Menger algebra of order \( m \). Let \( \varphi \) be the map of Theorem 4 associated with the \( n \)-place algebra. The map
\[ \varphi(a_1, a_2, \ldots, a_{n+1}) = a_1 a_2 \cdots a_{n+1} \]
is easily seen to satisfy the conditions
ditions of Theorem 4. And it follows by induction that there exist n-place grouplike Menger algebras of every order $2^k m$, where $k > 1$. Summarizing these results we obtain the following

**Theorem 6.** There exist n-place grouplike Menger algebras of every finite order if $n$ is odd; if $n$ is even there exist grouplike Menger algebras of every order not of the form $2p$ where $p$ is an odd prime. There exist no 2-place grouplike Menger algebras of order 2p.

The existence of even place grouplike Menger algebras of order 2p for the place number greater than 2 is as yet undecided.

A subset $B$ of a Menger algebra $A$ is called a subalgebra if it is closed with respect to the operation $\cdot$ on $A$.

**Theorem 7.** Any finite subalgebra of a grouplike Menger algebra is also grouplike.

Let $B$ be a finite subalgebra of a grouplike Menger algebra $A$. Then

$$o(b_1, ..., b_{n-1}, x, b_{n+1}, ..., b) \neq o(b_1, ..., b_{n-1}, y, b_{n+1}, ..., b)$$

for $x \neq y$ since $A$ is grouplike. Thus

$$o(b_1, ..., b_{n-1}, x, b_{n+1}, ..., b): x \in B)$$

equals $B$ for each $i$ and hence Condition (3') is satisfied and $B$ is grouplike.

**Theorem 8.** The order of any subalgebra $B$ of a finite grouplike Menger algebra $A$ divides the order of $A$.

It is easily shown that $B^*$ is a subgroup of $A^*$. Hence the order of $B^*$ divides the order of $A^*$. But $A^*$ and $A$, and, respectively, $B^*$ and $B$ have the same order.

Any grouplike Menger algebra of prime order therefore has no subalgebras. A composite order, however, does not guarantee the existence of subalgebras. For example the 2-place algebra $H_2$ of order 2k defined above has no subalgebras. The following 2-place grouplike Menger algebra of order 15 also has no subalgebras. Its diagonal group is the set of integers modulo 15 and we denote its elements by 0 through 14, and define the algebra by means of the map of Theorem 4:

$$
\begin{align*}
\varphi(0) &= 0, & \varphi(5) &= 11, & \varphi(10) &= 2, \\
\varphi(1) &= 3, & \varphi(6) &= 14, & \varphi(11) &= 8, \\
\varphi(2) &= 5, & \varphi(7) &= 1, & \varphi(12) &= 13, \\
\varphi(3) &= 7, & \varphi(8) &= 6, & \varphi(13) &= 12, \\
\varphi(4) &= 9, & \varphi(9) &= 4, & \varphi(14) &= 10.
\end{align*}
$$

References
