Grouplike Menger algebras

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Abstract. This paper generalizes the concept of group to include sets with n-ary operations. An n-place Menger algebra is a set A with an $(n+1)$-ary operation o satisfying the superassociative law: $o(o(a_0, a_1, \ldots, a_n), b_1, \ldots, b_n) = o(a_0, o(a_1, b_1, \ldots, b_n), \ldots$ $\ldots, o(a_n, b_1, \ldots, b_n))$. A is said to be *grouplike* if for any sequence $a_0, a_1, \ldots, a_n, b$ of elements from A there exist unique elements $x_0, x_1, \ldots, x_n$ of A such that $o(a_0, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n) = b$ for $i = 0, 1, \ldots, n$. It is proved that there exist n-place grouplike Menger algebras of every finite order if n is odd; if n is even there exist grouplike Menger algebras of every order not of the form $2p$ where p is an odd prime. There exist no 2-place grouplike Menger algebras of order $2p$. Alternate conditions for a Menger algebra to be grouplike are presented. The existence of grouplike subalgebras is also studied.

The study of functions of many variables gives rise to a natural extension of the concept of associativity for n-ary operations. An *n-place function f over a set S* is any mapping of $S^n = S \times \ldots \times S$ (the cross product of S with itself n times) into S. If $f_0, f_1, \ldots, f_n$ are n-place functions over S, the composite $f_0(f_1, \ldots, f_n)$ is defined in the usual way:

$$\big(f_0(f_1, \ldots, f_n)\big)(s_1, \ldots, s_n) = f_0\big(f_1(s_1, \ldots, s_n), \ldots, f_n(s_1, \ldots, s_n)\big)$$

for each n-tuple $(s_1, \ldots, s_n)$ from S^n. If $g_1, \ldots, g_n$ are also n-place functions over S then it is easily verified that

$$(1) \qquad \big(f_0(f_1, \ldots, f_n)\big)(g_1, \ldots, g_n) = f_0\big(f_1(g_1, \ldots, g_n), \ldots, f_n(g_1, \ldots, g_n)\big).$$

For the case $n = 1$, Equation (1) reduces to the associative law for transformations of a set into itself.

A set A of n-place functions is said to be an *algebra of functions* if the composite of any $n+1$ functions from A is also in A. Such algebras have been extensively studied in [1]-[12]. Generalizing the above concepts we define an *n-place Menger algebra* to be a set A with an $(n+1)$-ary operation o satisfying what has been called the *superassociative law*:

$$(2) \qquad o\big(o(a_0, a_1, \ldots, a_n), a'_1, \ldots, a'_n\big) = o\big(a_0, o(a_1, a'_1, \ldots, a'_n), \ldots, o(a_n, a'_1, \ldots, a'_n)\big),$$

for any elements a_i, a'_j, $i = 0, 1, \ldots, n$, $j = 1, \ldots, n$, from A. The case $n = 1$ in just the ordinary associative law and hence any 1-place Menger algebra is a semigroup.

It follows immediately from Equation (1) that any algebra of functions is a Menger algebra. Conversely, it was shown by Dikker (cf. [2]) that any Menger algebra is isomorphic to an algebra of functions over some set. Hence, in particular, any semigroup is isomorphic to a set of transformations over some set.

In this paper we study Menger algebras which satisfy certain solvability criteria, thus extending the group concept to include sets with n-ary operations.

An n-place Menger algebra is said to be *grouplike* if for any sequence $a_0, a_1, \ldots, a_n, b$ of elements from A there exist unique elements $x_0, x_1, \ldots, x_n$ in A such that

(3) $$o(a_0, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n) = b \quad \text{for} \quad i = 0, 1, \ldots, n.$$

Thus any 1-place grouplike Menger algebra is a group.

Some examples of grouplike Menger algebras are the following:

(i) The set of all n-place functions f_r over the reals defined by $f_r(x_1, \ldots, x_n) = (x_1 + \ldots + x_n + r)/n$ for each real number r.

(ii) The set I_{2n+1} of integers modulo $2k+1$ with the $(2n+1)$-ary operation $o(i_0, i_1, \ldots, i_{2n}) = i_0 - i_1 + \ldots + i_{2n-2} - i_{2n-1} + 2i_{2n} (\mathrm{mod}\, 2k+1)$.

(iii) The set I_{2k} of integers modulo $2k$ with the $2n$-ary operation $o(i_0, i_1, \ldots, i_{2n-1}) = i_0 + i_1 + i_2 - i_3 + \ldots + i_{2n-2} - i_{2n-1} \ (\mathrm{mod}\, 2k)$.

THEOREM 1. *An n-place Menger algebra A is grouplike if and only if A contains an element e (called the identity of A) such that*

(3a) $$o(e, x, \ldots, x) = x = o(x, e, \ldots, e) \quad \textit{for each } x \textit{ in } A;$$

and

(3b) *for any sequence $a_0, a_1, \ldots, a_n$ of elements of A, there exist unique elements $x_0, x_1, \ldots, x_n$ such that*

$$o(a_0, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n) = e \quad \textit{for} \quad i = 0, 1, \ldots, n.$$

First suppose A is grouplike. Let a be any element of A and let e be the element of A such that $o(e, a, \ldots, a) = a$. For any element x in A, there exists an element y such that $o(a, y, a, \ldots, a) = x$. Hence

$$x = o\big(o(e, a, \ldots, a), y, a, \ldots, a\big)$$
$$= o\big(e, o(a, y, a, \ldots, a), \ldots, o(a, y, a, \ldots, a)\big) = o(e, x, \ldots, x).$$

To prove the second equality of Condition (3a), choose elements $y_1, \ldots, y_n$ such that $o(x, y_1, \ldots, y_n) = e$. Then

$$o\big(o(e, x, \ldots, x), y_1, \ldots, y_n\big) = o\big(e, o(x, y_1, \ldots, y_n), \ldots, o(x, y_1, \ldots, y_n)\big)$$
$$= o(e, e, \ldots, e) = e.$$

Therefore by the unique solvability criterion x must equal $o(e, x, \ldots, x)$ and Condition (3a) follows. Clearly Condition (3b) is satisfied.

Conversely suppose A is a Menger algebra satisfying Conditions (3a) and (3b). Let $a_0, a_1, \ldots, a_n, b$ be any sequence of elements from A. We first prove there exists a unique element x_0 such that $o(x_0, a_1, \ldots, a_n) = b$. By Condition (3b) there exist elements $b_1, \ldots, b_n$ such that $o(b, b_1, \ldots, b_n) = e$. Let x_0 be the element such that $o\big(x_0, o(a_1, b_1, \ldots, b_n), \ldots, o(a_n, b_1, \ldots, b_n)\big) = e$. The left-hand side of this equality is just $o\big(o(x_0, a_1, \ldots, a_n), b_1, \ldots, b_n\big)$ and it follows from the uniqueness stipulation in Condition (3b) that $o(x_0, a_1, \ldots, a_n) = b$. Now suppose there exists another element x_0' such that $o(x_0', a_1, \ldots, a_n) = b$. Then also

$$o\big(x_0', o(a_1, b_1, \ldots, b_n), \ldots, o(a_n, b_1, \ldots, b_n)\big) = e.$$

Again, the uniqueness stipulation of Condition (3b) implies that $x_0 = x_0'$.

For $i = 1, \ldots, n$, let y_i be the element of A such that

$$o\big(a_0, o(a_1, b_1, \ldots, b_n), \ldots, y_i, \ldots, o(a_n, b_1, \ldots, b_n)\big) = e.$$

By the above, there exist elements x_i such that $o(x_i, b_1, \ldots, b_n) = y_i$, whence

$$e = o\big(a_0, o(a_1, b_1, \ldots, b_n), \ldots, o(x_i, b_1, \ldots, b_n), \ldots, o(a_n, b_1, \ldots, b_n)\big)$$
$$= o\big(o(a_0, a_1, \ldots, x_i, \ldots, a_n), b_1, \ldots, b_n\big).$$

And by the uniqueness stipulation in Condition (3b), $o(a_0, a_1, \ldots, x_i, \ldots, a_n) = b$. If also $o(a_0, a_1, \ldots, x_i', \ldots, a_n) = b$ for another element x_i', then $o(x_i, b_1, \ldots, b_n) = o(x_i', b_1, \ldots, b_n)$ and, from the above, it follows that $x_i = x_i'$.

We remark that the uniqueness stipulations of Conditions (3) and (3b) cannot be dropped. For example, the set of integers is a 2-place Menger algebra if we define

$$o(i, j, k) = \begin{cases} i + \frac{1}{2}(j+k), & \text{if } j+k \text{ is even}, \\ i + \frac{1}{2}(j+k-1), & \text{if } j+k \text{ is odd}. \end{cases}$$

Clearly 0 is the identity of this algebra and the equality $o(i, j, k) = m$ always has a solution if m and two of the three integers i, j, k are given. But $o(0, 0, 0) = o(0, 1, 0) = 0$. We prove however the following

THEOREM 2. *A finite Menger algebra A is grouplike if*

(3′) for any sequence $a_0, a_1, \ldots, a_n, b$ of elements from A, there exist elements $x_0, x_1, \ldots, x_n$ such that $b = o(a_0, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n)$; or if Condition (3a) and the following condition holds:

(3b′) for any sequence $a_0, a_1, \ldots, a_n$ of elements from A there exist elements $x_0, x_1, \ldots, x_n$ such that $e = o(a_0, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n)$.

Let $A = \{z_1, \ldots, z_k\}$. By Condition (3′) there exist elements c_{ij}, $i = 1, \ldots, k$, $j = 0, 1, \ldots, n$ such that $o(a_0, \ldots, a_{j-1}, c_{ij}, a_{j+1}, \ldots, a_n) = z_i$. Since $z_s \neq z_t$ for $s \neq t$, $c_{sj} \neq c_{tj}$ for $s \neq t$. Thus $A = \{c_{1j}, \ldots, c_{kj}\}$ for each j and it follows that the elements postulated in Condition (3′) are unique.

Now suppose A satisfies Conditions (3a) and (3b′). We first show that for any element a of A $o(a, x, \ldots, x) \neq o(a, y, \ldots, y)$ for $x \neq y$. For suppose that equality holds. By Condition (3b′) there exists an element a' such that $o(a', a, \ldots, a) = e$. But then

$$\begin{aligned} x = o(e, x, \ldots, x) &= o\big(o(a', a, \ldots, a), x, \ldots, x\big) \\ &= o\big(a', o(a, x, \ldots, x), \ldots, o(a, x, \ldots, x)\big) \\ &= o\big(a', o(a, y, \ldots, y), \ldots, o(a, y, \ldots, y)\big) \\ &= o\big(o(a', a, \ldots, a), y, \ldots, y\big) \\ &= o(e, y, \ldots, y) = y , \end{aligned}$$

whence $x = y$.

We now prove that Condition (3′) holds. Let $a_0, a_1, \ldots, a_n, b$ be elements from A. By Condition (3b′) there exists an element y_0 such that $o(y_0, a_1, \ldots, a_n) = e$. Hence

$$\begin{aligned} b = o(b, e, \ldots, e) &= o\big(b, o(y_0, a_1, \ldots, a_n), \ldots, o(y_0, a_1, \ldots, a_n)\big) \\ &= o\big(o(b, y_0, \ldots, y_0), a_1, \ldots, a_n\big) . \end{aligned}$$

Hence there exists an element x_0, namely $x_0 = o(b, y_0, \ldots, y_0)$ such that $o(x_0, a_1, \ldots, a_n) = b$.

For $i > 0$ let b' be the element of A such that $e = o(b', b, \ldots, b)$. By Condition (3b′) there exist elements x_i such that

$$\begin{aligned} e &= o\big(o(b', a_0, \ldots, a_0), a_1, \ldots, x_i, \ldots, a_n\big) \\ &= o\big(b', o(a_0, a_1, \ldots, x_i, \ldots, a_n), \ldots, o(a_0, a_1, \ldots, x_i, \ldots, a_n)\big) . \end{aligned}$$

On the grouplike Menger algebra A we define a binary operation $*$ by $a * b = o(a, b, \ldots, b)$. It is easily verified that, since the operation o is superassociative, $*$ is associative. By Condition (3a), $a * e = e * a = a$ and hence e is an identity of A with respect to $*$. Condition (3b) implies the existence of left inverses, whence A, with the operation $*$, is a group, which we will call the *diagonal group* of A and denote by $A*$. By direct calculation one easily shows that

$$(4) \qquad o(a * b, c_1, \ldots, c_n) = a * \big(o(b, c_1, \ldots, c_n)\big)$$

for any elements $a, b, c_1, \ldots, c_n$ from A.

THEOREM 3. *For any sequence $a_1, \ldots, a_n$ of elements from a grouplike Menger algebra A there exists an element a such that $o(x, a_1, \ldots, a_n) = x * a$ for every x in A.*

Let b be an element of A and $o(b, a_1, \ldots, a_n) = c$. Since A^* is a group there exists an element a such that $b * a = c$. We show $o(x, a_1, \ldots, a_n) = x * a$ for every x in A. Now $x = x * b^{-1} * b$, where b^{-1} is the inverse of b in A^*. Hence by Equation (4),

$$\begin{aligned} o(x, a_1, \ldots, a_n) &= o(x * b^{-1} * b, a_1, \ldots, a_n) \\ &= x * b^{-1} * \big(o(b, a_1, \ldots, a_n)\big) = x * b^{-1} * c = x * a . \end{aligned}$$

It follows from Theorem 3 that any grouplike Menger algebra A is completely determined by its diagonal group and the set

$$\big\{\big(x_1, \ldots, x_n, o(a, x_1, \ldots, x_n)\big)\big\}$$

for any particular element a. For if b is in A then $o(b, x_1, \ldots, x_n) = b * a^{-1} * \big(o(a, x_1, \ldots, x_n)\big)$.

That nonisomorphic grouplike Menger algebras may have isomorphic diagonal groups is seen by the following examples. On the set I_5 of integers modulo 5 we define three superassociative operations:

$$\begin{aligned} o(i, j, k) &= i - j + 2k \pmod 5 , \\ o'(i, j, k) &= i - 2j + 3k \pmod 5 , \\ o''(i, j, k) &= i - 3j + 4k \pmod 5 . \end{aligned}$$

That I_5 with each of these operations is a grouplike Menger algebra, no two of which are isomorphic, is easily verified. But the diagonal group of I_5 with respect to each of these operations is just the cyclic group of order 5.

Though there exist groups of every finite order, this result is not true for grouplike Menger algebras. For example, there exist no 2-place grouplike Menger algebras of order $2p$, where p is an odd prime. Before showing this, we prove the following

THEOREM 4. *In order that a finite group G be isomorphic to the diagonal group of an n-place Menger algebra, $n > 1$, it is necessary and sufficient that there exist a map φ of G^{n-1} onto G such that*

(5a) $\varphi(e, \ldots, e) = e$, *where e is the identity of G*,

(5b) $\varphi(a_1, \ldots, a_k, \ldots, a_{n-1}) \neq \varphi(a_1, \ldots, a'_k, \ldots, a_{n-1})$ *for* $a_k \neq a'_k$,

(5c) $\varphi(a_1 b, \ldots, a_{n-1} b) \neq \varphi(a_1, \ldots, a_{n-1}) b$ *for $b \neq e$ where the operation of G is denoted by juxtaposition.*

We first prove that the diagonal group of any grouplike Menger algebra A satisfies the conditions given in the theorem. Setting $\varphi(a_1, \ldots, a_{n-1}) = o(e, e, a_1, \ldots, a_{n-1})$ one immediately verifies Conditions (5a) and (5b). Moreover,

$$\begin{aligned} \varphi(a_1 * b, \ldots, a_{n-1} * b) &= o(e, e, a_1 * b, \ldots, a_{n-1} * b) \\ &= o(e, e * b^{-1}, a_1, \ldots, a_{n-1}) * b . \end{aligned}$$

Since

$$o(e, e*b^{-1}, a_1, \ldots, a_{n-1}) \neq o(e, e, a_1, \ldots, a_{n-1}) \quad \text{for} \quad b \neq e,$$

it follows that

$$\varphi(a_1*b, \ldots, a_{n-1}*b) \neq o(e, e, a_1, \ldots, a_{n-1})*b = \varphi(a_1, \ldots, a_{n-1})*b .$$

Conversely suppose the finite group G satisfies the conditions of the theorem. We define the $(n+1)$-ary operation as follows:

$$o(a_0, a_1, \ldots, a_n) = a_0\varphi(a_2a_1^{-1}, \ldots, a_na_1^{-1})a_1 .$$

It is easily verifed by direct calculation that the operation o is superassociative. Now let b be any element of A. Then the equality

$$b = o(x_0, a_1, \ldots, a_n) = x_0\varphi(a_2a_1^{-1}, \ldots, a_2a_1^{-1})a_1$$

is satisfied for $x_0 = ba_1^{-1}\big(\varphi(a_2a_1^{-1}, \ldots, a_2a_1^{-1})\big)^{-1}$. Because G is finite, in order to prove there exist unique elements $x_1, \ldots, x_n$ from G such that $o(a_0, \ldots, x_i, \ldots, a_n) = b$ it suffices to show that

$$\varphi(a_2x^{-1}, \ldots, a_nx^{-1})x \neq \varphi(a_2y^{-1}, \ldots, a_ny^{-1})y$$

and

$$\varphi(a_2a_1^{-1}, \ldots, xa_1^{-1}, \ldots, a_na_1^{-1}) \neq \varphi(a_2a_1^{-1}, \ldots, ya_1^{-1}, \ldots, a_na_1^{-1})$$

for $x \neq y$. The second inequality follows immediately from Condition (5b). In order to verify the first inequality we suppose

$$\varphi(a_2x^{-1}, \ldots, a_nx^{-1})x = \varphi(a_2y^{-1}, \ldots, a_ny^{-1})y .$$

Setting $c_i = a_ix^{-1}$ we obtain

$$\varphi(c_2, \ldots, c_n)xy^{-1} = \varphi(c_2xy^{-1}, \ldots, c_nxy^{-1})$$

and it follows from Condition (5c) that $xy^{-1} = e$, that is $x = y$. Hence G with the operation defined above is a grouplike Menger algebra. Moreover, since $a*b = o(a, b, \ldots, b) = a\varphi(e, \ldots, e)b = ab$ by Condition (5a), it follows that G is isomorphic to the diagonal group of the above defined Menger algebra.

THEOREM 5. *If G is a finite group and n an odd integer, there exists an n-place grouplike Menger algebra whose diagonal group is isomorphic to G; if n is even and the order of G odd, there exists an n-place grouplike Menger algebra whose diagonal group is isomorphic to G.*

For $n = 2m+1$, we define $\varphi(a_1, \ldots, a_{2m}) = a_1a_2^{-1}\ldots a_{2m-1}a_{2m}^{-1}$. For $n = 2m$ and the order of G odd, we define $\varphi(a_1, \ldots, a_{2m-1}) = a_1a_2^{-1}\ldots$ $\ldots a_{2m-3}a_{2m-2}^{-1}a_{2m-1}^2$. The maps φ are easily seen to satisfy the conditions of Theorem 4.

If both n and the order of G is even there need not exist n-place grouplike Menger algebras whose diagonal group is isomorphic to G.

For example the cyclic groups of even order and the dihedral groups of order twice an odd integer cannot be the diagonal group of any 2-place grouplike Menger algebra. In order to show this we let G be one of the above groups and assume there is a map φ satisfying the conditions of Theorem 4. Hence $\varphi(a) \neq \varphi(b)$ for $a \neq b$ and $\varphi(ab) \neq \varphi(a)b$ for $b \neq e$. Setting $c = ab$, we obtain from the second condition that $\varphi(c)c^{-1} \neq \varphi(a)a^{-1}$ for $a \neq c$. Hence the sets $\{\varphi(x): x \text{ in } G\}$ and $\{\varphi(x)x^{-1}: x \text{ in } G\}$ are both equal to G itself. Hence if $G = \{a_1, \ldots, a_{2k}\}$, the product $a_1a_2\ldots a_{2k} = b$ equals the product of $\varphi(a_1)a_1^{-1}, \ldots, \varphi(a_{2k})a_{2k}^{-1}$ in some order. But since $G = \{\varphi(x): x \text{ in } G\} = \{x^{-1}: x \text{ in } G\}$ b must therefore equal the product in some order of all the elements of G taken twice. If G is the cyclic group of order $2k$ let it be generated by, say, a_1. The product of all the elements of G is therefore a_1^k, while the product of all the elements of G taken twice is $a_1^{2k} = e \neq a_1^k$. If G is the dihedral group of order $2k$, where k is odd, let a_1 and a_2 be its generators such that $a_1^k = a_2^2 = e$. We write every element of G in the form $a_1^sa_2^t$ where $0 \leqslant s < k$ and $0 \leqslant t \leqslant 1$. Since k is odd the product of all the elements of G in any order is of the form $a_1^ia_2$ for some integer i; but the product of all elements of G taken twice and in any order is of the form a_1^j for some integer j. Hence neither of the above groups can be the diagonal groups of a grouplike Menger algebra.

Since the only groups of order $2p$ where p is a prime are the cyclic and the dihedral group, it follows that there exist no 2-place Menger algebras of order $2p$, where p is an odd prime.

There do exist 2-place grouplike Menger algebras of every other order, however. First, let H_k denote the group of order 2^k in which every element has order 2. Hence H_k is generated by k elements, say, $a_1, \ldots, a_k$. We define the sequence $x_1, x_2, \ldots, x_{2^k-1}$ of elements from H_k in the following manner: $x_i = a_i$ for $i = 1, \ldots, k$ and $x_i = x_{i-k}x_{i-k+1}$ for $i > k$. The elements x_i are all distinct and unequal the identity. Moreover, for $k > 1$, the map $\varphi(e) = e$, $\varphi(x_i) = x_{i+1}$, $\varphi(x_{2^k-1}) = x_1$ satisfies the conditions of Theorem 4. Hence there exists a 2-place grouplike Menger algebra of order 2^k for every integer $k > 1$.

If A and B are n-place grouplike Menger algebras then the direct product AB is also an n-place grouplike Menger algebra with the operation

$$o\big((a_0, b_0), \ldots, (a_n, b_n)\big) = \big(o(a_0, \ldots, a_n), o(b_0, \ldots, b_n)\big) .$$

Hence, since there exists 2-place grouplike Menger algebras of every odd order and of order 2^k for $k > 1$, there exist 2-place grouplike Menger algebras of order 2^km where m is odd and $k > 1$.

Let A be an n-place grouplike Menger algebra of order m. We show there exists an $(n+2)$-place grouplike Menger algebra of order m. Let φ be the map of Theorem 4 associated with the n-place algebra. The map $\varphi'(a_1, a_2, \ldots, a_{n+1}) = a_1a_2^{-1}\varphi(a_3, \ldots, a_{n+1})$ is easily seen to satisfy the conditions of Theorem 4. And it follows by induction that there exist n-place grouplike Menger algebras of every order $2^k m$, where $k > 1$. Summarizing these results we obtain the following

THEOREM 6. *There exist n-place grouplike Menger algebras of every finite order if n is odd; if n is even there exist grouplike Menger algebras of every order not of the form $2p$ where p is an odd prime. There exist no 2-place grouplike Menger algebras of order $2p$.*

The existence of even place grouplike Menger algebras of order $2p$ for the place number greater than 2 is as yet undecided.

A subset B of a Menger algebra A is called a *subalgebra* if it is closed with respect to the operation on A.

THEOREM 7. *Any finite subalgebra of a grouplike Menger algebra is also grouplike.*

Let B be a finite subalgebra of a grouplike Menger algebra A. Then

$$o(b_0, \ldots, b_{i-1}, x, b_{i+1}, \ldots, b) \neq o(b_0, \ldots, b_{i-1}, y, b_{i+1}, \ldots, b_n) \quad \text{for} \quad x \neq y$$

since A is grouplike. Thus $\{o(b_0, \ldots, b_{i-1}, x, b_{i+1}, \ldots, b_n): x \text{ in } B\}$ equals B for each i and hence Condition (3′) is satisfied and B is grouplike.

THEOREM 8. *The order of any subalgebra B of a finite grouplike Menger algebra A divides the order of A.*

It is easily shown that B^* is a subgroup of A^*. Hence the order of B^* divides the order of A^*. But A^* and A, and, respectively, B^* and B have the same order.

Any grouplike Menger algebra of prime order therefore has no subalgebras. A composite order, however, does not guarantee the existence of subalgebras. For example the 2-place algebra H_k of order 2^k defined above has no subalgebras. The following 2-place grouplike Menger algebra of order 15 also has no subalgebras. Its diagonal group is the set of integers modulo 15 and we denote its elements by 0 through 14, and define the algebra by means of the map of Theorem 4:

$$\begin{array}{lll}
\varphi(0) = 0, & \varphi(5) = 11, & \varphi(10) = 2, \\
\varphi(1) = 3, & \varphi(6) = 14, & \varphi(11) = 8, \\
\varphi(2) = 5, & \varphi(7) = 1, & \varphi(12) = 13, \\
\varphi(3) = 7, & \varphi(8) = 6, & \varphi(13) = 12, \\
\varphi(4) = 9, & \varphi(9) = 4, & \varphi(14) = 10.
\end{array}$$

References

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