# Some characterizations of smooth continua

by

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**Abstract.** A continuum $X$ is smooth if there is a point $p$ in $X$ such that (1) given any point $x$ in $X$, there is a unique subcontinuum $px$ which is irreducible between $p$ and $x$, and (2) for each sequence of points $a_n \in X$ which is convergent to the point $a$, the sequence of irreducible continua $pa_n$ is convergent to the continuum $pa$. The paper contains various characterizations of smooth continua, in particular by the structure of some decomposition space, by the weak cutpoint order, by the $T$-relation and some mapping properties. These characterizations have the form of necessary and sufficient conditions under which a continuum has property (2) if it is assumed to have property (1).

**§ 1. Introduction.** All the spaces considered in this paper are metric. A continuum is said to be *unicoherent* if for any decomposition onto two subcontinua the intersection of those subcontinua is connected. A continuum is said to be *hereditarily unicoherent* if each of its subcontinua is unicoherent. It is known that a continuum $X$ is hereditarily unicoherent if and only if, given any set $Z \subset X$, there exists a unique subcontinuum $I(Z)$ of $X$ which is irreducible with respect to containing $Z$ (see [2], T1, p. 189). It is also known ([14], Theorem 1.1, p. 179) that

(1.1) A continuum $X$ is hereditarily unicoherent if and only if for any two points $x, y \in X$ there exists a unique subcontinuum $I(x, y)$ which is irreducible between $x$ and $y$.

A *dendroid* is a hereditarily unicoherent, arcwise connected continuum. A point $p$ of a dendroid $X$ is called a *ramification point* (in the classical sense) if it is the common end-point of three (or more) arcs in $X$ whose only common point is $p$. A dendroid having exactly one ramification point is called a *fan*. A dendroid $X$ is said to be *smooth* if there exists a point $p \in X$ such that for every sequence of points $a_n \in X$ convergent to the point $a$ the sequence of arcs $pa_n$ is convergent (in the topological sense) to the arc $pa$.

The notion of smoothness was first introduced and investigated in [3] and [6] for fans, and next it was generalized to dendroids in [4], from another point of view it was investigated in [10]. Some characterizations of smooth dendroids are also contained in [15]. Recently this concept has been generalized to continua hereditarily unicoherent at a point.

A continuum $X$ is said to be *hereditarily unicoherent at a point* $p \in X$ if the intersection of any two subcontinua, each of which contains $p$, is connected (see [7], p. 52).

Similarly to (1.1) it is proved ([7], Theorem 1.3, p. 52) that

(1.2) A continuum $X$ is hereditarily unicoherent at a point $p$ if and only if, given any point $x \in X$, there exists a unique subcontinuum which is irreducible between $p$ and $x$.

This subcontinuum will be denoted by $px$.

A continuum $X$ is said to be *smooth at a point* $p$ if $X$ is hereditarily unicoherent at $p$ and, for each sequence of points $a_n \in X$ which is convergent to the point $a$, the sequence of irreducible continua $pa_n$ is convergent (in the topological sense) to the continuum $pa$. A continuum $X$ is said to be *smooth* if there exists a point $p \in X$ such that $X$ is smooth at $p$.

The aim of the paper is to investigate smooth continua by finding properties which characterize them. Most of those properties are patterned after those characterizing smooth dendroids (see [4]).

Section 2 plays an auxiliary role and contains some general facts about continua which are hereditarily unicoherent at a point and about smooth continua. In particular, Theorem (2.6) gives a characterization of dendroids. In Section 3 properties equivalent to smoothness in continua are formulated and several theorems on characterizations of smooth continua are proved. The section is divided into five parts. Part A contains a characterization of smooth continua which is given by the structure of the decomposition space of the canonical decomposition introduced by Gordh in [7]. The notion of the weak cutpoint order is used to give another characterization in part B, which is needed to prove the next theorems. Part C contains some characterizations of the smoothness of continua based upon the $T$-relation of non-aposyndeticity. The characterizations in parts B and C are closely related to the results obtained by Gordh in [9]. In part D we study various kinds of mappings and we obtain several characterizations of smooth continua in this way. In particular, the concepts of a radially convex mapping and of an order-preserving multivalued one are introduced and applied to obtain characterizations of smooth continua. Finally, two other properties equivalent to smoothness are given in part E. They concern the inner structure of smooth continua.

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**§ 2. The smoothness of continua.** The definition of the hereditary unicoherence at a point implies immediately (see [7], Theorem 1.1) that



(2.1) A continuum is hereditarily unicoherent if and only if it is hereditarily unicoherent at every point.

Let a continuum $X$ be hereditarily unicoherent at a point $p \in X$. We define the following equivalence relation $\varrho_p$ on $X$:

(a) $$x \,\varrho_p\, y \quad \text{if and only if} \quad px = py\,.$$

The equivalence class of a point $x \in X$ with respect to the relation $\varrho_p$ will be denoted by $[x]$. Notice that

(2.2) If $z \in px$, then $[z] \subset px$.

Indeed, $z \in px$ implies $pz \subset px$. If $y \in [z]$, i.e. $py = pz$, then $y \in pz$; thus $y \in px$.

The set of all points of a topological space $X$ which can be joined with the point $p$ by a closed, connected and proper subset of the space $X$ is called the composant of the point $p$. It is well known (see [11], p. 239 and [12], Theorem 3, § 48, p. 210) that

(2.3) If $C$ is a composant of a continuum $X$, then the set $X \setminus C$ is connected.

(2.4) Lemma. *Let a continuum $X$ be hereditarily unicoherent at a point $p$. For every $x \in X$ the equivalence classes $[x]$ are connected.*

Proof. Let $x \in X$. By (2.3) it remains to show that the composant $C$ of the point $p$ in the continuum $px$ is equal to $px \setminus [x]$. Indeed, if $z \in px \setminus [x]$, then $px \setminus [x]$ by (2.2) contains $pz$; thus $px \setminus [x] \subset C$. If $z \in [x]$, then, by definition, we have $pz = px$; thus $z \in px \setminus C$ and hence $C \subset px \setminus [x]$.

Let $\varphi\colon X \to X/\varrho_p$ denote the canonical map from a continuum $X$ onto the quotient space $X/\varrho_p$. Lemma (2.4) and Theorem 9 in [12], § 46, I, p. 131 imply the following

(2.5) Corollary. *If a continuum $X$ is hereditarily unicoherent at a point $p$ and if the canonical map $\varphi\colon X \to X/\varrho_p$ is closed, then $\varrho$ is monotone.*

We shall now prove (cf. [9], Theorems 2.2 and 2.3)

(2.6) Theorem. *A continuum $X$ is a dendroid if and only if $X$ is hereditarily unicoherent at some point $p$ and for every $x \in X$ the continuum $px$ is an arc.*

Proof. If $X$ is a dendroid, then the condition follows immediately from (2.1) and from the definition of a dendroid. To prove the other implication, let $x$ and $y$ be points of a continuum $X$ which satisfies the condition assumed. Since the continua $px$ and $py$ are arcs by hypothesis, their union $px \cup py$ contains an arc $xy$. Thus $X$ is arcwise connected. We shall prove that $X$ is hereditarily unicoherent, i.e., that $xy$ is the only continuum irreducible between $x$ and $y$ (see (1.1)). For this purpose let $I(x, y)$ be an arbitrary continuum irreducible between $x$ and $y$. Consider two cases.

1. either $y \in px$ or $x \in py$. Assume $y \in px$ (if $x \in py$, the proof is the same). Then $py \subset px$ and $xy \subset px$; hence $py \cap xy = \{y\}$. Since the union $py \cup I(x, y)$ is a continuum containing points $p$ and $x$, we have $px \subset py \cup I(x, y)$ and hence $xy \subset py \cup I(x, y)$. The arc $xy$ has only one point $y$ in common with the arc $py$; therefore $xy \setminus \{y\} \subset I(x, y)$, and hence taking closure we have $xy \subset I(x, y)$; thus $xy = I(x, y)$ by the irreducibility of $I(x, y)$.

2. $x \in X \setminus py$ and $y \in X \setminus px$. Since the continuum $X$ is hereditarily unicoherent at the point $p$, the intersection $px \cap py$ is a continuum; therefore it is the arc $pz$. Take arcs $xz \subset px$ and $yz \subset py$. Then the union $xz \cup yz$ is equal to the arc $xy$ and we have $yz \cap px = \{z\}$ and $xz \cap py = \{z\}$. As in case 1 we conclude that $px \subset py \cup I(x, y)$ and $py \subset px \cup I(x, y)$; hence $xz \subset I(x, y)$ and $yz \subset I(x, y)$. Therefore $xz \cup yz \subset I(x, y)$, i.e., $xy \subset I(x, y)$; thus as before $xy = I(x, y)$, which completes the proof.

For an arbitrary continuum $X$ the set $P(X) = \{p \in X\colon X \text{ is smooth at } p\}$ is called the *initial set* of $X$. The definition of a smooth continuum given in § 1 can now be formulated as follows: a continuum $X$ is called smooth if $P(X) \neq \emptyset$.

Since dendroids, in particular fans, are hereditarily unicoherent (thus hereditarily unicoherent at every point by (2.1)), the definitions of smoothness formulated in [3] and [4] for those continua do agree with the above definition. A spiral winding up the circle is an example of a smooth continuum which is neither hereditarily unicoherent nor arcwise connected.

It follows from Theorem (2.6) that

(2.7) If a continuum is smooth and arcwise connected, then it is a smooth dendroid.

(2.8) If a continuum is smooth and $p$ is an initial point of it, then any subcontinuum of it containing the point $p$ is smooth at $p$.

In fact, let a continuum $X$ be smooth at a point $p$, and let $A$ be a continuum such that $p \in A \subset X$. If $x \in A$, then $px \subset A$ by the hereditary unicoherence of $X$ at $p$. Let $x_n \to x$ and let $x_n \in A$. Then $px_n$ and $px$ are contained in $A$ and we have $\operatorname{Lim}_{n\to\infty} px_n = px$ by the smoothness of $X$; thus $A$ is smooth at $p$.

More on smooth continua can be found in [7], [8] and [9]. In particular, the following theorem is an immediate consequence of Theorem 5.2 in [7], p. 58 and of the fact that the decomposition space $X/\varrho_p$ is metric separable and that the decomposition of $X$ into equivalence classes of the relation $\varrho_p$ is upper semi-continuous (see [17], Theorem (2.2), p. 123).

(2.9) Let a continuum $X$ be smooth at a point $p$, let $\varrho_p$ be the equivalence relation defined by (a), and let $\varphi\colon X \to X/\varrho_p$ be the canonical map. Then the decomposition $\mathfrak{D} = \{\varphi^{-1}(t)\colon t \in X/\varrho_p\}$ has the following properties:

(i) $\mathfrak{D}$ is upper semi-continuous,
(ii) the elements of $\mathfrak{D}$ are continua,
(iii) the decomposition space of $\mathfrak{D}$ is arcwise connected, and
(iv) if $\mathcal{E}$ is a decomposition satisfying (i), (ii) and (iii), then $\mathfrak{D} \leqslant \mathcal{E}$ (i.e., $\mathfrak{D}$ refines $\mathcal{E}$).

Moreover, the decomposition space $X/\varrho_p$ of $\mathfrak{D}$ is a smooth dendroid, $\varphi(p)$ is an initial point of $X/\varrho_p$ and each element of $\mathfrak{D}$ has a void interior.

The following corollary can be drawn from (2.9).

(2.10) If a continuum $X$ is smooth at a point $p$, then the canonical map $\varphi\colon X \to X/\varrho_p$ is closed.

Further,

(2.11) If a continuum $X$ is smooth at a point $p$, and $I(x, y)$ is an irreducible continuum between $x$ and $y$ in $X$, then $I(x, y) \subset px \cup py$.

Indeed, if $N$ is a subcontinuum of $X$ such that $p \in N$, then for each subcontinuum $H$ of $X$ the intersection $N \cap H$ is a continuum (see [7], Theorem 3.3, p. 55). Thus if we take $N = px \cup py$ and $H = I(x, y)$, we see that $N \cap H$ is a continuum contained in $I(x, y)$. Moreover, $N \cap H$ contains both $x$ and $y$; thus by the irreducibility of $I(x, y)$ we have $I(x, y) = N \cap H$, since $N \cap H \subset px \cup py$, and hence $I(x, y) \subset px \cup py$.

## § 3. Characterizations of smooth continua.

In this section we study various properties of continua hereditarily unicoherent at a point which are equivalent to the smoothness of those continua.

A. The quotient space. It follows from Theorem (2.9) that if a continuum $X$ is hereditarily unicoherent at a point $p$ and if $\varrho_p$ is the equivalence relation on $X$ defined by (a) in § 2, then the smoothness of $X$ in $p$ implies that the quotient space $X/\varrho_p$ is a smooth dendroid having $\varphi(p)$ as an initial point. It can be proved that this condition is not only necessary but also sufficient for $X$ to be smooth at $p$. Namely we have the following

(3.1) THEOREM. *Let a continuum $X$ be hereditarily unicoherent at a point $p$. $X$ is smooth at $p$ if and only if the quotient space $X/\varrho_p$ is a smooth dendroid and $\varphi(p) \in P(X/\varrho_p)$.*

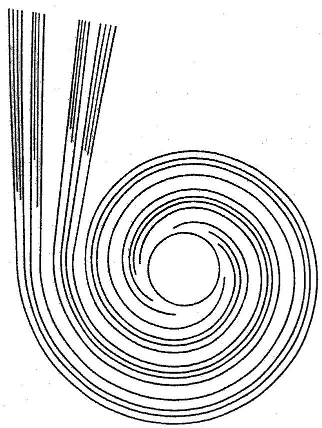
Proof. If $X$ is smooth at $p$, then $X/\varrho_p$ is a smooth dendroid and $\varphi(p) \in P(X/\varrho_p)$ by Theorem (2.9).

Conversely, suppose that $X/\varrho_p$ is a smooth dendroid at the point $\varphi(p)$. Then the canonical map $\varphi$ is closed and hence monotone by Corollary (2.5); therefore for each $x \in X$ we have $\varphi(px) = \varphi(p)\varphi(x)$ (see [7], Theorem 4.1, (ii), p. 56). Suppose that $X$ is not smooth at the point $p$. Then there exist sequences $\{a_m\}$ and $\{b_m\}$ such that (i) $\{a_m\}$ converges to $a$, (ii) $\{b_m\}$ converges to $b$, (iii) $b_m \in pa_m$ for each $m$ and (iv) $b \in X \setminus pa$ (see [7], Theorem 2.3, p. 53). Since the map $\varphi$ is continuous, properties (i) and (ii) imply $\lim_{m\to\infty} \varphi(a_m) = \varphi(a)$ and $\lim_{m\to\infty} \varphi(b_m) = \varphi(b)$. Property (iii) gives $\varphi(b_m) \in \varphi(pa_m) = \varphi(p)\varphi(a_m)$. This leads to $\varphi(b) \in \varphi(p)\varphi(a) = \varphi(pa)$ because the dendroid $X/\varrho_p$ is smooth at the point $\varphi(p)$. Thus $\varphi^{-1}\varphi(b) \subset \varphi^{-1}\varphi(pa)$, and hence $[b] \subset pa$ by (2.2), and $b \in [b]$ implies $b \in pa$. This is a contradiction of (iv).

(3.2) Example. Joining the points of the Cantor set $C$ which lay in a natural way in the unit segment $[0, 1]$ with the point $t = (\frac{1}{2}, \frac{1}{2})$ we obtain the Cantor fan. The Cantor fan is a quotient space $X/\varrho_p$ of a continuum $X$ for which the canonical map $\varphi\colon X \to X/\varrho_p$ is defined by the properties $\varphi(p) = (0, 0) \in C$ and

$$\varphi^{-1}(y) = \begin{cases} \text{a circle}, & \text{if} \quad y = t, \\ \text{one-point set}, & \text{if} \quad y \neq t \end{cases}$$

(see the figure).



Obviously the continuum $X$ is not smooth, although $X/\varrho_p$ is a smooth dendroid (at the point $t$, but not at $\varphi(p)$). Hence we see that the assumption $\varphi(p) \in P(X/\varrho_p)$ in Theorem (3.1) is essential.

B. The weak cutpoint order. A relation $\leqslant$ on a set $X$ is said to be a *quasi-order* if it is reflexive and transitive. A quasi-order $\leqslant$ on a topological space $X$ is said to be *monotone* if the set $L(x) = \{y \in X\colon y \leqslant x\}$ is connected for each point $x \in X$. A quasi-order $\leqslant$ on a topological space $X$ is called *closed* if the set $\{(x, y) \in X \times X\colon x \leqslant y\}$ is closed in $X \times X$.



If a continuum $X$ is hereditarily unicoherent at a point $p$, then the quasi-order $\leqslant_p$ on $X$ defined by

$$x \leqslant_p y \quad \text{if and only if} \quad x \in py$$

is said to be a *weak cutpoint order* with respect to $p$ (see [9], Section 1).

(3.3) The weak cutpoint order $\leqslant_p$ is monotone. Moreover, the point $p$ is the unique minimal element in $X$ with respect to $\leqslant_p$.

Indeed, since $L(x) = \{y \in X\colon y \leqslant_p x\} = \{y \in X\colon y \in px\} = px$, $L(x)$ is connected for each $x \in X$, i.e., $\leqslant_p$ is monotone. Let $q$ be a minimal element in $X$ with respect to $\leqslant_p$. It follows by the definition of a minimal element that if $x \leqslant_p q$, then $q \leqslant_p x$. Take $p = x$. Since $p \in pq$, we have $p \leqslant_p q$, which implies $q \leqslant_p p$ and hence $q = p$.

The following theorem, proved in [9] as Theorem 3.1, gives a characterization of smooth continua in terms of the weak cutpoint order.

(3.4) Theorem (Gordh). *Let a continuum $X$ be hereditarily unicoherent at a point $p$. $X$ is smooth at $p$ if and only if the weak cutpoint order $\leqslant_p$ is closed.*

C. The $T$-relation. Let $X$ be a continuum. For points $x$ and $y$ of $X$ define $x\,T\,y$ if and only if every subcontinuum of $X$ which contains $y$ in its interior also contains $x$. Put $T_x = \{y \in X\colon x\,T\,y\}$. It is known ([5], p. 115) that

(3.5) The relation $T$ is closed, and the set $T_x$ is a continuum for every $x \in X$.

The following is proved in [4], Theorem 4, p. 301.

(3.6) Let a monotone, closed quasi-order $\leqslant$ on a continuum $X$ be given such that for each two minimal elements $p$ and $q$ of $X$ (with respect to $\leqslant$) the conditions $p \leqslant q$ and $q \leqslant p$ hold. Then $x \leqslant y$ for each $x \in X$ and each $y \in T_x$.

We have the following characterization of smooth continua.

(3.7) Theorem. *Let a continuum $X$ be hereditarily unicoherent at a point $p$. $X$ is smooth at $p$ if and only if $x \in py$ for each $x \in X$ and each $y \in T_x$.*

Proof. If $X$ is smooth at $p$, then $\leqslant_p$ is a closed, monotone quasi-order with the unique minimal element by (3.3) and (3.4). It follows by (3.6) that $x \in py$ for each $x \in X$ and each $y \in T_x$. Conversely, suppose that $x \in py$ for each $x \in X$ and each $y \in T_x$. To show that the continuum $X$ is smooth at the point $p$, it suffices by Theorem (3.4) to prove that the weak cutpoint order $\leqslant_p$ is closed, i.e., that the complement of the set $\{(x, y)\colon x \leqslant_p y\}$ is open in $X \times X$. Let $x \not\leqslant_p y$, i.e., let $x \in X \setminus py$. Therefore, by hypothesis, $y \in X \setminus T_x$. Hence there is a continuum $K$ containing $y$ in its interior which fails to contain $x$. Then $K' = K \cup py$ is also a continuum containing $y$ in its interior which fails to contain $x$.

Hence $V = (X \setminus K') \times \operatorname{Int} K$ is an open set in $X \times X$ which contains $(x, y)$. Let $(z, w) \in V$. Then $w \in \operatorname{Int} K \subset K \subset K'$ and $p \in K'$; thus $pw \subset K'$. Since $z \in X \setminus K'$, we have $z \in X \setminus pw$, i.e., $z \not\leqslant_p w$, which completes the proof.

As a consequence of Theorem (3.7) we have the following corollary (see [9], Corollary 3.5).

(3.8) Corollary. *Let a continuum $X$ be hereditarily unicoherent at a point $p$. $X$ is smooth at $p$ if and only if*

(b) $$px \cap T_x \subset [x] \quad \text{for each } x \in X .$$

Proof. Let $X$ be smooth at $p$. If $y \in px \cap T_x$, then in particular $y \in T_x$, whence $x \in py$ by Theorem (3.7). Since $y \in px$, we conclude that $py = px$ and hence $y \in [x]$.

Conversely, suppose that condition (b) holds. By Theorem (3.7) it suffices to prove that if $y \in T_x$, then $x \in py$. Let $y \in T_x$. Hence $T_x \cap py \neq \emptyset$, and thus the union $T_x \cup py$ is a continuum which contains the points $p$ and $x$ by (3.5). This leads to $px \subset T_x \cup py$. Therefore $px \subset (T_x \cap px) \cup (py \cap px)$. The inverse inclusion being obvious, we have the equality $px = (T_x \cap px) \cup (py \cap px)$. Since $px$ is connected and the sets $T_x \cap px$ and $py \cap px$ are closed, there is a point $z \in X$ such that $z \in T_x \cap py \cap px$. Hence according to (b) we have $z \in [x]$ and since $z \in py$, we have $[x] \subset py$ by (2.2), i.e., $x \in py$, which completes the proof.

(3.9) Corollary. *Let a continuum $X$ be smooth at a point $p$. If $px \cap T_y \neq \emptyset$, then $y \in px$.*

In fact, if $z \in px$, then $pz \subset px$. If $z \in T_y$, then $y \in pz$ by (3.7). Hence $z \in px \cap T_y$ implies $y \in pz \subset px$.

Theorem (3.7) and Corollary (3.8) imply

(3.10) Let a continuum $X$ be smooth at a point $p$. If $T_x = T_y$, then $x \varrho_p y$.

(3.11) Let a continuum $X$ be hereditarily unicoherent at a point $p$. If $T_x \subset [x]$ for each point $x \in X$, then $X$ is smooth at $p$.

(3.12) Theorem. *If a continuum $X$ is smooth at a point $p$, then*

(c) *for any two points $x, y \in X$ and for each continuum $I(x, y)$ irreducible between $x$ and $y$ we have either $I(x, y) \cap T_x \subset [x]$ or $I(x, y) \cap T_y \subset [y]$.*

Proof. Let $x$ and $y$ be points of $X$ and let $I(x, y)$ be an arbitrary continuum irreducible between $x$ and $y$. It follows from (2.11) that $I(x, y) \subset px \cup py$. Consider two cases.

1. If $y \in px$, then $py \subset px$; thus $px \cup py = px$, and therefore $I(x, y) \subset px$. Hence $I(x, y) \cap T_x \subset px \cap T_x \subset [x]$ by (3.8).

2. If $y \in X \setminus px$, then $px \cap T_y = \emptyset$ by (3.9). Hence $I(x, y) \cap T_y \subset (px \cup py) \cap T_y = (px \cap T_y) \cup (py \cap T_y) = py \cap T_y \subset [y]$ by (3.8).



Condition (c) in Theorem (3.12) does not characterize smooth continua. The continuum $X$ need not be smooth even if this condition is satisfied for each point $p$ at which the continuum is hereditarily unicoherent, and, moreover, if the inclusions in (c) are replaced by equalities. This can be seen from Example (3.2). Namely the continuum $X$ defined there is hereditarily unicoherent at each point $q$ which does not belong to the circle contained in $X$. If we admit $[x]$ to be the equivalence class of the point $x$ with respect to the equivalence relation $\varrho_q$, condition (c) holds for each $q$, but $X$ is not smooth at any of its points. However, if the continuum $X$ is a dendroid, then condition (c) implies the smoothness of $X$ (see [4], Theorem 6, p. 302).

D. Mapping characterizations. We recall that a metric $d$ on a dendroid $X$ is said to be *radially convex* with respect to a point $p \in X$ provided that $x \in py$ and $x \neq y$ implies $d(p, x) < d(p, y)$. This is proved in [4], Theorem 10, p. 310 (see also [1], p. 229) that

(3.13) A dendroid $X$ is smooth with an initial point $p$ if and only if $X$ has a metric which is radially convex with respect to $p$.

Let a continuum $X$ be hereditarily unicoherent at a point $p$ and let $\varrho_p$ be the equivalence relation on $X$ defined by (a) in § 2. A continuous map $f_p\colon X \to [0, \infty)$ is said to be *radially convex* with respect to $p$ if the following conditions are satisfied:

(i) if $x \in py \setminus [y]$, then $f_p(x) < f_p(y)$,

(ii) if $y \in [x]$, then $f_p(x) = f_p(y)$.

(3.14) Theorem. *Let a continuum $X$ be hereditarily unicoherent at a point $p$. $X$ is smooth at $p$ if and only if there exists on $X$ a mapping which is radially convex with respect to $p$.*

Proof. To begin with assume that the continuum $X$ is smooth at $p$. Then by Theorem (3.1) the quotient space $X/\varrho_p$ is a smooth dendroid, thus according to Theorem (3.13) $X/\varrho_p$ has a metric $d$ which is radially convex with respect to the point $\varphi(p)$. Put $f_p(x) = d\big(\varphi(p), \varphi(x)\big)$. The map $f_p$ is continuous. Indeed, if $x_n \to x$ in $X$, then $\varphi(x_n) \to \varphi(x)$ in $X/\varrho_p$; hence $d\big(\varphi(p), \varphi(x_n)\big) \to d\big(\varphi(p), \varphi(x)\big)$, i.e., $f_p(x_n) \to f_p(x)$. Obviously, the map $f_p$ satisfies conditions (i) and (ii).

Next, assume that there exists on $X$ a radially convex map $f_p$ with respect to $p$. We show that $X$ is smooth at $p$. According to Theorem (3.8) it suffices to prove that $px \cap T_x \subset [x]$ for each $x \in X$. Let $x \in X$ and suppose that $q \in px \setminus [x]$. Then $f_p(q) < f_p(x)$ by condition (i). Define $K = \{z \in X\colon f_p(z) \leqslant f_p(q) + \varepsilon\}$, where $\varepsilon = \frac{1}{2}\big(f_p(x) - f_p(q)\big) > 0$. Take $z \in K$. If $y \in pz$, then $f_p(y) \leqslant f_p(z)$ by (i) and (ii). Hence, by the definition of the set $K$, we have $y \in K$; thus $pz \subset K$, and we conclude that the set $K$ is connected. We show that it is also closed. Indeed, if $z_n \in K$, then $f_p(z_n) \leqslant f_p(q) + \varepsilon$; hence if $z_n \to z$, then — by the continuity of $f_p$ — we have

$f_p(z) \leqslant f_p(q)+\varepsilon$, i.e. $z \in K$. Thus the set $K$ is a continuum. Moreover, $K$ contains $q$ in its interior and does not contain $x$. Indeed, suppose that $q \in X \setminus \operatorname{Int} K$. Then there exists a sequence $q_n \to q$ with $q_n \in X \setminus K$, i.e., $f_p(q_n) > f_p(q)+\varepsilon$; hence by the continuity of $f_p$ we have $f_p(q) \geqslant f_p(q)+\varepsilon$, a contradiction. Further, since $f_p(x)-f_p(q) = 2\varepsilon$, we have $f_p(x) = f_p(q)+ +2\varepsilon > f_p(q)+\varepsilon$, and thus $x \in X \setminus K$. Therefore, by the definition of the set $T_x$, we conclude that $q \in X \setminus T_x$, and the proof is complete.

Let $F\colon X \to Y$ be a multivalued function defined on a space $X$ with values in a space $Y$, i.e., a function which assigns to each $x \in X$ a closed set $F(x) \subset Y$. We assume that $F$ maps $X$ onto $Y$, i.e., that for each $y \in Y$ there exists a point $x \in X$ such that $y \in F(x)$. Adopt the following notation, for which $A \subset X$ and $B \subset Y$: $F(A) = \bigcup \{F(x)\colon x \in A\}$, $F^{-1}(B) = \{x \in X\colon F(x) \cap B \neq \emptyset\}$ and $F_d^{-1}(B) = \{x \in X\colon F(x) \subset B\}$.

By the definition of $F(A)$ we have

(3.15) Lemma. *If $A$, $A_1$ and $A_2$ are subsets of $X$, then*

(i) $Y \setminus F(X \setminus A) \subset F(A)$,

(ii) *if* $A_1 \subset A_2$, *then* $F(A_1) \subset F(A_2)$.

A multivalued function $F\colon X \to Y$ is said to be *upper semi-continuous* if for each open set $B \subset Y$ the set $F_d^{-1}(B)$ is open in $X$. In other words, $F$ is upper semi-continuous if and only if for each closed set $B \subset Y$ the set $F^{-1}(B)$ is closed in $X$.

Let continua $X$ and $Y$ be hereditarily unicoherent at points $p$ and $q$ respectively. Let an upper semi-continuous function $F\colon X \to Y$ map $X$ onto $Y$ such that $F(p) = \{q\}$. The mapping $F$ is called *order-preserving with respect to $p$* (or simply *$\leqslant_p$-preserving*) if and only if $x \leqslant_p y$ implies that $r \leqslant_q s$ for each $s \in F(x)$ and each $s \in F(y)$.

Denote by $C$ the Cantor fan with the top $t$. The following theorems are proved in [4], p. 309–311.

(3.16) If a dendroid $X$ is smooth at $p$, then there is an $\leqslant_t$-preserving (single-valued) function $f$ from $C$ onto $X$ such that $f(t) = p$.

(3.17) If $f$ is $\leqslant_p$-preserving (single-valued) function of a dendroid $X$ onto a dendroid $Y$ and $p$ is an initial point of $X$, then $f(p)$ is an initial point of $Y$.

We will prove similar theorems for continua hereditarily unicoherent at some point and for multivalued functions.

(3.18) Theorem. *Let a continuum $X$ be smooth at a point $p$. There exists a multivalued function $F$ which maps $C$ onto $X$ and is such that*

(i) $F(t) = \{p\}$,

(ii) *if* $z \in F(x)$, *then* $F(x) = [z]$,

(iii) *$F$ is $\leqslant_t$-preserving.*



Proof. Since the continuum $X$ is smooth at $p$, according to Theorem (3.1) the quotient space $X/\varrho_p$ is a smooth dendroid at the point $\varphi(p)$. Therefore there exists by (3.18) a $\leqslant_t$-preserving function $f\colon C \to X/\varrho_p$ such that $f(t) = \varphi(p)$. Put $F(x) = \varphi^{-1}(f(x))$ for each $x \in C$. The multivalued function $F$ maps $C$ onto $X$ and satisfies the required conditions. Indeed, $F(t) = \varphi^{-1}(f(t)) = \varphi^{-1}(\varphi(p)) = \{p\}$. For each point $x \in C$ its image $f(x)$ is a point in $X/\varrho_p$; thus $\varphi^{-1}(f(x))$ is an equivalence class of the relation $\varrho_p$ in $X$, thus (ii) follows by the definition of $F(x)$. If $A$ is a closed subset of $X$, then by (2.10) the set $F^{-1}(A) = f^{-1}(\varphi(A))$ is closed in $C$. Hence $F$ is upper semi-continuous. We show now that $F$ is an $\leqslant_t$-preserving function. Let $x \leqslant_t y$, $r \in F(x)$ and $s \in F(y)$. Therefore by (ii) it follows that $\varphi(r) = f(x)$ and $\varphi(s) = f(y)$. Since $f$ is an $\leqslant_t$-preserving function, $x \leqslant_t y$ implies $f(x) \leqslant_{f(t)} f(y)$, i.e. $f(x) \leqslant_{\varphi(p)} f(y)$; thus $\varphi(r) \leqslant_{\varphi(p)} \varphi(s)$. It follows that $\varphi(r) \in \varphi(p)\varphi(s)$. Since $X$ is smooth at $p$, the canonical map $\varphi$ is closed by (2.10); thus it is monotone by (2.5). Hence $\varphi(p)\varphi(s) = \varphi(ps)$ (see [7] Theorem 4.1, (ii), p. 56), and thus $\varphi(r) \in \varphi(ps)$. Therefore $r \in \varphi^{-1}\varphi(r) \subset \varphi^{-1}\varphi(ps) = ps$ by (2.2), whence $r \leqslant_p s$.

(3.19) Theorem. *Let continua $X$ and $Y$ be hereditarily unicoherent at points $p$ and $q$ respectively. Let $F$ be a multivalued function of $X$ onto $Y$ such that*

(i) $F(p) = \{q\}$,

(ii) *$F(x)$ is connected for each $x \in X$,*

(iii) *$F$ is $\leqslant_p$-preserving.*

*If $p$ is an initial point of the continuum $X$, then $q$ is an initial point of the continuum $Y$.*

Proof. To show that $q$ is an initial point of $Y$ it suffices to prove, according to Theorem (3.8), that for each $y \in Y$ we have $qy \cap T_y \subset [y]$, where $[y]$ denotes an equivalence class of the weak cutpoint order with respect to the point $q$ in $Y$. Since $p$ is an initial point in $X$, by (3.4) the graph of the relation $\leqslant_p$, i.e., the set $W = \{(a, b) \in X \times X\colon a \leqslant_p b\}$, is closed in $X \times X$. Let $z \in qy \setminus [y]$. Then $y \in Y \setminus qz$. Let $Q = F^{-1}(y)$ and $Z = F^{-1}(z)$. By the upper semi-continuity of $F$ the sets $Q$ and $Z$ are closed, and thus compact. We show that $(Q \times Z) \cap W = \emptyset$. Indeed, if $(s, t) \in Q \times Z$ and $s \leqslant_p t$, then $y \in F(s)$, $z \in F(t)$ and $y \leqslant_q z$, and thus $y \in zq$. Therefore there exist sets $U$ and $V$ which are open in $X$ and such that $Q \times Z \subset U \times V$ and $(\overline{U} \times \overline{V}) \cap W = \emptyset$. Let $K = \{x \in X\colon x \leqslant_p v$ for some $v \in \overline{V}\}$. Let $x \in K$ and $t \in px$. Then $t \leqslant_p x \leqslant_p v$ for some $v \in \overline{V}$, which implies the inclusion $px \subset K$. Hence the set $K$ is connected. We show that it is closed. Let $x_n \in K$ and $x_n \to x$. Then $x_n \leqslant_p v_n$ for some $v_n \in \overline{V}$. Take a convergent subsequence $\{v_{n_m}\}$ of the sequence $\{v_n\}$. This subsequence converges to some $v \in \overline{V}$ and we have $x_{n_m} \leqslant_p v_{n_m}$. Since the weak cutpoint order $\leqslant_p$ is closed by Theorem (3.4), we have $x = \lim_{m\to\infty} x_{n_m} \leqslant_p \lim_{m\to\infty} v_{n_m} = v$. Therefore $K$ is a continuum. Moreover, $Z \subset V \subset \overline{V} \subset K$ and thus $Z \subset \operatorname{Int} K$. If $t \in K$, then $t \leqslant_p v$ for some $v \in \overline{V}$; hence $(t, v) \in W$, if $t \in Q$, $(t, v) \in \overline{U} \times \overline{V} \cap W$. This contradiction proves that $Q \cap K = \emptyset$.

It follows by [16], Lemma 3, p. 161, that $F(K)$ is a continuum. We will show that $F(K)$ contains $z$ in its interior and $y \in Y \setminus F(K)$. Since the set $V$ is open, $X \setminus V$ is compact, and thus $F(X \setminus V)$ is compact by [13], Corollary 9.6, p. 180. If $Z \subset V$, then by (3.15) (ii) we have $F(X \setminus V) \subset F(X \setminus Z) = F(\{x \in X\colon z \notin F(x)\}) \subset Y \setminus \{z\}$, and thus $z \notin F(X \setminus V)$. Therefore there exists an open set $G$ such that $z \in G$ and $G \cap F(X \setminus V) = \emptyset$; thus, because $V \subset K$, by (3.15) we have $z \in G \subset Y \setminus F(X \setminus V) \subset F(V) \subset F(K)$. This implies that $z \in \operatorname{Int} F(K)$. Since $\emptyset = Q \cap K = K \cap F^{-1}(y) = K \cap \{x \in X\colon y \in F(x)\}$, we have $y \notin F(K)$. Thus we have proved that $F(K)$ is a continuum which contains $z$ in its interior and fails to contain $y$, whereby $z \notin T_y$. Therefore the inclusion $qy \cap T_y \subset [y]$ is proved.

(3.20) COROLLARY. *Let a continuum $X$ be hereditarily unicoherent at a point $p$. $X$ is smooth at $p$ if and only if there exists a multivalued function $F$ from $C$ onto $Y$ such that*

(i) $F(t) = \{p\}$,

(ii) *$F(x)$ is connected for each $x \in C$,*

(iii) *$F$ is $\leqslant_p$-preserving.*

Indeed, according to Theorem (3.18) the smoothness of $X$ at $p$ implies that there is such a function $F$, because the equivalence classes $[z]$ are continua for each $z \in X$ (see (2.9)). If $X$ is an image of $C$ by such a function $F$, then it is smooth at $p$ by Theorem (3.19).

E. Other characterizations. The two characterizations of smoothness which we give here are connected with the structure of some subsets of the continuum.

(3.21) THEOREM. *Let a continuum $X$ be hereditarily unicoherent at a point $p$. $X$ is smooth at $p$ if and only if for each sequence $x_n \to x$ the following conditions are satisfied:*

(i) $\bigcap_{n=1}^{\infty} px_n \subset px$,

(ii) *the set $px \cup \bigcup_{n=1}^{\infty} px_n$ is a continuum.*

Proof. Let $X$ be smooth at $p$. If $x_n \to x$, then $\bigcap_{n=1}^{\infty} px_n \subset \operatorname{Lim}_{n\to\infty} px_n = px$; thus condition (i) holds. Put $K = px \cup \bigcup_{n=1}^{\infty} px_n$. Then the set $K$ is connected, because each member of the union contains the point $p$. We will prove that $K$ is closed. Let $y_n \to y$ and $y_n \in K$. Consider two cases:



1. There exists a subsequence $\{y_{k_n}\}$ of sequence $\{y_n\}$ such that $y_{k_n} \in px_{k_n}$. Then $y = \lim_{n\to\infty} y_{k_n} \in \operatorname{Lim}_{n\to\infty} px_{k_n} \subset \operatorname{Lim}_{n\to\infty} px_n = px \subset K$.

2. There exists a subsequence $\{y_{k_n}\}$ of sequence $\{y_k\}$ such that $y_{k_n} \in px_{n_0}$. Then obviously $y \in px_{n_0} \subset K$.

Conversely, suppose that $X$ is not smooth at $p$. Then there exist some sequences $\{a_m\}$ and $\{b_m\}$ such that $a_m \to a$, $b_m \to b$, $b_m \in pa_m$ and $b \in X \setminus pa$ (see [7], Theorem 2.3). Take $M = \{m\colon b \in pa_m\}$. If the set $M$ is finite, then we take a subsequence $\{a_{n_m}\}$, where $n_m$ is not in $M$. Obviously $a_{n_m} \to a$, and thus the set $K = pa \cup \bigcup_{m=1}^{\infty} pa_{n_m}$ is a continuum by (ii); therefore the conditions $b_{n_m} \to b$ and $b_{n_m} \in pa_{n_m} \subset K$ imply $b \in K$; hence $b \in K \setminus \bigcup_{m=1}^{\infty} pa_{n_m}$ by the definition of the set $M$. Thereby $b \in pa$, a contradiction. If $M$ is infinite, take a subsequence $\{a_{n_m}\}$, where $n_m \in M$. Obviously $a_{n_m} \to a$ and $b \in \bigcap_{m=1}^{\infty} pa_{n_m}$ by the definition of $M$. Thereby condition (i) implies $b \in pa$ and we obtain a contradiction as above.

A continuum $X$ is said to be *countably generated* provided $X$ is irreducible about a countable closed subset $A$ of $X$. If the set $A$ has $n$ cluster points, where $n$ is either finite or countably infinite, then $X$ is called *$n$-countably generated* (cf. [4], p. 303).

(3.22) THEOREM. *Let a continuum $X$ be hereditarily unicoherent at a point $p$. $X$ is smooth at $p$ if and only if*

(d) *every 1-countably generated subcontinuum of $X$ containing $p$ is smooth at the point $p$.*

Proof. The smoothness of $X$ at $p$ gives (d) according to (2.8). Conversely, suppose that (d) is satisfied. Take a sequence $x_n \to x$ and put $K = px \cup \bigcup_{n=1}^{\infty} px_n$. Then $\overline{K}$ is a continuum which is irreducible about a set $\{p, x_1, x_2, x_3, \ldots\}$, and thus it is 1-countably generated; thereby $\overline{K}$ is smooth at $p$ by (d). Applying Theorem (3.21) to the continuum $\overline{K}$ and to the sequence $\{x_n\}$, we conclude that conditions (i) and (ii) are satisfied in $\overline{K}$, and thus they hold in $X$. Using Theorem (3.21) once more, we obtain the smoothness of $X$ at $p$.

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