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On essential cluster sets

by

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Abstract. Let f be a real function defined in the open half plane H bounded by a line L. The fine cluster set of f at a point x in L, designated by $W_0(f, x)$, is the set of all y such that for every $\varepsilon > 0$, x is a point of positive lower density for the set $f^{-1}\{(y-\varepsilon, y+\varepsilon)\}$. The fine cluster set of f at x in the direction θ, designated by $W_0(f, x, \theta)$, is defined analogously by restricting f on a line $L_\theta(x)$ in H emanating from x and making angle θ with L. It is shown that each of the sets $\{x\colon x \in L;\ W_0(f, x, \theta) \not\subset W_0(f, x)\}$ and $\{\theta\colon \theta \in (0, \pi);\ W_0(f, x, \theta) \not\subset W_0(f, x)\}$ is of measure zero when f is measurable and is of the first category when f is continuous, and some consequences are studied.

1. In a recent paper [3] Goffman and Sledd have obtained certain interesting relations between the total essential cluster sets and the directional essential cluster sets. They have proved that if a measurable function f is defined in the upper half plane above the x-axis and if θ is a direction then except a set of points x of measure zero the essential cluster set of f at x is a subset of the essential cluster set of f at x in the direction θ. If further f is continuous then this exceptional set is also of the first category. Regarding the ordinary cluster sets there is an analogous result [2]. In this paper we study further properties of these sets by weakening the density conditions. We have defined the fine cluster sets and obtained certain relations between the fine cluster sets, essential cluster sets and ordinary cluster sets.

2. The function f is taken to be defined in the open half plane H above a line L, which, in particular may be taken to be the x-axis. The point on the line L, viz $(x, 0)$, will be denoted simply by x while any other point in H will be denoted by p. $\mu(A)$ and $\mu^*(A)$ will denote the Lebesgue measure and the Lebesgue outer measure, respectively, for the set A, linear or planar, according as A is linear or planar, which will be clear from the context. For $\delta > 0$, $S_\delta(x)$ will denote the set of all points p in H, whose distance, $|p-x|$, from x is less than δ. For $0 < \theta < \pi$, $L_\theta(x)$ denotes the half ray in H, in the direction θ, terminating at x and $L_\theta(x, h)$ is the open line segment in H in the direction θ, of length h, and having x as one of its end points.

If $E\subset H$ is measurable then the upper and the lower densities of E at x are

$$\bar{d}(E,x)=\limsup_{\delta\to 0}\frac{\mu(E\cap S_\delta(x))}{\mu(S_\delta(x))}$$

and

$$\underline{d}(E,x)=\liminf_{\sigma\to 0}\frac{\mu(E\cap S_\sigma(x))}{\mu(S_\sigma(x))}$$

respectively. If $E\cap L_\theta(x)$ is measurable then the upper and the lower densities of E at x in the direction θ are

$$\bar{d}_\theta(E,x)=\limsup_{h\to 0}\frac{\mu(E\cap L_\theta(x,h))}{\mu(L_\theta(x,h))}$$

and

$$\underline{d}_\theta(E,x)=\liminf_{h\to 0}\frac{\mu(E\cap L_\theta(x,h))}{\mu(L_\theta(x,h))}$$

respectively.

Let $f\colon H\to R$ be measurable. The fine cluster set of f at x is the set of all points y such that for every open set G containing y the set $f^{-1}(G)$ has positive lower density at x. The essential cluster set of f at x is, as usual, the set of y such that for every open set G containing y the set $f^{-1}(G)$ has positive upper density at x. The fine cluster set and the essential cluster set of f at x are denoted by $W_0(f,x)$ and $W(f,x)$ respectively. The cluster set of f at x, denoted by $C(f,x)$, is the set of all y such that for every open set G containing y, x is a limiting point of the set $f^{-1}(G)$. The fine cluster set, the essential cluster set, and the cluster set of f at x in the direction θ are defined analogously by restricting the relevant sets to $L_\theta(x)$ and are denoted by $W_0(f,x,\theta)$, $W(f,x,\theta)$ and $C(f,x,\theta)$ respectively. It is clear from the definition that

$$W_0(f,x)\subset W(f,x)\subset C(f,x)\,,\qquad W_0(f,x,\theta)\subset W(f,x,\theta)\subset C(f,x,\theta)$$

and

$$\limsup_{p\to x}f(p)=\sup C(f,x)\,,\qquad \limsup_{p\to x,\theta}f(p)=\sup C(f,x,\theta)\,,$$
$$\limsup_{p\to x}\operatorname{ap}f(p)=\sup W(f,x)\,,\qquad \limsup_{p\to x,\theta}\operatorname{ap}f(p)=\sup W(f,x,\theta)\,.$$

3. LEMMA 1. *Let $E\subset H$ be measurable. Then for fixed θ, the set*

$$S_{mn}=\left\{x\colon\ \mu(E\cap L_\theta(x,h))\geqslant\frac{h}{m}\ \text{for}\ 0<h<\frac{1}{n}\right\}$$

is measurable for all positive integers m and n.

Proof. Let

$$E_h=\left\{x\colon\ \mu(E\cap L_\theta(x,h))\geqslant\frac{h}{m}\right\}.$$

Then

$$S_{mn}=\bigcap_{0<h<1/n}E_h\,.$$

Since $\mu(E\cap L_\theta(x,h))$, is a continuous function of h,

(1) $$S_{mn}=\bigcap_{0<h_r<1/n}E_{h_r}$$

where h_r are the rational values of h in $(0,1/n)$. Since $\mu(E\cap L_\theta(x,h))$ is a measurable function of x for fixed h, the set E_{h_r} is measurable for each h_r and so the measurability of S_{mn} follows by (1).

LEMMA 2. *Let $E\subset H$ be measurable. Then $\underline{d}(E,x)$, $\bar{d}(E,x)$, $\underline{d}(E,x,\theta)$, $\bar{d}(E,x,\theta)$, are measurable functions of x.*

Proof. Let

$$u(x,r)=\frac{\mu(E\cap S_r(x))}{\mu(S_r(x))}\,.$$

Then

$$\underline{d}(E,x)=\lim_{k\to 0}\ \inf_{0<r<k}u(x,r)\,.$$

Since $u(x,r)$ is a continuous function of r in $(0,k)$ the infimum is the same if only rational values of r are considered. Since $u(x,r)$ is a measurable function of x for each r, $\inf_{0<r<k}u(x,r)$ is measurable for fixed k. Since $\inf_{0<r<k}u(x,r)$ is a monotone function of k, $\underline{d}(E,x)$ is measurable.

LEMMA 3. *Let $E\subset H$ be measurable. Then for fixed θ, the set*

$$S=\{x\colon\ \underline{d}(E,x)=0;\ \underline{d}_\theta(E,x)>0\}$$

is of measure zero.

Proof. Let

$$S_{mn}=\left\{x\colon\ \underline{d}(E,x)=0;\ \mu(E\cap L_\theta(x,h))\geqslant\frac{h}{m}\ \text{for}\ 0<h<\frac{1}{n}\right\}.$$

Then

(1) $$S\subset\bigcup S_{mn}$$

where the union is taken over the set of all positive integers m and n. By Lemma 1 and 2 the set S_{mn} is measurable for each m and n. Suppose $\mu(S_{mn})>0$. Let x_0 be a point of S_{mn} which is also a point of density of S_{mn}. Let $\varepsilon>0$. Then there is δ_0, $0<\delta_0<1/2n$ such that for $0<\delta<\delta_0$

(2) $$\mu\big(S_{mn}\cap(x_0-\delta,x_0+\delta)\big)>2\delta(1-\varepsilon)\,.$$

Let δ, $0<\delta<\delta_0$, be fixed. For $x\in(x_0-\frac{1}{2}\delta, x_0+\frac{1}{2}\delta)$ let $h(x)$ denote the length of the line segment in the direction θ joining x to the circumference of $S_\delta(x_0)$. Then writing $F=S_{mn}\cap(x_0-\frac{1}{2}\delta, x_0+\frac{1}{2}\delta)$

$$\mu\big(E\cap S_\delta(x_0)\big)\geqslant\sin\theta\int_F\mu\big(E\cap L_\theta(x, h(x))\big)dx$$

$$\geqslant\frac{\sin\theta}{m}\int_F h(x)\,dx$$

$$\geqslant\frac{\sin\theta}{m}\cdot\frac{\delta}{2}\cdot\mu(F)\,,\quad h(x)\geqslant\frac{\delta}{2}\ \text{for}\ x\in\left(x_0-\frac{\delta}{2}, x_0+\frac{\delta}{2}\right)$$

$$\geqslant\frac{\sin\theta}{m}\cdot\frac{\delta}{2}\cdot\delta(1-\varepsilon)\quad\text{by (2)}.$$

Hence

$$\frac{\mu\big(E\cap S_\delta(x_0)\big)}{\mu\big(S_\delta(x_0)\big)}\geqslant\frac{\sin\theta}{m\pi}(1-\varepsilon)\,.$$

Letting $\delta\to0$,

$$\underline{d}(E, x_0)\geqslant\frac{\sin\theta}{m\pi}(1-\varepsilon)\,.$$

This contradicts the fact that $x_0\in S_{mn}$. Hence $\mu(S_{mn})=0$ and the proof is completed by (1).

LEMMA 4. *Let $E\subset H$ be measurable. Then the set*

$$S=\{x\colon\ \bar{d}_{\theta_1}(E, x)<\underline{d}_{\theta_2}(E, x)\}$$

is of measure zero, where θ_1 and θ_2 are any two fixed directions.

This is proved in [3].

LEMMA 5. *Let $E\subset H$ be measurable. Then for any two directions θ_1 and θ_2 the set*

$$S=\{x\colon\ \bar{d}_{\theta_1}(E, x)>0;\ x\notin\overline{E\cap L_{\theta_2}(x)}\}$$

is of measure zero, where, for example, $\bar{A}$ denotes the closure of A.

Proof. For each n, let

$$S_n=\left\{x\colon\ \bar{d}_{\theta_1}(E, x)>\frac{1}{n};\ E\cap L_{\theta_2}\left(x, \frac{1}{n}\right)=\Phi\right\}.$$

Let $\theta_1>\theta_2$. Suppose $\mu^*(S_n)>0$. Let x_0 be a point of S_n such that x_0 is a point of outer density of S_n. Since $L_{\theta_1}(x)\cap E$ is measurable for almost all $x\in L$, we may suppose that $L_{\theta_1}(x_0)\cap E$ is measurable. Then for each $x\in S_n$, $x<x_0$, let $L_{\theta_2}(x)$ intersect $L_{\theta_1}(x_0)$ at the point $q(x)$. Since x_0 is a point of outer density for the set S_n, x_0 is also a point of outer density (one sided) for the set

$$Q=\{q(x)\colon\ x\in S_n;\ x<x_0\}\,.$$

Since $x_0\in S_n$, we conclude from the above assertion that there is an interval $J\subset L_{\theta_1}(x_0)$ with x_0 as one of its end points such that

$$\mu(J)<\frac{1}{n}\cdot\frac{\sin\theta_2}{\sin\theta_1}\,,\tag{1}$$

$$\mu(J\cap E)>\mu(J)\cdot\frac{3}{4n}\,,\tag{2}$$

$$\mu^*(J\cap Q)>\mu(J)\cdot\left(1-\frac{1}{2n}\right).\tag{3}$$

From (2) and (3) we conclude that $J\cap E$ and $J\cap Q$ have common points. Let $p_0\in J\cap Q\cap E$. Then there is $x_1<x_0$, $x_1\in S_n$, such that $q(x_1)=p_0$. So, $p_0\in L_{\theta_2}(x_1, 1/n)$ by (1). Since $x_1\in S_n$, this contradicts the definition of S_n. Thus $\mu(S_n)=0$ for each n. Since

$$S\subset\bigcup_{n=1}^{\infty}S_n\,,$$

S is of measure zero. The case $\theta_1<\theta_2$ can be similarly treated.

LEMMA 6. *Let $E\subset H$ be measurable. If for a point x, $\underline{d}(E, x)=0$ then the set of directions*

$$S=\{\theta\colon\ 0<\theta<\pi;\ \underline{d}_\theta(E, x)>0\}$$

is of measure zero.

Proof. Denote

$$S_{mn}=\left\{\theta\colon\ 0<\theta<\pi;\ \mu\big(E\cap L_\theta(x, h)\big)\geqslant\frac{h}{m}\ \text{ for }\ 0<h<\frac{1}{n}\right\}.$$

Then $S\subset\bigcup S_{mn}$ where m, n run over the set of positive integers. It can be shown as in Lemma 1 that the sets S_{mn} are measurable. If possible, suppose $\mu(S_{mn})=\sigma>0$. Choose $0<\delta<1/n$. Then

$$\mu\big(E\cap S_\delta(x)\big)\geqslant\tfrac{1}{2}\int_{S_{mn}}\big[\mu\big(E\cap L_\theta(x, \delta)\big)\big]^2d\theta$$

$$\geqslant\frac{\delta^2}{2m^2}\int_{S_{mn}}d\theta=\frac{\delta^2}{2m^2}\mu(S_{mn})=\frac{\sigma\delta^2}{2m^2}\,.$$

So,

$$\frac{\mu(E\cap S_\delta(x))}{\mu(S_\delta(x))}\geqslant\frac{\sigma}{\pi m^2}.$$

Letting $\delta\to 0$,

$$\underline{d}(E,x)\geqslant\frac{\sigma}{\pi m^2}.$$

But this contradicts our hypothesis.

THEOREM 1. *Let $f\colon H\to R$ be measurable and let θ be any direction. Then the set*

$$S=\{x\colon W_0(f,x,\theta)\not\subset W_0(f,x)\}$$

is of measure zero. If f is continuous then S is of the first category.

Proof. For fixed $r,s,r<s$, we write

$$E=f^{-1}\big((r,s)\big),\qquad S_{rs}=\{x\colon \underline{d}(E,x)=0;\ \underline{d}_\theta(E,x)>0\}.$$

Then $S\subset\bigcup S_{rs}$ where r,s run over the set of all rational numbers. By Lemma 3, $\mu(S_{rs})=0$ for all r and s and hence the first part of the theorem follows.

For fixed $r,s,r<s,m,n,m\geqslant 2$, let F_{rsmn} denote the set of all x for which the following relations are true:

(1) $$\underline{d}\big(\{p\colon r\leqslant f(p)\leqslant s\},x\big)=0,$$

(2) $$\mu\big(\{p\colon p\in L_\theta(x,h);\ r\leqslant f(p)\leqslant s\}\big)\geqslant\frac{h}{m}\quad\text{for}\quad 0<h<\frac{1}{n}.$$

Then

(3) $$S\subsetneq\bigcup F_{rsmn}$$

where the union is taken over the set of all pairs of rationals $r,s,r<s$, and over the set of all positive integers $m,n,m\geqslant 2$. We shall show that F_{rsmn} is nondense for all r,s,m,n.

Suppose that F_{rsmn} is dense in an interval (a,b). We may suppose that $\frac{1}{2}(a+b)\in F_{rsmn}$ and $b-a<1/n$. Hence by (1) there is a semi circular region C in H bounded by a semi circular arc Γ with centre $\frac{1}{2}(a+b)$ and radius $R<\frac{1}{2}(b-a)$ such that

(4) $$\mu\big(\{p\colon p\in C;\ r\leqslant f(p)\leqslant s\}\big)<\frac{R^2}{4m}(\sin 2\theta_0+2\theta_0)$$

where $\theta_0=\theta$ if $0<\theta\leqslant\frac{1}{2}\pi$ and $\theta_0=\pi-\theta$ if $\frac{1}{2}\pi<\theta<\pi$. Let I_0 be the diameter of C. Then from (4) we conclude that there is a point x_0 in the interior of I_0 such that

$$\mu\Big(\big\{p\colon p\in L_\theta\big(x_0,h(x_0)\big);\ r\leqslant f(p)\leqslant s\big\}\Big)<\frac{h(x_0)}{2m}$$

where $h(x)$ denotes for each $x\in I_0$ the length of the line segment joining x to Γ in the direction θ. Let $r'<r<s<s'$ be such that

$$\mu\Big(\big\{p\colon p\in L_\theta\big(x_0,h(x_0)\big);\ r'<f(p)<s'\big\}\Big)<\frac{h(x_0)}{2m}.$$

Choose $0<\sigma<h(x_0)/2m$, such that

$$\mu\big(\{p\colon p\in L_\theta(x_0);\ \sigma\leqslant|p-x_0|\leqslant h(x_0);\ r'<f(p)<s'\}\big)<\frac{1}{2m}\big(h(x_0)-\sigma\big).$$

Let $r'<r''<r<s<s''<s'$. Then

$$\mu\big(\{p\colon p\in L_\theta(x_0);\ \sigma\leqslant|p-x_0|\leqslant h(x_0);\ f(p)\notin[r'',s'']\}\big)>\Big(1-\frac{1}{2m}\Big)\big(h(x_0)-\sigma\big).$$

Since f is continuous, it is uniformly continuous in any bounded closed region in H and hence there is $\delta>0$ such that $|x-x_0|<\delta$ implies

$$\begin{aligned}\mu\big(\{p\colon p\in L_\theta(x);\ \sigma\leqslant|p-x|\leqslant h(x_0);\ f(p)\notin[r,s]\}\big)&>\Big(1-\frac{1}{2m}\Big)\big(h(x_0)-\sigma\big)\\&>\Big(1-\frac{1}{m}\Big)h(x_0).\end{aligned}$$

Hence

(5) $$\mu\Big(\big\{p\colon p\in L_\theta\big(x,h(x_0)\big);\ r\leqslant f(p)\leqslant s\big\}\Big)<\frac{h(x_0)}{m}.$$

But since (2) holds on a dense subset of I_0 and since $h(x_0)<1/n$, (5) provides with a contradiction. Hence F_{rsmn} is nondense and by (3), S is of the first category.

THEOREM 2. *Let $f\colon H\to R$ be measurable and let x be any fixed point in L. Then the set of directions*

$$S=\{\theta\colon 0<\theta<\pi;\ W_0(f,x,\theta)\not\subset W_0(f,x)\}$$

is of measure zero.

If f is continuous, then S is of the first category.

Proof. As in Theorem 1, denote

$$E=f^{-1}\big((r,s)\big),$$

$$S_{rs}=\{\theta\colon 0<\theta<\pi;\ \underline{d}(E,x)=0;\ \underline{d}_\theta(E,x)>0\}.$$

Then $S\subset\bigcup S_{rs}$ where the union is taken over the set of all pairs of rationals $r,s,r<s$. By Lemma 6, $\mu(S_{rs})=0$ and hence $\mu(S)=0$.

To show that S is of the first category, consider for fixed rationals r, s, $r < s$, and for fixed positive integers m, n, $m \geqslant 2$, the set F_{rsmn} of all θ for which

$$(1) \qquad \mu(\{p\colon p \in L_\theta(x, h);\ r \leqslant f(p) \leqslant s\}) \geqslant \frac{h}{m} \quad \text{for} \quad 0 < h < \frac{1}{n}.$$

Then

$$(2) \qquad S \subset \bigcup F_{rsmn}$$

where the union extends over the set of all rationals r, s, $r < s$, for which

$$(3) \qquad \underline{d}(\{p\colon p \in H;\ r \leqslant f(p) \leqslant s\}, x) = 0$$

holds and over the set of all positive integers m, n, $m \geqslant 2$. Suppose F_{rsmn} is dense in an interval $I \subset (0, \pi)$. By (3) there is a semi circular region C in H with centre x and radius $R < 1/n$ such that

$$(4) \qquad \mu(\{p\colon p \in C;\ r \leqslant f(p) \leqslant s\}) < \frac{R^2}{8m^2}\mu(I).$$

From (4) there is at least one θ_0 in the interior of I such that

$$\mu(\{p\colon p \in L_{\theta_0}(x, R);\ r \leqslant f(p) \leqslant s\}) < \frac{R}{2m}.$$

Let $r' < r < s < s'$ be such that

$$\mu(\{p\colon p \in L_{\theta_0}(x, R);\ r' < f(p) < s'\}) < \frac{R}{2m}.$$

Choose $0 < \sigma < R/2m$ such that

$$\mu(\{p\colon p \in L_{\theta_0}(x);\ \sigma \leqslant |p - x| \leqslant R;\ r' < f(p) < s'\}) < \frac{1}{2m}(R - \sigma).$$

Let $r' < r'' < r < s < s'' < s'$. Then

$$\mu(\{p\colon p \in L_{\theta_0}(x);\ \sigma \leqslant |p - x| \leqslant R;\ f(p) \notin [r'', s'']\}) > \left(1 - \frac{1}{2m}\right)(R - \sigma).$$

Since f is continuous, it is uniformly continuous in any bounded closed region in H and hence there is $\delta > 0$ such that $|\theta - \theta_0| < \delta$ implies

$$\mu(\{p\colon p \in L_\theta(x);\ \sigma \leqslant |p - x| \leqslant R;\ f(p) \notin [r, s]\}) > \left(1 - \frac{1}{2m}\right)(R - \sigma)$$
$$> \left(1 - \frac{1}{m}\right)R.$$

So,

$$(5) \qquad \mu(\{p\colon p \in L_\theta(x, R);\ r \leqslant f(p) \leqslant s\}) < \frac{R}{m}.$$

Since (1) holds on a dense subset of I and $R < 1/n$, the relation (5) is a contradiction. Hence F_{rsmn} is nondense and from (2) S is of the first category.

Remark 1. The analogue of Theorem 2 considering $W(f, x)$ and $W(f, x, \theta)$ is not true. For, consider the closed subsets $[\frac{1}{6}\pi, \frac{1}{3}\pi]$ and $[\frac{2}{3}\pi, \frac{5}{6}\pi]$ of $(0, \pi)$ and consider a continuous f in H with values in $[0, 1]$ such that

$$f(p) = 1 \quad \text{for} \quad p \in \{p\colon p \in L_\theta(x);\ 0 < |p - x| \leqslant 1;\ \theta \in [\tfrac{1}{6}\pi, \tfrac{1}{3}\pi]\}$$

and

$$f(p) = 0 \quad \text{for} \quad p \in \{p\colon p \in L_\theta(x);\ 0 < |p - x| \leqslant 1;\ \theta \in [\tfrac{2}{3}\pi, \tfrac{5}{6}\pi]\}.$$

Clearly

$$[\tfrac{1}{6}\pi, \tfrac{1}{3}\pi] \cup [\tfrac{2}{3}\pi, \tfrac{5}{6}\pi] \subset \{\theta\colon W(f, x) \not\subset W(f, x, \theta)\}.$$

Theorem 2 yields the following result:

THEOREM 3. *Let $f\colon H \to R$ be measurable. Then for every $x \in L$ there exists a set of directions $\Phi(x) \subset (0, \pi)$ such that $\mu(\Phi(x)) = \pi$ and*

$$\bigcup \{W_0(f, x, \theta);\ \theta \in \Phi(x)\} \subset W_0(f, x).$$

If f is continuous then the above is true with $\Phi(x)$ residual in $(0, \pi)$.

THEOREM 4. *Let $f\colon H \to R$ be measurable and let θ_1 and θ_2 be fixed directions. Then the set*

$$S = \{x\colon W_0(f, x, \theta_2) \not\subset W(f, x, \theta_1)\}$$

is of measure zero.

If f is continuous then S is of the first category.

Proof. If $x \in S$, then there are rational numbers r, s, $r < s$ such that

$$\underline{d}_{\theta_2}(E, x) > 0, \qquad \bar{d}_{\theta_1}(E, x) = 0$$

where $E = f^{-1}((r, s))$ and hence the first part follows by an application of Lemma 4.

To prove the second part consider for fixed rationals r, r', s', s, $r < r' < s' < s$, and for fixed positive integers m, n, $m \geqslant 2$, the set $F_{rr's'smn}$ of all points x such that $0 < h < 1/n$ implies

$$(1) \qquad \mu(\{p\colon p \in L_{\theta_1}(x, h);\ r < f(p) < s\}) \leqslant \frac{h}{3m}$$

and

$$(2) \qquad \mu(\{p\colon p \in L_{\theta_2}(x, h);\ r' \leqslant f(p) \leqslant s'\}) > \frac{h}{m}.$$

Then

$$(3) \qquad S \subset \bigcup F_{rr's'smn}$$

where the union is taken over the set of all quardruple of rationals r, r', s', s, $r < r' < s' < s$, and over the set of all pairs of positive integers m, n, $m \geqslant 2$. We shall show that $F_{rr's'smn}$ is nondense for every r, r', s', s, m, n.

Suppose that one of the sets $F_{rr's'smn}$ is dense in an interval I. Since f is continuous, the set of points x for which (1) holds is closed and hence (1) holds for all x in I. Let us consider a triangle T with base $[a, b] \subset I$ and other two sides, L_1 and L_2, are in the directions θ_1 and θ_2 respectively. We take the triangle T in such a way that the length of the sides L_1 and L_2 are less than $1/n$. For each $x \in (a, b)$ we denote by $h_1(x)$ (resp. $h_2(x)$) the length of the line segment in the direction θ_1 (resp. θ_2) joining x to L_2 (resp. L_1). Since (1) holds for all $x \in (a, b)$ we conclude

$$\mu(\{p:\ p \in T;\ r < f(p) < s\}) \leqslant \frac{\mu(T)}{3m}. \tag{4}$$

From (4) we assert that there is $x_0 \in (a, b)$ such that

$$\mu\big(\{p:\ p \in L_{\theta_2}(x_0, h_2(x_0));\ r < f(p) < s\}\big) < \frac{h_2(x_0)}{2m}.$$

Choose $0 < \sigma < h_2(x_0)/2m$, such that

$$\mu(\{p:\ p \in L_{\theta_2}(x_0);\ \sigma \leqslant |p - x_0| \leqslant h_2(x_0);\ r < f(p) < s\}) < \frac{1}{2m}\big(h_2(x_0) - \sigma\big).$$

Choose $r < r'' < r' < s' < s'' < s$. Then

$$\mu(\{p:\ p \in L_{\theta_2}(x_0);\ \sigma \leqslant |p - x_0| \leqslant h_2(x_0);\ f(p) \notin [r'', s'']\}) > \left(1 - \frac{1}{2m}\right)\big(h(x_0) - \sigma\big).$$

From the uniform continuity of f we conclude that there is $\delta > 0$ such that $|x - x_0| < \delta$ implies

$$\mu(\{p:\ p \in L_{\theta_2}(x);\ \sigma \leqslant |p - x| \leqslant h_2(x_0);\ f(p) \notin [r', s']\}) > \left(1 - \frac{1}{2m}\right)\big(h_2(x_0) - \sigma\big)$$
$$> \left(1 - \frac{1}{m}\right) h_2(x_0).$$

Hence

$$\mu\big(\{p:\ p \in L_{\theta_2}(x, h_2(x_0));\ r' \leqslant f(p) \leqslant s'\}\big) < \frac{h_2(x_0)}{m}. \tag{5}$$

But since (2) holds on a dense subset of I, (5) provides with a contradiction. Hence the sets $F_{rr's'smn}$ are nondense and the proof is complete by (3).

Remark 2. It is known [3] that there is a continuous function f in H such that for every $x \in L$ the upper approximate limit of f at x is 1 in the vertical direction and in any other direction the upper approximate limit of f at x is 0. From this we conclude that $W_0(f, x, \theta_2)$ cannot be replaced by $W(f, x, \theta_2)$ in the above theorem.

THEOREM 5. *Let f: $H \to R$ be measurable and let θ_1 and θ_2 be fixed directions. Then the set*

$$S = \{x:\ W(f, x, \theta_1) \not\subset C(f, x, \theta_2)\}$$

is of measure zero.

If f is continuous then S is of the first category.

Proof. If $x \in S$, then there are rational numbers r, s, $r < s$ such that

$$\bar{d}_{\theta_1}(E, x) > 0, \quad x \notin \overline{E \cap L_{\theta_2}(x)}$$

where $E = f^{-1}((r, s))$ and hence the first part follows by an application of Lemma 5.

To complete the proof suppose f is continuous. Then by a known result [2] the set

$$Z = \{x:\ C(f, x, \theta_1) \neq C(f, x, \theta_2)\}$$

is of the first category. Since $S \subset Z$, we conclude that S is of the first category.

COROLLARY. *Let f: $H \to R$ be measurable and θ_1, θ_2 be directions. Then the set*

$$S = \{x:\ \limsup_{p \to x, \theta_2} f(p) < \limsup_{p \to x, \theta_1} \operatorname{ap} f(p)\}$$

is of measure zero.

If f is continuous, then S is of the first category.

Remark. These are proved in [4] in less general form.

4. In this section we shall consider a result of Belna [1] on the directional essential cluster sets concerning real functions. We require the following simple lemma.

LEMMA 7. *Let f: $H \to R$ be measurable. Then the sets $W(f, x)$ and $W(f, x, \theta)$ are nonvoid.*

Proof. Suppose that $W(f, x) = \Phi$. Then since $\pm\infty \notin W(f, x)$, there is $M > 0$ such that

$$\bar{d}\big(\{p:\ p \in H;\ f(p) > M\}, x\big) = 0, \tag{1}$$

$$\bar{d}\big(\{p:\ p \in H;\ f(p) < -M\}, x\big) = 0. \tag{2}$$

Also for each $y \in [-M, M]$, there is $\varepsilon_y > 0$ such that

$$\bar{d}\big(\{p:\ p \in H;\ y - \varepsilon_y < f(p) < y + \varepsilon_y\}, x\big) = 0.$$

So, there are finite number of points y, say, $y_1, y_2, \ldots, y_k$ such that

$$[-M, M] \subset \bigcup_{r=1}^{k} (y_r - \varepsilon_{y_r}, y_r + \varepsilon_{y_r})$$

and

$$\text{(3)} \quad \bar{d}(\{p\colon p \in H;\ y_r - \varepsilon_{y_r} < f(p) < y_r + \varepsilon_{y_r}\}, x) = 0 \quad \text{for} \quad r = 1, 2, \ldots, k.$$

Let

$$A = \{p\colon p \in H;\ f(p) > M\},$$
$$B = \{p\colon p \in H;\ f(p) < -M\},$$
$$C_r = \{p\colon p \in H;\ y_r - \varepsilon_{y_r} < f(p) < y_r + \varepsilon_{y_r}\}, \quad r = 1, 2, \ldots, k.$$

Then

$$H = A \cup (\bigcup_{r=1}^{k} C_r) \cup B.$$

Hence

$$1 = \bar{d}(H, x) = \bar{d}\big(A \cup (\bigcup_{r=1}^{k} C_r) \cup B, x\big)$$
$$= \limsup_{\delta \to 0} \frac{\mu\Big(\big(A \cup (\bigcup_{r=1}^{k} C_r) \cup B\big) \cap S_\delta(x)\Big)}{\mu(S_\delta(x))}$$
$$\leqslant \limsup_{\delta \to 0} \left(\frac{\mu(A \cap S_\delta(x))}{\mu(S_\delta(x))} + \sum_{r=1}^{k} \frac{\mu(C_r \cap S_\delta(x))}{\mu(S_\delta(x))} + \frac{\mu(B \cap S_\delta(x))}{\mu(S_\delta(x))}\right)$$
$$\leqslant \bar{d}(A, x) + \sum_{r=1}^{k} \bar{d}(C_r, x) + \bar{d}(B, x) = 0, \quad \text{by (1), (2) and (3).}$$

Thus we arrive at a contradiction and hence the result follows. The proof for $W(f, x, \theta)$ is similar.

THEOREM 6. *Let $f\colon H \to R$ be measurable and let $\{\theta_n\}$ be any countable collection of direction. Then the set*

$$S = \{x\colon \bigcap_n W(f, x, \theta_n) = \emptyset\}$$

is of measure zero.

If f is continuous then S is also of the first category.

Proof. From [3, Theorem 2] it follows that for each n the set

$$S_n = \{x\colon W(f, x) \not\subset W(f, x, \theta_n)\}$$

is of measure zero when f is measurable and is of the first category when f is continuous. Hence we conclude that the set

$$\{x\colon W(f, x) \not\subset \bigcap_n W(f, x, \theta_n)\}$$

is of measure zero when f is measurable and is of the first category when f is continuous. Since by Lemma 7 the set $W(f, x)$ is nonvoid, the proof is complete.

From Theorem 6 we deduce the following known result

COROLLARY. *If $f\colon H \to R$ is measurable and θ_1, θ_2 are directions then the set*

$$S = \{x\colon \limsup_{p \to x, \theta_1} \operatorname{ap} f(p) < \liminf_{p \to x, \theta_2} \operatorname{ap} f(p)\}$$

is of measure zero.

If f is continuous then S is also of the first category.

These are proved in [3] and [2] respectively.

References

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