On positions of sets in spaces

by

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Abstract. The notion of position $\mathrm{Pos}(X, A)$ of a set A in a space X is introduced in order to exhibit the most important, global similarities and differences between the placements of different sets in different spaces. In particular, it is shown that for every plane continuum A with the first Betti number equal to 1 there exists in the Euclidean 3-space E^3 a continuum $\hat{A}$ homeomorphic to A and such that its position in E^3 is the same as the position in E^3 of an arbitrarily given polygonal knot K.

§ 1. Introduction. One says that a subset A of a space X has in X the same *topological position* as a set B in a space Y if there exists a homeomorphism h of X onto Y such that $h(A) = B$.

This concept has a purely qualitative character and it does not allow us to discover any similarity between two different positions. However, from the intuitive point of view, one can observe some similarities between the position of A in X and that of B in Y, even if A is not homeomorphic to B. For instance, if A is a geometric circle lying in the Euclidean 3-space $X = E^3$ and B is a geometric torus lying in $Y = E^3$, then it seems reasonable to consider the position of A in X as similar to the position of B in Y. In some sense we can also speak about similar knots tied on a simple closed curve A (in E^3) and on a set $B \subset E^3$ homeomorphic to a geometric torus. More generally, it seems possible to speak in ma cases about the position of A in X being more or less complicated an that of B in Y and to classify the positions into some classes, ju as one classifies the (metrizable) spaces into classes called shapes (see [2], p. 131 and [3], p. 80).

In order to obtain such a classification, it seems to be reasonable to use some notions of the theory of shape. However, the situation is now more delicate, because in the theory of shape we can always assume that the space X in question is a closed subset of an $\mathrm{AR}(\mathfrak{M})$-space M. But if A is not a closed subset of X, then it is not closed in M either, and thus we are confronted with a situation in which it is difficult to operate with the notion of a fundamental sequence, basic for the theory of shape. Thus, instead of this notion, we use here a more general notion of a *weak fundamental sequence* (abbreviated to *W-sequence*).

§ 2. **W-sequences.** Let X be an arbitrary subset of a space $M \in \mathrm{AR}(\mathfrak{M})$ and let Y be an arbitrary subset of a space $N \in \mathrm{AR}(\mathfrak{M})$. By a *$W$-sequence from X to Y in M, N* we understand a system consisting of sets X, Y, M, N and of a sequence of maps

$$f_k \colon M \to N, \quad k = 1, 2, \dots,$$

satisfying the following condition:

(2.1) *For every compactum $C \subset X$ there is a compactum $D \subset Y$ such that for every neighbourhood V of D (in N) there is a neighbourhood U of C (in M) such that $f_k/U \simeq f_{k+1}/U$ in V for almost all k.*

We denote this W-sequence by $\{f_k, X, Y\}_{M,N}$, or shortly by $\underline{f}$, and we write $\underline{f}\colon X \to Y$ in M, N. Every compactum D satisfying (2.1) is said to be *$\underline{f}$-assigned* to the compactum C.

It is clear that if X and Y are compacta, then the W-sequences $\underline{f}\colon X \to Y$ in M, N are the same as the fundamental sequences.

A W-sequence $\underline{f} = \{f_k, X, Y\}_{M,N}$ is said to be *generated* by a map $f\colon X \to Y$ if $f_k(x) = f(x)$ for every point $x \in X$ and for $k = 1, 2, \dots$ It is clear that there exist W-sequences generated by a given map $f\colon X \to Y$ if and only if there exists a map $\bar{f}$ of the closure $\bar{X}$ of X in M into the closure $\bar{Y}$ of Y in N satisfying the condition $\bar{f}(x) = f(x)$ for every point $x \in X$. In particular, if $X = Y$ and $M = N$ and if i denotes the identity map of M onto itself, then $\{i, X, X\}_{M,M}$ is a W-sequence generated by the identity map of X onto itself. We denote this W-sequence by $\underline{i}_{X,M}$ and we call it the *identity W-sequence* for X in M.

If $\underline{f} = \{f_k, X, Y\}_{M,N}$ and $\underline{g} = \{g_k, Y, Z\}_{N,P}$ are W-sequences, then one can easily see that $\{g_k f_k, X, Z\}_{M,P}$ is a W-sequence; we denote it by $\underline{gf}$ and call it the *composition* of $\underline{f}$ and of $\underline{g}$.

§ 3. **Homotopy of W-sequences.** Two W-sequences $\underline{f} = \{f_k, X, Y\}_{M,N}$ and $\underline{f}' = \{f'_k, X, Y\}_{M,N}$ are said to be *homotopic* (notation: $\underline{f} \simeq \underline{f}'$) if for every compactum $C \subset X$ there is a compactum $D \subset Y$ such that for every neighbourhood V of D (in N) there is a neighbourhood U of C (in M) such that

(3.1) $$f_k/U \simeq f'_k/U \text{ in } V \quad \text{for almost all } k.$$

Every compactum D satisfying this condition is said to be *$(\underline{f}, \underline{f}')$-assigned* to the compactum C.

It is clear that

(3.2) *The relation of homotopy for W-sequences is reflexive, symmetric and transitive.*

Two W-sequences $\underline{f} = \{f_k, X, Y\}_{M,N}$ and $\underline{g} = \{g_k, X, Y\}_{M,N}$ are said to be *associated* if $f_k(x) = g_k(x)$ for every point $x \in X$. Just as for fundamental sequences, one proves that

(3.3) *Two associated W-sequences are homotopic.*

Moreover, one easily proves that

(3.4) *If $\underline{f}, \underline{f}'\colon X \to Y$ in M, N and $\underline{g}, \underline{g}'\colon Y \to Z$ in N, P are W-sequences, then the homotopies $\underline{f} \simeq \underline{f}'$ and $\underline{g} \simeq \underline{g}'$ imply the homotopy $\underline{gf} \simeq \underline{g'f'}$.*

Repeating the proof given for fundamental sequences in [1], p. 242, one easily shows that

(3.5) *Every W-sequence $\underline{f}\colon X \to Y$ in M, N induces a homomorphism $\underline{f}_{hl}\colon H_n(X, \mathfrak{A}) \to H_n(Y, \mathfrak{A})$ for $n = 0, 1, \dots$ and for every abelian group $\mathfrak{A}$. The homomorphism $\underline{f}_{hl}$ depends covariantly on $\underline{f}$. If $\underline{f} \simeq \underline{f}'$, then $\underline{f}_{hl} = \underline{f}'_{hl}$. If $\underline{f} = \underline{i}_{X,M}$, then $\underline{f}_{hl}$ is the identity isomorphism.*

§ 4. **Pointed W-sequences.** If (X, x_0) is an arbitrary pointed subset of a space $M \in \mathrm{AR}(\mathfrak{M})$ and (Y, y_0) is an arbitrary pointed subset of a space $N \in \mathrm{AR}(\mathfrak{M})$, then by a *pointed W-sequence* from (X, x_0) to (Y, y_0) in M, N we understand the system consisting of (X, x_0), (Y, y_0), M, N and of a sequence of maps

$$f_k\colon (M, x_0) \to (N, y_0), \quad k = 1, 2, \dots,$$

satisfying the condition:

(4.1) *For every pointed compactum $(C, x_0) \subset (X, x_0)$ there is a pointed compactum $(D, y_0) \subset (Y, y_0)$ such that for every neighbourhood V of D (in N) there is a neighbourhood U of C (in M) satisfying the condition $f_k/(U, x_0) \simeq f_{k+1}/(U, x_0)$ in (V, y_0) for almost all k.*

We denote this pointed W-sequence by $\{f_k, (X, x_0), (Y, y_0)\}_{M,N}$, or shortly by $\underline{f}$, and we write $\underline{f}\colon (X, x_0) \to (Y, y_0)$ in M, N. Every compactum D satisfying (4.1) is said to be $\underline{f}$-assigned to the compactum C.

It is clear that if $\{f_k, (X, x_0), (Y, y_0)\}_{M,N}$ is a pointed W-sequence, then $\{f_k, X, Y\}_{M,N}$ is a W-sequence and that every pointed fundamental sequence is a pointed W-sequence. Moreover, if X, Y are compacta, then the notion of the pointed W-sequence is the same as the notion of the pointed fundamental sequence.

If $(X, x_0) = (Y, y_0)$ and $M = N$, and if $i\colon M \to M$ denotes the identity map, then $\{i, (X, x_0), (X, x_0)\}_{M,M}$ is said to be the *identity pointed W-sequence* for (X, x_0) in M. It will be denoted by $\underline{i}_{(X,x_0),M}$.

The definition of the composition of W-sequences can be transferred directly to pointed W-sequences.

Two pointed W-sequences

$$\underline{f} = \{f_k, (X, x_0), (Y, y_0)\}_{M,N} \quad \text{and} \quad \underline{f}' = \{f'_k, (X, x_0), (Y, y_0)\}_{M,N}$$

are said to be *homotopic* (notation: $\underline{f} \simeq \underline{f}'$) if for every compactum $C \subset X$ there is a compactum $D \subset Y$ (which is said to be $(\underline{f}, \underline{f}')$-*assigned* to C) such that for every neighbourhood V of D (in N) there is a neighbourhood U of C (in M) satisfying the condition

(4.2) $$f_k/(U, x_0) \simeq f'_k/(U, x_0) \text{ in } (V, y_0) \quad \text{for almost all } k.$$

It is clear that this relation is reflexive, symmetric and transitive and that if we replace, in the composition $\underline{g}\underline{f}$ of two pointed W-sequences, $\underline{f}$ by $\underline{f}' \simeq \underline{f}$ and $\underline{g}$ by $\underline{g}' \simeq \underline{g}$, then $\underline{g}'\underline{f}' \simeq \underline{g}\underline{f}$.

One easily shows (by repeating the proof given in [1], p. 252 for pointed, fundamental sequences) that

(4.3) *Every pointed W-sequence $\underline{f}$: $(X, x_0) \to (Y, y_0)$ in M, N induces a homomorphism $\underline{f}_*$ of the n-th fundamental group $\underline{\pi}_n(X, x_0)$ into the n-th fundamental group $\underline{\pi}_n(Y, y_0)$. This homomorphism depends covariantly on $\underline{f}$ and if $\underline{f} \simeq \underline{f}'$, then $\underline{f}_* = \underline{f}'_*$. If $\underline{f} = \underline{i}_{(X,x_0),M}$, then $\underline{f}_*$ is the identity isomorphism.*

§ 5. W-equivalence and W-domination. Let X, Y be arbitrary sets lying in spaces $M, N \in \mathrm{AR}(\mathfrak{M})$ respectively. The sets X, Y are said to be *W-equivalent* in M, N (notation: $X \underset{W}{\simeq} Y$ in M, N) if there exist two W-sequences $\underline{f} = \{f_k, X, Y\}_{M,N}$, $\underline{g} = \{g_k, Y, X\}_{N,M}$ satisfying the following conditions:

(5.1) $$\underline{g}\underline{f} \simeq \underline{i}_{X,M} \quad \text{and} \quad \underline{f}\underline{g} \simeq \underline{i}_{Y,M}.$$

If we assume only that $\underline{f}, \underline{g}$ satisfy the condition

(5.2) $$\underline{f}\underline{g} \simeq \underline{i}_{Y,N},$$

then we say that X *W-dominates* Y in M, N and we write $X \underset{W}{\geqslant} Y$ in M, N, or $Y \underset{W}{\leqslant} X$ in M, N.

One easily sees that

(5.3) *If $X \underset{W}{\simeq} Y$ in M, N, then $X \underset{W}{\leqslant} Y$ in M, N.*

(5.4) *If $X \underset{W}{\simeq} Y$ in M, N and $Y \underset{W}{\simeq} Z$ in N, P, then $X \underset{W}{\simeq} Z$ in M, P.*

(5.5) *If $X \underset{W}{\leqslant} Y$ in M, N and $Y \underset{W}{\leqslant} Z$ in N, P, then $X \underset{W}{\leqslant} Z$ in M, P.*

(5.6) *If X is a retract of Y and Y is a closed subset of a space $N \in \mathrm{AR}(\mathfrak{M})$, then $X \underset{W}{\leqslant} Y$ in N, N.*

Replacing in the definitions of the W-equivalence and of the W-domination the sets X, Y by pointed sets (X, x_0), (Y, y_0) and the W-sequences $\underline{f}, \underline{g}$ by pointed W-sequences and the condition (5.1) by the condition

$$\underline{g}\underline{f} \simeq \underline{i}_{(X,x_0),M}, \quad \underline{f}\underline{g} \simeq \underline{i}_{(Y,y_0),N},$$

and the condition (5.2) by the condition

$$\underline{f}\underline{g} \simeq \underline{i}_{(Y,y_0),N}$$

respectively, one gets the notion of the *W-equivalence* in M, N of (X, x_0) with (Y, y_0) (notation: $(X, x_0) \underset{W}{\simeq} (Y, y_0)$ in M, N), or the notion of the *W-domination* in M, N of (X, x_0) over (Y, y_0) (notation: $(X, x_0) \underset{W}{\geqslant} (Y, y_0)$ in M, N, or $(Y, y_0) \underset{W}{\leqslant} (X, x_0)$ in M, N) respectively. It is clear that $(X, x_0) \underset{W}{\simeq} (Y, y_0)$ in M, N implies $X \underset{W}{\simeq} Y$ in M, N and that $(X, x_0) \underset{W}{\geqslant} (Y, y_0)$ in M, N implies $X \underset{W}{\geqslant} Y$ in M, N.

If there exist spaces $M, N \in \mathrm{AR}(\mathfrak{M})$ containing X and Y respectively and such that $X \underset{W}{\leqslant} Y$ in M, N (or $(X, x_0) \underset{W}{\leqslant} (Y, y_0)$ in M, N), then we say that X is *pseudo-dominated by* Y and we write $X \underset{P}{\leqslant} Y$ (or that (X, x_0) is *pseudo-dominated by* (Y, y_0) and we write $(X, x_0) \underset{P}{\leqslant} (Y, y_0)$). If the spaces M, N can be selected so that $X \underset{W}{\simeq} Y$ in M, N (or that $(X, x_0) \underset{W}{\simeq} (Y, y_0)$ in M, N), then we say that X and Y (or (X, x_0) and (Y, y_0)) are *pseudo-equivalent* and we write $X \underset{P}{\simeq} Y$ (or $(X, x_0) \underset{P}{\simeq} (Y, y_0)$ respectively).

(5.7) THEOREM. *If $X \underset{P}{\simeq} Y$ then the homology groups of X are isomorphic to the corresponding homology groups of Y.*

Proof. Since $X \underset{P}{\simeq} Y$, there exist spaces $M, N \in \mathrm{AR}(\mathfrak{M})$ containing X and Y respectively and such that there are W-sequences $\underline{f}$: $X \to Y$ in M, N and $\underline{g}$: $Y \to X$ in N, M such that $\underline{g}\underline{f} \underset{W}{\simeq} \underline{i}_{X,M}$ and $\underline{f}\underline{g} \underset{W}{\simeq} \underline{i}_{Y,N}$. Then (3.5) and (5.1) imply that the compositions $g_{hl}f_{hl}$ and $f_{hl}g_{hl}$ of the homomorphisms $\underline{f}_{hl}$: $H_n(X, \mathfrak{A}) \to H_n(Y, \mathfrak{A})$ and of $\underline{g}_{hl}$: $H_n(Y, \mathfrak{A}) \to H_n(X, \mathfrak{A})$ are identity-isomorphisms.

By the same argument one gets

(5.8) THEOREM. *If $X \underset{P}{\leqslant} Y$, then each homology group of X is an r-image of the corresponding homology group of Y.*

(5.9) Corollary. *If $X \underset{P}{\leqslant} Y$, then each homology group of X is a direct factor of the corresponding homology group of Y.*

Similarly, using (4.3), one gets:

(5.10) Theorem. *If $(X, x_0) \underset{P}{\simeq} (Y, y_0)$, then the fundamental group $\pi_n(X, x_0)$ is isomorphic to the fundamental group $\pi_n(Y, y_0)$ for $n = 1, 2, \ldots$*

(5.11) Theorem. *If $(X, x_0) \underset{P}{\leqslant} (Y, y_0)$, then the group $\pi_n(X, x_0)$ is an r-image of the group $\pi_n(Y, y_0)$ for $n = 1, 2, \ldots$*

§ 6. Homotopy similarity and W-similarity of pairs. Let A be an arbitrary subset of a space X and let B be an arbitrary subset of a space Y. The pairs (X, A) and (Y, B) are said to be *homotopically similar* (notation: $(X, A) \underset{H}{\leftrightarrow} (Y, B)$) if there exist two maps

$$f\colon X \to Y, \qquad g\colon Y \to X$$

such that

(6.1) $\quad f(A) \subset B, \quad f(X \setminus A) \subset Y \setminus B, \quad g(B) \subset A, \quad g(Y \setminus B) \subset X \setminus A$

and that

(6.2)
$$\begin{gathered} gf/A \simeq i_A \text{ in } A, \quad fg/B \simeq i_B \text{ in } B, \\ gf/(X \setminus A) \simeq i_{X \setminus A} \text{ in } X \setminus A, \quad fg/(Y \setminus B) \simeq i_{Y \setminus B} \text{ in } Y \setminus B. \end{gathered}$$

It is clear that if A has in X the same topological position as B in Y (notation: $(X, A) \underset{T}{\leftrightarrow} (Y, B)$), then $(X, A) \underset{H}{\leftrightarrow} (Y, B)$, but not conversely. For instance, if X, Y are disks, A is an arc lying in the boundary of X and B consists of only one point b lying on the boundary of Y, then one easily sees that $(X, A) \underset{H}{\leftrightarrow} (Y, B)$ but the relation $(X, A) \underset{T}{\leftrightarrow} (Y, B)$ does not hold. It is clear that $(X, A) \underset{H}{\leftrightarrow} (Y, B)$ implies that A and $X \setminus A$ are homotopically equivalent to B and to $Y \setminus B$ respectively. However, it is rather problematic whether the concept of the homotopy similarity furnishes a sufficient base for a classification of pairs (X, A) from the point of view of their most conspicuous, global properties, adequate to the needs of our intuition.

Thus let us replace this concept by another concept of similarity, based on the notion of the W-sequence.

Consider an arbitrary subset A of a space X and an arbitrary subset B of a space Y and assume that X, Y are closed subsets of two spaces $M, N \in \mathrm{AR}(\mathfrak{M})$ respectively. Let us say that (X, A), (Y, B) are *W-similar* in M, N (notation: $(X, A) \underset{W}{\leftrightarrow} (Y, B)$ in M, N) if there exist two sequences of maps:

6.3) $\quad f_k\colon M \to N \quad$ and $\quad g_k\colon N \to M, \quad k = 1, 2, \ldots$

such that

(6.4)
$$\begin{aligned} \underline{f}' &= \{f_k, A, B\}_{M,N}; & \underline{f}'' &= \{f_k, X \setminus A, Y \setminus B\}_{M,N}, \\ \underline{g}' &= \{g_k, B, A\}_{N,M}, & \underline{g}'' &= \{g_k, Y \setminus B, X \setminus A\}_{N,M}, \end{aligned}$$ *are W-sequences*

and that

(6.5)
$$\begin{aligned} \underline{g}'\underline{f}' &\simeq \underline{i}_{A,M}, & \underline{g}''\underline{f}'' &\simeq \underline{i}_{(X \setminus A),M} \\ \underline{f}'\underline{g}' &\simeq \underline{i}_{B,N}, & \underline{f}''\underline{g}'' &\simeq \underline{i}_{(Y \setminus B),N}. \end{aligned}$$

It is clear that

(6.6) *$(X, A) \underset{W}{\leftrightarrow} (Y, B)$ in M, N implies that $A \underset{W}{\simeq} B$ in M, N and $X \setminus A \underset{W}{\simeq} Y \setminus B$ in M, N.*

In the special case when $A = B = \emptyset$, the pairs (X, A), (Y, B) are considered as identical with X and Y respectively, and then the relation $X = (X, \emptyset) \underset{W}{\leftrightarrow} (Y, \emptyset) = Y$ in M, N is the same as the relation $X \underset{W}{\simeq} Y$ in M, N.

Let us prove that the relation $(X, A) \underset{W}{\leftrightarrow} (Y, B)$ in M, N does not depend on the choice of spaces $M, N \in \mathrm{AR}(\mathfrak{M})$ containing X, Y as closed subsets respectively. That is, let us prove the following

(6.7) Theorem. *Let A be a subset of a space X and let B be a subset of a space Y. Assume that X is a closed subset of a space $M \in \mathrm{AR}(\mathfrak{M})$ and also a closed subset of another space $M' \in \mathrm{AR}(\mathfrak{M})$, and that Y is a closed subset of a space $N \in \mathrm{AR}(\mathfrak{M})$ and also a closed subset of a space $N' \in \mathrm{AR}(\mathfrak{M})$. If $(X, A) \underset{W}{\leftrightarrow} (Y, B)$ in M, N, then $(X, A) \underset{W}{\leftrightarrow} (Y, B)$ in M', N'.*

Proof. Since $M, M', N, N' \in \mathrm{AR}(\mathfrak{M})$, there exist four maps

$$\alpha\colon M \to M', \quad \alpha'\colon M' \to M, \quad \beta\colon N \to N', \quad \beta'\colon N' \to N$$

such that $\alpha(x) = \alpha'(x) = x$ for every point $x \in X$ and $\beta(y) = \beta'(y) = y$ for every point $y \in Y$. It is clear that setting $\alpha_k = \alpha$, $\alpha'_k = \alpha'$, $\beta_k = \beta$, $\beta'_k = \beta'$ for $k = 1, 2, \ldots$, one gets for every set $H \subset M \cap M'$ and for every set $K \subset N \cap N'$ the following W-sequences:

$$\begin{aligned} \underline{\alpha}_H &= \{\alpha_k, H, H\}_{M,M'}, & \underline{\alpha}'_H &= \{\alpha'_k, H, H\}_{M',M}, \\ \underline{\beta}_K &= \{\beta_k, K, K\}_{N,N'}, & \underline{\beta}'_K &= \{\beta'_k, K, K\}_{N',N}. \end{aligned}$$

It follows by (3.3) that

(6.8)
$$\begin{aligned} \underline{\alpha}_H\underline{\alpha}'_H &\simeq \underline{i}_{H,M'}, & \underline{\alpha}'_H\underline{\alpha}_H &\simeq \underline{i}_{H,M}, \\ \underline{\beta}_K\underline{\beta}'_K &\simeq \underline{i}_{K,N'}, & \underline{\beta}'_K\underline{\beta}_K &\simeq \underline{i}_{K,N}. \end{aligned}$$

Now let us assume that $(X, A) \underset{W}{\leftrightarrow} (Y, B)$ in M, N, that is that there exist two sequences of maps $f_k\colon M \to N$ and $g_k\colon N \to M$, $k = 1, 2, \ldots$ satisfying (6.4) and (6.5). Setting

$$\hat{f}_k = \beta f_k \alpha', \qquad \hat{g}_k = \alpha g_k \beta' \qquad \text{for} \qquad k = 1, 2, \ldots,$$

we infer by § 3 that

$$\underline{\hat{f}}' = \{\hat{f}_k, A, B\}_{M', N'} = \underline{\beta}_B \underline{f}' \underline{\alpha}'_A,$$
$$\underline{\hat{g}}' = \{\hat{g}_k, B, A\}_{N', M'} = \underline{\alpha}_A \underline{g}' \underline{\beta}'_B,$$
$$\underline{\hat{f}}'' = \{\hat{f}_k, X \setminus A, Y \setminus B\}_{M', N'} = \underline{\beta}_{Y \setminus B} \underline{f}'' \underline{\alpha}'_{X \setminus A},$$
$$\underline{\hat{g}}'' = \{\hat{g}_k, Y \setminus B, X \setminus A\}_{N', M'} = \underline{\alpha}_{X \setminus A} \underline{g}'' \underline{\beta}'_{Y \setminus B}$$

are W-sequences. Moreover, (6.5), (6.8) and (3.4) imply that

$$\underline{\hat{g}}' \underline{\hat{f}}' = \underline{\alpha}_A \underline{g}' \underline{\beta}'_B \underline{\beta}_B \underline{f}' \underline{\alpha}'_A \simeq \underline{\alpha}_A \underline{g}' \underline{f}' \underline{\alpha}'_A \simeq \underline{\alpha}_A \underline{\alpha}'_A \simeq \underline{i}_{A, M'},$$
$$\underline{\hat{f}}' \underline{\hat{g}}' = \underline{\beta}_B \underline{f}' \underline{\alpha}'_A \underline{\alpha}_A \underline{g}' \underline{\beta}'_B \simeq \underline{\beta}_B \underline{f}' \underline{g}' \underline{\beta}'_B \simeq \underline{\beta}_B \underline{\beta}'_B \simeq \underline{i}_{B, N'},$$
$$\underline{\hat{g}}'' \underline{\hat{f}}'' = \underline{\alpha}_{X \setminus A} \underline{g}'' \underline{\beta}'_{Y \setminus B} \underline{\beta}_{Y \setminus B} \underline{f}'' \underline{\alpha}'_{X \setminus A} \simeq \underline{\alpha}_{X \setminus A} \underline{g}'' \underline{f}'' \underline{\alpha}'_{X \setminus A} \simeq \underline{\alpha}_{X \setminus A} \underline{\alpha}'_{X \setminus A} \simeq \underline{i}_{X \setminus A, M'},$$
$$\underline{\hat{f}}'' \underline{\hat{g}}'' = \underline{\beta}_{Y \setminus B} \underline{f}'' \underline{\alpha}'_{X \setminus A} \underline{\alpha}_{X \setminus A} \underline{g}'' \underline{\beta}'_{Y \setminus B} \simeq \underline{\beta}_{Y \setminus B} \underline{f}'' \underline{g}'' \underline{\beta}'_{Y \setminus B} \simeq \underline{\beta}_{Y \setminus B} \underline{\beta}'_{Y \setminus B} \simeq \underline{i}_{Y \setminus B, N'}.$$

It follows that $(X, A) \underset{W}{\leftrightarrow} (Y, B)$ in M', N'. Thus Theorem (6.7) is proved.

Theorem (6.7) allows us in the sequel to omit in the relation "$\underset{W}{\leftrightarrow}$" the words "in M, N", that is to write $(X, A) \underset{W}{\leftrightarrow} (Y, B)$ instead of $(X, A) \underset{W}{\leftrightarrow} (Y, B)$ in M, N.

Let us observe that

(6.9) *If* $(X, A) \underset{H}{\leftrightarrow} (Y, B)$ *then* $(X, A) \underset{W}{\leftrightarrow} (Y, B)$.

In fact, if $(X, A) \underset{H}{\leftrightarrow} (Y, B)$, then there exist maps $f\colon X \to Y$ and $g\colon Y \to X$ satisfying (6.1) and (6.2). Let $M, N \in \mathrm{AR}(\mathfrak{M})$ be spaces containing X and Y respectively as closed subsets. Then there exist maps

$$\hat{f}\colon M \to N, \qquad \hat{g}\colon N \to M$$

such that $\hat{f}(x) = f(x)$ for every point $x \in X$ and $\hat{g}(y) = g(y)$ for every point $y \in Y$. Setting $f_k = \hat{f}$ and $g_k = \hat{g}$ for $k = 1, 2, \ldots$, one gets two W-sequences and one easily sees that the conditions (6.4) and (6.5) are satisfied.

§ 7. Position of a set A in a space X. It is clear that the relation "$\underset{W}{\leftrightarrow}$" is reflexive and symmetric. Let us show that it is also transitive, that is let us prove the following

(7.1) THEOREM. *If* $(X, A) \underset{W}{\leftrightarrow} (Y, B)$ *and* $(Y, B) \underset{W}{\leftrightarrow} (Z, C)$, *then* $(X, A) \underset{W}{\leftrightarrow} (Z, C)$.

Proof. Consider space $M, N, P \in \mathrm{AR}(\mathfrak{M})$ containing X, Y, Z as closed subsets respectively. Assume that $(X, A) \underset{W}{\leftrightarrow} (Y, B)$ and $(Y, B) \underset{W}{\leftrightarrow} (Z, C)$. Then there exist sequences of maps $f_k\colon M \to N$, $g_k\colon N \to M$, $k = 1, 2, \ldots$ such that $\underline{f}', \underline{g}', \underline{f}'', \underline{g}''$, given by the formulas (6.4), are W-sequences satisfying conditions (6.5). Moreover, there exist sequences of maps $\varphi_k\colon N \to P$, $\psi_k\colon P \to N$, $k = 1, 2, \ldots$ such that the formulas

$$\underline{\varphi}' = \{\varphi_k, B, C\}_{N, P}, \qquad \underline{\psi}' = \{\psi_k, C, B\}_{P, N},$$
$$\underline{\varphi}'' = \{\varphi_k, Y \setminus B, Z \setminus C\}_{N, P}, \qquad \underline{\psi}'' = \{\psi_k, Z \setminus C, Y \setminus B\}_{P, N}$$

define W-sequences satisfying the conditions

$$\text{(7.2)} \qquad \begin{aligned} \underline{\psi}' \underline{\varphi}' &\simeq \underline{i}_{B, N}, & \underline{\varphi}' \underline{\psi}' &\simeq \underline{i}_{C, P}, \\ \underline{\psi}'' \underline{\varphi}'' &\simeq \underline{i}_{Y \setminus B, N}, & \underline{\varphi}'' \underline{\psi}'' &\simeq \underline{i}_{Z \setminus C, P}. \end{aligned}$$

Then

$$\underline{\chi}' = \{\varphi_k f_k, A, C\}_{M, P} = \underline{\varphi}' \underline{f}', \qquad \underline{\vartheta}' = \{g_k \psi_k, C, A\}_{P, M} = \underline{g}' \underline{\psi}',$$
$$\underline{\chi}'' = \{\varphi_k f_k, X \setminus A, Z \setminus C\}_{M, P} = \underline{\varphi}'' \underline{f}'', \qquad \underline{\vartheta}'' = \{g_k \psi_k Z \setminus C, X \setminus A\}_{P, M} = \underline{g}'' \underline{\psi}''$$

are W-sequences. Moreover, we infer by (6.5), (7.2) and (3.4) that

$$\underline{\vartheta}' \underline{\chi}' = \underline{g}' \underline{\psi}' \underline{\varphi}' \underline{f}' \simeq \underline{g}' \underline{f}' \simeq \underline{i}_{A, M},$$

and similarly that

$$\underline{\chi}' \vartheta' \simeq \underline{i}_{C, P}, \qquad \underline{\vartheta}'' \underline{\chi}'' \simeq \underline{i}_{X \setminus A, M}, \qquad \underline{\chi}'' \underline{\vartheta}'' \simeq \underline{i}_{Z \setminus C, P}.$$

Hence $(X, A) \underset{W}{\leftrightarrow} (Z, C)$ and Theorem (7.1) is proved.

It follows by Theorem (7.1) that the collection of all pairs (X, A), where A is a subset of a space X, decomposes into disjoint classes of pairs, where (X, A) and (X', A') belong to one of these classes if and only if $(X, A) \underset{W}{\leftrightarrow} (X', A')$. The class containing (X, A) will be denoted by $\mathrm{Pos}(X, A)$ and called the *position of the set A in the space X*. Thus $\mathrm{Pos}(X, A) = \mathrm{Pos}(Y, B)$ means that $(X, A) \underset{W}{\leftrightarrow} (Y, B)$.

A direct consequence of this definition is that

(7.3) *If* $\mathrm{Pos}(X, A) = \mathrm{Pos}(Y, B)$, *then* $A \underset{P}{\simeq} B$ *and* $X \setminus A \underset{P}{\simeq} Y \setminus B$.

EXAMPLE. A map f of a space X into another space Y is said to be *homologically trivial* if there exists a compactum $D \subset Y$ such that

$$f(\underline{\gamma}) \sim 0 \text{ in } D \text{ for every true cycle } \underline{\gamma} \text{ lying in } X.$$

Consider in the space E^3 a geometric circle A and a trefoil B. Let us recall the following two well-known facts:

(7.4) *Every homologically trivial map* $\varphi\colon A \to E^3 \setminus A$ *is null-homotopic.*

(7.5) *There exists a map* $\psi\colon A \to E^3 \setminus B$ *which is homologically trivial, but it is not null-homotopic.*

Now let us show that $\operatorname{Pos}(E^3, A) \neq \operatorname{Pos}(E^3, B)$. Otherwise there exist two W-sequences

$$\underline{f} = \{f_k, E^3 \setminus A, E^3 \setminus B\}_{E^3, E^3}, \quad \underline{g} = \{g_k, E^3 \setminus B, E^3 \setminus A\}_{E^3, E^3}$$

satisfying the relations

(7.6) $$\underline{gf} \simeq \underline{i}_{E^3 \setminus A} \quad \text{and} \quad \underline{fg} \simeq \underline{i}_{E^3 \setminus B}.$$

Let $\psi\colon A \to E^3 \setminus B$ be a map satisfying (7.5) and let $D \subset E^3 \setminus B$ be a compactum such that

(7.7) $\psi(\underline{\gamma}) \sim 0$ *in* D *for every true cycle* $\underline{\gamma}$ *lying in* A.

Since $\underline{g}$ is a W-sequence, there exists a compactum $C \subset E^3 \setminus A$ such that

(7.8) $$g_k(D) \subset C \quad \text{for almost all } k,$$

and that the map $g_k\psi\colon A \to E^3$ assigns to every true cycle $\underline{\gamma}$ lying in A a true cycle $g_k\psi(\underline{\gamma}) \sim 0$ in C. Hence the map $g'_k\colon A \to E^3 \setminus A$ given by the formula $g'_k(x) = g_k\psi(x)$ for every point $x \in A$, is homologically trivial and we infer by (7.4) that

(7.9) g'_k *is null-homotopic.*

Since $\underline{f}$ is a W-sequence, there exists a compactum $\hat{D} \subset E^3 \setminus B$ such that $f_k(C) \subset \hat{D}$ for almost all k. One infers by (7.8) and (7.9) that

(7.10) *For almost all* k *the map* $f_k g_k \psi\colon A \to E^3$ *is null-homotopic in* $\hat{D}$.

But the second of the relations (7.6) implies that for almost all k the map $f_k g_k / \hat{D}$ is homotopic in $E^3 \setminus B$ to the map $i/\hat{D}$. It follows that $\psi \simeq f_k g_k \psi$ in $E^3 \setminus B$, and we infer by (7.10) that the map ψ is null-homotopic in $E^3 \setminus B$, contrary to (7.5). Thus the supposition that $\operatorname{Pos}(E^3, A) = \operatorname{Pos}(E^3, B)$ leads to a contradiction; hence $\operatorname{Pos}(E^3, A) \neq \operatorname{Pos}(E^3, B)$.

(7.11) Remark. The following example shows that in general the relations $(X, A) \underset{W}{\leftrightarrows} (Y, B)$ and $(A, A_0) \underset{W}{\leftrightarrows} (B, B_0)$ do not imply the relation $(X, A_0) \underset{W}{\leftrightarrows} (Y, B_0)$. Let $X = Y = E^2$ and let $A = B$ be the annulus consisting of all points $(x, y) \in E^2$ satisfying the inequality $1 \leqslant x^2 + y^2 \leqslant 9$. Let C denote the circle given in E^2 by the equation $x^2 + y^2 = 4$ and let $p = (\frac{3}{2}, 0)$, $q = (\frac{5}{2}, 0)$. Setting $A_0 = C \cup (p)$, $B_0 = C \cup (q)$, one can readily see that $(A, A_0) \underset{T}{\leftrightarrows} (B, B_0)$, whence also $(A, A_0) \underset{W}{\leftrightarrows} (B, B_0)$. However, (E^2, A_0) and (E^2, B_0) are not W-similar, because for each component of the set $E^2 \setminus A_0$ the first Betti number is positive and $E^2 \setminus B_0$ contains a component with the first Betti number vanishing.

(7.12) Remark. It follows by (7.3) and by Theorem (5.7) that $\operatorname{Pos}(X, A) = \operatorname{Pos}(Y, B)$ implies that all homology groups of A are isomorphic to the corresponding groups of B, and all homology groups of $X \setminus A$ are isomorphic to the corresponding groups of $Y \setminus B$. The following problems remain open:

Does $\operatorname{Pos}(X, A) = \operatorname{Pos}(Y, B)$ *imply that all homology groups of* X *are isomorphic to the corresponding groups of* Y?

Does $\operatorname{Pos}(X, A) = \operatorname{Pos}(Y, B)$ *imply that there are points* $a \in A$, $b \in B$, $x_0 \in X \setminus A$, $y_0 \in Y \setminus B$ *such that* $\underline{\pi}_n(A, a) \simeq \underline{\pi}_n(B, b)$ *and* $\underline{\pi}_n(X \setminus A, x_0) \simeq \underline{\pi}_n(Y \setminus B, y_0)$ *for every* $n = 1, 2, \ldots$?

§ 8. Similar decreasing sequences of sets.

Let $A_1 \supset A_2 \supset \ldots$ be a sequence of subsets of a space X and let $B_1 \supset B_2 \supset \ldots$ be a sequence of subsets of another space Y. These sequences are said to be *similar* (in X, Y) if there exists a sequence of homeomorphisms

$$f_k\colon X \xrightarrow[\text{onto}]{} Y, \quad k = 1, 2, \ldots$$

satisfying the following conditions:

(8.1) $$f_k(A_k) = B_k \quad \text{for} \quad k = 1, 2, \ldots,$$

(8.2) $$f_k / A_k \simeq f_{k'} / A_k \text{ in } B_k \quad \text{for every } k \leqslant k',$$

(8.3) $$f_k / (X \setminus A_k) = f_{k'} / (X \setminus A_k) \quad \text{for every } k \leqslant k'.$$

A sequence $Z_1, Z_2, \ldots$ of subsets of a space Z is said to be *strongly decreasing* in Z if Z_{k+1} lies in the interior $\mathring{Z}_k$ of Z_k (interior relative to the space Z) for every $k = 1, 2, \ldots$

Let us prove the following

(8.4) THEOREM. *Let* X, Y *be two homeomorphic* $\operatorname{ANR}(\mathfrak{M})$*-spaces, let* $\{A_k\}$ *be a sequence, strongly decreasing (in* X*), of compacta lying in* X *and let* $\{B_k\}$ *be a sequence of subsets of* Y *similar to* $\{A_k\}$ *(in* X, Y*). Then* $\operatorname{Pos}(X, \bigcap_{k=1}^{\infty} A_k) = \operatorname{Pos}(Y, \bigcap_{k=1}^{\infty} B_k)$.

Proof. We may assume that X is a closed subset of a space $M \in \mathrm{AR}(\mathfrak{M})$ and Y is a closed subset of a space $N \in \mathrm{AR}(\mathfrak{M})$. Setting

$$\hat{M} = (X \times \langle 0, 1\rangle) \cup (M \times (1)), \qquad \hat{N} = (Y \times \langle 0, 1\rangle) \cup (N \times (1)),$$

one readily sees that $\hat{M}$ and $\hat{N}$ are contractible $\mathrm{ANR}(\mathfrak{M})$-spaces, and hence $\mathrm{AR}(\mathfrak{M})$-spaces. Let us identify each point $x \in X$ with the point $(x, 0) \in \hat{M}$ and each point $y \in Y$ with the point $(y, 0) \in \hat{N}$. Hence X is a closed subset of $\hat{M}$ and Y is a closed subset of $\hat{N}$.

Since $\{B_k\}$ is similar to $\{A_k\}$ (in X, Y), we infer that $\{B_k\}$ is a strongly decreasing (in Y) sequence of compacta. Setting:

$$A = \bigcap_{k=1}^{\infty} A_k, \qquad B = \bigcap_{k=1}^{\infty} B_k,$$

$$\hat{A}_k = A_k \times \left\langle 0, \frac{1}{k+1}\right\rangle, \qquad \hat{B}_k = B_k \times \left\langle 0, \frac{1}{k+1}\right\rangle \quad \text{for} \quad k = 1, 2, \ldots,$$

one readily sees that the sets $\hat{A}_k$ are neighbourhoods of the compactum A (in $\hat{M}$) shrinking to A and the sets $\hat{B}_k$ are neighbourhoods of the compactum B (in $\hat{N}$) shrinking to B.

Let f_k: $X \to Y$, $k = 1, 2, \ldots$, be a sequence of homeomorphisms satisfying the conditions (8.1), (8.2) and (8.3), and let $g_k = f_k^{-1}$ for every $k = 1, 2, \ldots$ Since X is closed in M and $N \in \mathrm{AR}(\mathfrak{M})$, there exists a map $\bar{f}_k$: $M \to N$ such that $\bar{f}_k(x) = f_k(x)$ for every point $x \in X$. Similarly one infers that there exists a map $\bar{g}_k$: $N \to M$ such that $\bar{g}_k(y) = g_k(y)$ for every point $y \in Y$. Setting

$$\hat{f}_k(\hat{x}) = \bar{f}_k(x) \quad \text{for every point } \hat{x} = (x, t) \in \hat{M},$$

$$\hat{g}_k(\hat{y}) = \bar{g}_k(y) \quad \text{for every point } \hat{y} = (y, t) \in \hat{N},$$

one gets maps $\hat{f}_k$: $\hat{M} \to \hat{N}$ and $\hat{g}_k$: $\hat{N} \to \hat{M}$. If we recall the conditions (8.2) and (8.3), then we infer that

(8.5) *If $k \leqslant k'$, then $\hat{f}_k/\hat{A}_k \simeq \hat{f}_{k'}/\hat{A}_k$ in $B_k \subset \hat{B}_k$ and $\hat{g}_k/\hat{B}_k \simeq \hat{g}_{k'}/\hat{B}_k$ in $A_k \subset \hat{A}_k$.*

(8.6) *If $k \leqslant k'$, then $\hat{f}_k(\hat{x}) = \hat{f}_{k'}(\hat{x})$ for every $\hat{x} = (x, t)$ with $x \in X \setminus A_k$ and $\hat{g}_k(\hat{y}) = \hat{g}_{k'}(\hat{y})$ for every $\hat{y} = (y, t)$ with $y \in Y \setminus B_k$.*

Consider a neighbourhood $\hat{V}$ of the set B in the space $\hat{N}$. Then there is an index k_0 such that $\hat{B}_{k_0} \subset \hat{V}$. Setting $\hat{U} = \hat{A}_{k_0}$, we get a neighbourhood $\hat{U}$ of the set A in $\hat{M}$ and we infer by (8.5) that

$$\hat{f}_k/\hat{U} \simeq \hat{f}_{k_0}/\hat{U} \text{ in } \hat{B}_{k_0} \subset \hat{V} \quad \text{for every } k \geqslant k_0.$$

Hence $\underline{\hat{f}}' = \{\hat{f}_k, A, B\}_{\hat{M}, \hat{N}}$ is a W-sequence.

In order to prove that $\underline{\hat{f}}'' = \{\hat{f}_k, X \setminus A, Y \setminus B\}_{\hat{M}, \hat{N}}$ is a W-sequence, consider a compactum $C \subset X \setminus A$. Since the sequence $\{A_k\}$ shrinks to A, there exists an index k_1 such that $C \subset X \setminus A_{k_1}$. Let G denote the subset of $\hat{M}$ consisting of all points $\hat{x} = (x, t)$ with $x \in X \setminus A_{k_1}$. Then G is a neighbourhood of C in $\hat{M}$ and we infer by (8.6) that

(8.7) $$\hat{f}_k/G = \hat{f}_{k'}/G \quad \text{for every } k \leqslant k'.$$

Setting $D = f_{k_1}(C)$, we get a compactum $D \subset Y \setminus B$. If $\hat{V}$ is a neighbourhood of D in the space $\hat{N}$, then there is a neighbourhood $\hat{U} \subset G$ of C in the space $\hat{M}$ such that $\hat{f}_{k_1}(\hat{U}) \subset \hat{V}$. It follows by (8.7) that $\hat{f}_k/\hat{U} = \hat{f}_{k_1}/\hat{U}$ for every $k \geqslant k_1$, and consequently $\hat{f}_k/\hat{U} \simeq \hat{f}_{k+1}/\hat{U}$ in $\hat{V}$ for almost all k. Hence $\underline{\hat{f}}''$ is a W-sequence.

By an analogous argument one proves that $\underline{\hat{g}}' = \{\hat{g}_k, B, A\}_{\hat{N}, \hat{M}}$ and $\underline{\hat{g}}'' = \{\hat{g}_k, Y \setminus B, X \setminus A\}_{\hat{N}, \hat{M}}$ are W-sequences. Moreover

$$\underline{\hat{g}}'\underline{\hat{f}}' = \{\hat{g}_k\hat{f}_k, A, A\}_{\hat{M}, \hat{M}} \simeq \underline{i}_{A, \hat{M}},$$

$$\underline{\hat{f}}'\underline{\hat{g}}' = \{\hat{f}_k\hat{g}_k, B, B\}_{\hat{N}, \hat{N}} \simeq \underline{i}_{B, \hat{N}},$$

$$\underline{\hat{g}}''\underline{\hat{f}}'' = \{\hat{g}_k\hat{f}_k, X \setminus A, X \setminus A\}_{\hat{M}, \hat{M}} \simeq \underline{i}_{X \setminus A, \hat{M}},$$

$$\underline{\hat{f}}''\underline{\hat{g}}'' = \{\hat{f}_k\hat{g}_k, Y \setminus B, Y \setminus B\}_{\hat{N}, \hat{N}} \simeq \underline{i}_{Y \setminus B, \hat{N}},$$

because $\hat{g}_k\hat{f}_k(x) = g_k f_k(x) = x$ for every point $x \in X$ and $\hat{f}_k\hat{g}_k(y) = y$ for every point $y \in Y$. Hence $(X, A) \underset{W}{\leftrightarrow} (Y, B)$ in $\hat{M}, \hat{N}$ and consequently $\mathrm{Pos}(X, A) = \mathrm{Pos}(Y, B)$.

(8.8) Problem. *Does Theorem* (8.4) *remain true if we omit the hypothesis that $X, Y \in \mathrm{ANR}(\mathfrak{M})$?*

Let us illustrate Theorem (8.4) by the following

(8.9) Example. *Let A denote the circle defined as the set of all points $x = (x_1, x_2, x_3) \in E^3$ satisfying the equations:*

$$x_1^2 + x_2^2 = 4 \quad \text{and} \quad x_3 = 0,$$

and let A_k denote, for $k = 1, 2, \ldots$, the torus consisting of all points $x \in E^3$ with $\varrho(x, A) \leqslant 1/k$. Then $A = \bigcap_{k=1}^{\infty} A_k$.

If B is an arbitrary continuum with $p_1(B) = 1$, lying in the plane E^2, then there exists in E^2 a strongly decreasing sequence of topological annuli $\{B'_k\}$ such that $B = \bigcap_{k=1}^{\infty} B'_k$. Assume that E^2 is a subset of E^3 defined by the equation $x_3 = 0$. Let B_k denote the set of all points $(x_1, x_2, x_3) \in E^3$ such that $(x_1, x_2, 0) \in B'_k$ and $-1/k \leqslant x_3 \leqslant 1/k$. It is clear that $\{B_k\}$ is a strongly decreasing (in E^3) sequence of compacta such that $\bigcap_{k=1}^{\infty} B_k = B$. Moreover, one easily sees that there exists a homeomorphism

$$h\colon E^3 \setminus A \xrightarrow[\text{onto}]{} E^3 \setminus B$$

such that for every $k = 1, 2, \ldots, h$ maps the boundary $\dot{A}_k$ of A_k onto the boundary $\dot{B}_k$ of B_k and that there is a homeomorphism

$$h_k\colon A_k \xrightarrow[\text{onto}]{} B_k$$

such that $h_k(x) = h(x)$ for every point $x \in \dot{A}_k$.

Setting, for every $k = 1, 2, \ldots,$

$$f_k(x) = h_k(x) \quad \text{for every point } x \in A_k,$$
$$f_k(x) = h(x) \quad \text{for every point } x \in E^3 \setminus A_k,$$

we get a sequence of homeomorphisms $f_k\colon E^3 \xrightarrow[\text{onto}]{} E^3$ satisfying (for $X = E^3$) the conditions (8.1), (8.2) and (8.3). It follows that

(8.9) *If $k \leqslant k'$, then $f_k^{-1}/B_k \simeq f_{k'}^{-1}/B_k$ in A_k and $f_k^{-1}/(E^3 \setminus B_k) = f_{k'}^{-1}/(E^3 \setminus B_k)$.*

Now let us consider a simple closed polygonal curve K lying in E^3. Then there exists a homeomorphism g mapping the torus A_1 onto a set $T \subset E^3$ so that $g(A) = K$. Setting

$$\hat{g}(x) = f_1 g^{-1}(x) \quad \text{for every point } x \in T,$$

we get a homeomorphism $\hat{g}\colon T \to B_1$ such that

$$\hat{g}(x) = hg^{-1}(x) \quad \text{for every point } x \in \dot{T},$$

where $\dot{T}$ denotes the boundary of T. Setting

$$\hat{A}_k = g(A_k) \quad \text{and} \quad \hat{B}_k = \hat{g}^{-1}(B_k) \quad \text{for} \quad k = 1, 2, \ldots,$$

we get two strongly decreasing sequences $\{\hat{A}_k\}$, $\{\hat{B}_k\}$ of compacta lying in E^3 and such that $\hat{A}_1 = g(A_1) = T = \hat{g}^{-1}(B_1) = \hat{B}_1$. Moreover,

$$\hat{A} = \bigcap_{k=1}^{\infty} \hat{A}_k = g(A) = K \quad \text{and} \quad \hat{B} = \bigcap_{k=1}^{\infty} \hat{B}_k = \hat{g}^{-1}(B).$$

Now let us set, for every $k = 1, 2, \ldots,$

$$\hat{f}_k(x) = g f_k^{-1} \hat{g}(x) \quad \text{for every point } x \in T,$$
$$\hat{f}_k(x) = x \quad \text{for every point } x \in E^3 \setminus T.$$

These formulas define a map $\hat{f}_k\colon E^3 \to E^3$, because if $x \in \dot{T}$, then

$$g f_k^{-1} \hat{g}(x) = g h^{-1} h g^{-1}(x) = x.$$

Moreover, one easily sees that $\hat{f}_k$ is a homeomorphism and that

$$\hat{f}_k(\hat{B}_k) = g f_k^{-1} \hat{g} \hat{g}^{-1}(B_k) = g f_k^{-1}(B_k) = g(A_k) = \hat{A}_k.$$

Finally, if $k \leqslant k'$, then $\hat{g}(\hat{B}_k) = \hat{g}\hat{g}^{-1}(B_k) = B_k$ and we infer by (8.10) that

$$\hat{f}_k/\hat{B}_k \simeq \hat{f}_{k'}/\hat{B}_k \text{ in } \hat{A}_k \quad \text{and} \quad \hat{f}_k/(E^3 \setminus \hat{B}_k) = \hat{f}_{k'}/(E^3 \setminus \hat{B}_k).$$

It follows that $\{\hat{A}_k\}$ and $\{\hat{B}_k\}$ are strongly decreasing sequences of compacta similar (in E^3, E^3) and we infer by Theorem (8.4) that

$$\mathrm{Pos}(E^3, K) = \mathrm{Pos}(E^3, \hat{B}).$$

Since the set $\hat{B} = \hat{g}^{-1}(B)$ is homeomorphic to B, we have shown that for any simple closed polygonal curve K lying in E^3 and for any plane continuum B with $p_1(B) = 1$ there exists a set $\hat{B} \subset E^3$ homeomorphic to B and such that $\mathrm{Pos}(E^3, \hat{B}) = \mathrm{Pos}(E^3, K)$. Thus, from the intuitive point of view, one may say that on *every plane continuum B with $p_1(B) = 1$ one can tie a knot similar to the given polygonal knot.*

§ 9. Positions of continua in the plane. As an application of Theorem (8.4), let us prove the following

(9.1) Theorem. *Let A, B be two continua lying in the plane E^2. Then $\mathrm{Pos}(E^2, A) = \mathrm{Pos}(E^2, B)$ if and only if $\mathrm{Sh}(A) = \mathrm{Sh}(B)$.*

Proof. If $\mathrm{Pos}(E^2, A) = \mathrm{Pos}(E^2, B)$, then (7.3) implies that $\mathrm{Sh}(A) = \mathrm{Sh}(B)$. It remains to prove the converse. We limit ourselves to the case where the number of components of the set $E^2 \setminus A$ and the number of components of the set $E^2 \setminus B$ are infinite.

Let us arrange the components of the set $E^2 \setminus A$ into a sequence $G_0, G_1, \ldots$, and the components of the set $E^2 \setminus B$ into a sequence $H_0, H_1, \ldots$ such that G_0 and H_0 are unbounded, and that $G_i \neq G_j$, $H_i \neq H_j$ for $i \neq j$. By a *k-perforated disk* we understand a subcompactum of E^2 with the boundary being the union of $k+1$ disjoint simple closed curves. One can easily see that for every $k = 1, 2, \ldots$ there exist two k-perforated disks: A_k with the boundary being the union of simple closed curves $C_{k,0}, C_{k,1}, \ldots, C_{k,k}$ and B_k with the boundary being the union of simple closed curves $D_{k,0}, D_{k,1}, \ldots, D_{k,k}$ satisfying the following conditions:

(9.2) $\quad C_{k,i} \subset G_i$ and $D_{k,i} \subset H_i$ for $i = 0, 1, \ldots, k$.

(9.3) *The sequences $\{A_k\}$ and $\{B_k\}$ are strongly decreasing and $A = \bigcap_{k=1}^{\infty} A_k$, $B = \bigcap_{k=1}^{\infty} B_k$.*

One readily sees that there exists a sequence of homeomorphisms

$$f_k\colon E^2 \xrightarrow[\text{onto}]{} E^2, \quad k = 1, 2, \ldots,$$

preserving the given orientation of E^2 and such that

(9.4) $$f_k(C_{k,i}) = D_{k,i} \quad \text{for} \quad i = 0, 1, \dots, k$$

and

(9.5) $$f_{k+1}/(E^2 \setminus A_k) = f_k/(E^2 \setminus A_k) \quad \text{for} \quad k = 1, 2, \dots$$

It follows that

(9.6) $$f_k(A_k) = B_k \quad \text{for} \quad k = 1, 2, \dots$$

and that

(9.7) $$f_k/A_k \simeq f_{k'}/A_k \text{ in } B_k \quad \text{for} \quad k \leqslant k'.$$

The relations (9.6), (9.7) and (9.5) are the same as the relations (8.1), (8.2) and (8.3). Thus it suffices to apply Theorem (8.4) in order to infer that $\mathrm{Pos}(E^2, A) = \mathrm{Pos}(E^2, B)$.

§ 10. Position and Cartesian product. Let us prove the following

(10.1) THEOREM. *If* $\mathrm{Pos}(X, A) = \mathrm{Pos}(Y, B)$ *and if* $\mathrm{Sh}_W(\hat{X}) = \mathrm{Sh}_W(\hat{Y})$, *then* $\mathrm{Pos}(X \times \hat{X}, A \times \hat{X}) = \mathrm{Pos}(Y \times \hat{Y}, B \times \hat{Y})$.

Proof. Assume that $X, Y, \hat{X}, \hat{Y}$ are closed subsets of spaces $M, N, \hat{M}, \hat{N} \in \mathrm{AR}(\mathfrak{M})$ respectively. By our hypotheses there exist four sequences of maps: $f_k\colon M \to N$, $g_k\colon N \to M$, $\hat{f}_k\colon \hat{M} \to \hat{N}$, $\hat{g}_k\colon \hat{N} \to \hat{M}$ $(k = 1, 2, \dots)$ such that

(10.2) $$\begin{aligned} &\underline{f}' = \{f_k, A, B\}_{M,N}, && \underline{g}' = \{g_k, B, A\}_{N,M}, \\ &\underline{f}'' = \{f_k, X\setminus A, Y\setminus B\}_{M,N}, && \underline{g}'' = \{g_k, Y\setminus B, X\setminus A\}_{N,M}, \\ &\underline{\hat{f}} = \{\hat{f}_k, \hat{X}, \hat{Y}\}_{\hat{M},\hat{N}} \quad \text{and} && \underline{\hat{g}} = \{\hat{g}_k, \hat{Y}, \hat{X}\}_{\hat{N},\hat{M}} \end{aligned}$$

are W-sequences

such that

(10.3) $$\underline{g}'\underline{f}' \simeq \underline{i}_{A,M}, \qquad \underline{f}'\underline{g}' \simeq \underline{i}_{B,N},$$

(10.4) $$\underline{g}''\underline{f}'' \simeq \underline{i}_{X\setminus A,M}, \qquad \underline{f}''\underline{g}'' \simeq \underline{i}_{Y\setminus B,N},$$

(10.5) $$\underline{\hat{g}}\underline{\hat{f}} \simeq \underline{i}_{\hat{X},\hat{M}}, \qquad \underline{\hat{f}}\underline{\hat{g}} \simeq \underline{i}_{\hat{Y},\hat{N}}.$$

Let us set:

(10.6) $$\begin{aligned} &\varphi_k(x, \hat{x}) = (f_k(x), \hat{f}_k(\hat{x})) && \text{for every point } (x, \hat{x}) \in M \times \hat{M}, \\ &\psi_k(y, \hat{y}) = (g_k(y), \hat{g}_k(\hat{y})) && \text{for every point } (y, \hat{y}) \in N \times \hat{N}. \end{aligned}$$

Then

$$\varphi_k\colon M \times \hat{M} \to N \times \hat{N} \quad \text{and} \quad \psi_k\colon N \times \hat{N} \to M \times \hat{M} \quad \text{for} \quad k = 1, 2, \dots$$

Let us show that $\underline{\varphi}' = \{\varphi_k, A \times \hat{X}, B \times \hat{Y}\}_{M\times\hat{M}, N\times\hat{N}}$ is a W-sequence. Consider a compactum $\check{C} \subset A \times \hat{X}$. Then there exist a compactum $C \subset A$

and a compactum $\hat{C} \subset \hat{X}$ such that $\check{C} \subset C \times \hat{C}$. Since $\underline{f}'$ and $\underline{\hat{f}}$ are W-sequences, there is a compactum $D \subset B$ and a compactum $\hat{D} \subset \hat{Y}$ such that for every neighbourhood V of D (in N) and for every naighbourhood $\hat{V}$ of $\hat{D}$ (in $\hat{N}$) there is a neighbourhood U of C (in M) and a neighbourhood $\hat{U}$ of $\hat{C}$ (in $\hat{M}$) such that

(10.7) $$f_k/U \simeq f_{k+1}/U \textit{ in } V \textit{ and } \hat{f}_k/\hat{U} \simeq \hat{f}_{k+1}/\hat{U} \textit{ in } \hat{V} \textit{ for almost all } k.$$

Let W be a neighbourhood (in $N \times \hat{N}$) of the compactum $D \times \hat{D}$. Then V and $\hat{V}$ can be selected so that $V \times \hat{V} \subset W$. Setting $\check{U} = U \times \hat{U}$, we get a neighbourhood of $\check{C}$ (in $M \times \hat{M}$) and we infer by (10.6) and (10.7) that

$$\varphi_k/\check{U} \simeq \varphi_{k+1}/\check{U} \text{ in } V \times \hat{V} \subset W \quad \text{for almost all } k.$$

Hence $\underline{\varphi}'$ is a W-sequence. Similarly one shows that

$$\begin{aligned} &\underline{\psi}' = \{\psi_k, B \times \hat{Y}, A \times \hat{X}\}_{N\times\hat{N}, M\times\hat{M}}, \\ &\underline{\varphi}'' = \{\varphi_k, (X\setminus A) \times \hat{X}, (Y\setminus B) \times \hat{Y}\}_{M\times\hat{M}, N\times\hat{N}} \quad \textit{and} \\ &\underline{\psi}'' = \{\psi_k, (Y\setminus B) \times \hat{Y}, (X\setminus A) \times \hat{X}\}_{N\times\hat{N}, M\times\hat{M}} \quad \textit{are W-sequences}. \end{aligned}$$

Let us prove that $\underline{\psi}'\underline{\varphi}' \simeq \underline{i}_{A\times\hat{X}, M\times\hat{M}}$.

Consider a compactum $\check{C} \subset A \times \hat{X}$. Then there is a compactum $C \subset A$ and a compactum $\hat{C} \subset \hat{X}$ such that $\check{C} \subset C \times \hat{C}$. By (10.3) and (10.5) there is a compactum $D \subset A$ and a compactum $\hat{D} \subset \hat{X}$ such that for every neighbourhood V of D (in M) and for every neighbourhood $\hat{V}$ of $\hat{D}$ (in $\hat{M}$) there is a neighbourhood U of D (in M) and a neighbourhood $\hat{U}$ of $\hat{D}$ (in $\hat{M}$) such that

(10.8) $$g_k f_k/U \simeq i/U \text{ in } V \quad \text{and} \quad \hat{g}_k\hat{f}_k/\hat{U} \simeq i/\hat{U} \text{ in } \hat{V} \quad \text{for almost all } k.$$

Let W be a neighbourhood (in $M \times \hat{M}$) of the compactum $C \times \hat{C}$. Then V and $\hat{V}$ may be selected so that $V \times \hat{V} \subset W$. Setting $\check{U} = U \times \hat{U}$, we get a neighbourhood of $\check{C}$ (in $M \times \hat{M}$) and we infer by (10.6) and (10.8) that

$$\psi_k\varphi_k/\check{U} \simeq i/\check{U} \text{ in } V \times \hat{V} \subset W \quad \text{for almost all } k.$$

Hence $\underline{\psi}'\underline{\varphi}' \simeq i_{A\times\hat{X}, M\times\hat{M}}$. Similarly one shows (using (10.4), (10.5) and (10.6)) that

$$\underline{\varphi}'\underline{\psi}' \simeq \underline{i}_{B\times\hat{Y}, N\times\hat{N}}, \ \underline{\psi}''\underline{\varphi}'' \simeq \underline{i}_{(X\setminus A)\times\hat{X}, M\times\hat{M}} \quad \text{and} \quad \underline{\varphi}''\underline{\psi}'' \simeq \underline{i}_{(Y\setminus B)\times\hat{Y}, N\times\hat{N}}.$$

Since $X \times \hat{X}$ is closed in $M \times \hat{M}$ and $Y \times \hat{Y}$ is closed in $N \times \hat{N}$, we infer that $\mathrm{Pos}(X \times \hat{X}, A \times \hat{X}) = \mathrm{Pos}(Y \times \hat{Y}, B \times \hat{Y})$ and the proof of Theorem (10.1) is finished.

(10.9) PROBLEM. *Is it true that if* $\mathrm{Pos}(X, A) \neq \mathrm{Pos}(Y, B)$ *and* $\mathrm{Pos}(\hat{X}, \hat{A}) = \mathrm{Pos}(\hat{Y}, \hat{B})$, *then* $\mathrm{Pos}(X \times \hat{X}, A \times \hat{A}) = \mathrm{Pos}(Y \times \hat{Y}, B \times \hat{B})$?

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On essential cluster sets

by

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Abstract. Let f be a real function defined in the open half plane H bounded by a line L. The fine cluster set of f at a point x in L, designated by $W_0(f, x)$, is the set of all y such that for every $\varepsilon > 0$, x is a point of positive lower density for the set $f^{-1}\{(y-\varepsilon, y+\varepsilon)\}$. The fine cluster set of f at x in the direction θ, designated by $W_0(f, x, \theta)$, is defined analogously by restricting f on a line $L_\theta(x)$ in H emanating from x and making angle θ with L. It is shown that each of the sets $\{x: x \in L;\ W_0(f, x, \theta) \not\subset W_0(f, x)\}$ and $\{\theta: \theta \in (0, \pi);\ W_0(f, x, \theta) \not\subset W_0(f, x)\}$ is of measure zero when f is measurable and is of the first category when f is continuous, and some consequences are studied.

1. In a recent paper [3] Goffman and Sledd have obtained certain interesting relations between the total essential cluster sets and the directional essential cluster sets. They have proved that if a measurable function f is defined in the upper half plane above the x-axis and if θ is a direction then except a set of points x of measure zero the essential cluster set of f at x is a subset of the essential cluster set of f at x in the direction θ. If further f is continuous then this exceptional set is also of the first category. Regarding the ordinary cluster sets there is an analogous result [2]. In this paper we study further properties of these sets by weakening the density conditions. We have defined the fine cluster sets and obtained certain relations between the fine cluster sets, essential cluster sets and ordinary cluster sets.

2. The function f is taken to be defined in the open half plane H above a line L, which, in particular may be taken to be the x-axis. The point on the line L, viz $(x, 0)$, will be denoted simply by x while any other point in H will be denoted by p. $\mu(A)$ and $\mu^*(A)$ will denote the Lebesgue measure and the Lebesgue outer measure, respectively, for the set A, linear or planar, according as A is linear or planar, which will be clear from the context. For $\delta > 0$, $S_\delta(x)$ will denote the set of all points p in H, whose distance, $|p-x|$, from x is less than δ. For $0 < \theta < \pi$, $L_\theta(x)$ denotes the half ray in H, in the direction θ, terminating at x and $L_\theta(x, h)$ is the open line segment in H in the direction θ, of length h, and having x as one of its end points.