[31] M. J. Russell, *Monotone decompositions of continua irreducible about a finite set*, Fund. Math. 72 (1971), pp. 255–264.
[32] E. S. Thomas, Jr., *Monotone decompositions of irreducible continua*, Dissert. Math. (Rozprawy Mat.) 50 (1966).
[33] G. T. Whyburn, *Analytic Topology*, AMS Colloq. Publ. 38, New Edition (1963).
[34] W. A. Wilson, *On the structure of a continuum, limited and irreducible between two points*, Amer. J. Math. 48 (1926), pp. 147–168.
[35] — *On upper semi-continuous decompositions of compact continua*, Amer. J. Math. 54 (1932), pp. 377–386.

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The Sorgenfrey plane in dimension theory

by

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Abstract. The Sorgenfrey plane S^2 is shown in several different ways to be strongly zero-dimensional. This eliminates it as a possible counterexample to the conjecture that the product of strongly zero-dimensional spaces is itself strongly zero-dimensional. Related properties of S^2 are demonstrated, and several allied conjectures in dimension theory are discussed.

One of the drawbacks of ordinary Lebesque covering dimension is that it does not behave well under products. One would like to have the inequality $\dim(X\times Y)\leqslant \dim X+\dim Y$ hold for arbitrary spaces X and Y. But, as Nagata points out in his text [5, p. 196], it does not even hold for the case where X and Y are paracompact spaces of dimension 0, and X is homeomorphic to Y. The counterexample there given by Nagata is the Sorgenfrey line S, which is the real line with upper half-open intervals $[a, b)$ as a base for the topology [7]. We have $\dim S=0$ because S is Lindelöf and the base given consists of clopen [closed-and-open] sets [cf. 2, 16.16]. But $\dim S\times S>0$ because a space of covering dimension zero is automatically normal [5, p. 196] and the Sorgenfrey plane, $S\times S$, is not normal [7].

A definition of covering dimension has been adopted by some authors [2, Chapter 16], [3, p. 97] which agrees with Lebesgue dimension for normal spaces and in some respects is more satisfactory for non-normal spaces. One simply replaces open sets by cozero sets at one point in the definition of Lebesgue dimension: $\dim X\leqslant n$ if every finite cover of X by cozero sets can be refined to an open cover of order n. (A cover $\mathfrak{U}$ of a space X is of *order* m if every point of X belongs to at most $m+1$ members of $\mathfrak{U}$.)

For $\dim X=0$ this is equivalent to the following simpler condition [2, 16.17]: given two disjoint zero-sets Z_1 and Z_2 of X, there is a clopen set C such that $Z_1\subset C$, $Z_2\cap C=\emptyset$. A Tychonoff space satisfying this property will be called a *strongly zero-dimensional space*.

In this paper, we show that the Sorgenfrey plane is strongly zero-dimensional. This keeps alive the conjecture that $\dim X\times Y\leqslant\dim X+\dim Y$ holds for arbitrary spaces for this kind of covering dimension. The following conjecture is also kept in good standing: given any set $\{X_a\}_{a\in\mathcal{A}}$ of strongly zero-dimensional spaces, the product $X = \prod_{a\in\mathcal{A}} X_a$ is also strongly zero-dimensional. Unfortunately, however, none of the proofs extend to higher products of S with itself. It is conceivable that one of them may be a counterexample ([1]).

1. We begin with an important characterization of strongly zero-dimensional spaces due to Heider:

LEMMA. *Let X be a Tychonoff space. The following are equivalent.*

(i) *The space X is strongly zero-dimensional.*

(ii) *Every zero-set of X is a countable intersection of clopen sets.*

(iii) *Every countable cover of X by cozero-sets can be refined to a partition of X into clopen sets.*

Proof. (i) implies (ii): Suppose X is strongly zero-dimensional. Let Z be a zero-set of X. We can write Z as a countable intersection of cozero-sets K_n, i.e. $Z = \bigcap_{n=1}^{\infty} K_n$. For each n let C_n be a clopen set containing Z and contained in K_n. Then $Z = \bigcap_{n=1}^{\infty} C_n$.

(ii) implies (iii): Let $\{K_n\}_{n=1}^{\infty}$ be a cover of X by cozero-sets. For each n let $K_n = \bigcup_{m=1}^{\infty} C_{mn}$, where each C_{mn} is a clopen set. Order the sets C_{mn} in a sequence, $\{C_n\}_{n=1}^{\infty}$, and let $C_n' = C_n \setminus \bigcup_{i=1}^{n-1} C_i$. Then $\{C_n'\}_{n=1}^{\infty}$ is a partition of X into clopen sets refining $\{K_n\}_{n=1}^{\infty}$.

Finally, the third condition trivially implies the first.

We will show that condition (ii) holds for the Sorgenfrey plane S^2. For the sake of brevity we will refer to a countable intersection of clopen sets as an $(FG)_\delta$-set. We will also refer to a countable intersection of open sets as a G_δ, a countable union of closed sets as an F_σ, and a countable union of clopen sets as an $(FG)_\sigma$.

Since the topology on S^2 is finer than the Euclidean topology, one class of zero-sets on S^2 is the family of Euclidean closed sets. The next result is part of the folklore.

2. LEMMA. *Every Euclidean-closed subset of S^2 is an $(FG)_\delta$-set in the Sorgenfrey topology.*

Proof. Let A be a Euclidean-closed subset of S^2. Since A is a Euclidean G_δ it is enough to show that, given any Euclidean-closed set B disjoint from A, there is an S^2-clopen set covering A and missing B.

([1]) Added in proof. Both S. Mrówka [4] and T. Teresawa [8] have shown that all powers of S are strongly zero-dimensional.

For each pair (m, n) of integers let $B_{(m,n;1)}$ be the S^2-clopen unit square with the point (m, n) at its lower left corner. That is, $B_{(m,n;1)}$ is a square with sides of length 1 parallel to the coordinate axes, excluding the sides at the top and to the right and including the other sides.

More generally, we define, for each ordered pair (x, y) of real numbers and each positive real number δ, the set $B_{(x,y;\delta)} = \{(z, w)\colon x \leqslant z < x+\delta,\ y \leqslant w < y+\delta\}$. Clearly $B_{(x,y;\delta)}$ is clopen in S^2 for all x, y, and δ. We now take the Euclidean closures of $A \cap B_{(m,n;1)}$ and $B \cap B_{(m,n;1)}$. They will be disjoint compact sets in the Euclidean topology, hence separated by some positive distance ε_{mn}. Divide $B_{(m,n;1)}$ into smaller squares of the same shape, with sides of length $\frac{1}{2}\varepsilon_{mn}$ or less. Let R_{mn} be the union of all subsquares of $B_{(m,n;1)}$ which meet A. Then $\bigcup_{m,n} R_{m,n}$ is a clopen set containing A and missing B.

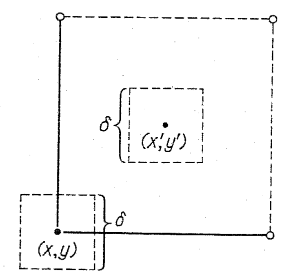
3. The next result, discovered independently by D. Lutzer, R. Heath, S. Mrówka, and the author, is of interest even aside from its usefulness in showing S^2 to be strongly zero-dimensional.

THEOREM. *Every cozero-set of S^2 is a Euclidean F_σ.*

Proof. Let $A = f^{-1}(0, +\infty)$, where we assume f is a non negative real-valued continuous function on S^2 (any cozero-set can be so expressed). We define $A_n = \{(x, y)\colon B_{(x,y;1/n)} \subset f^{-1}[1/n, +\infty)\}$. Clearly, $A = \bigcup_{n=1}^{\infty} A_n$, and $A_n \subset A_{n+1}$ for all n. We now show that A_n is closed in the Euclidean topology for all n.

Let (x, y) be a Euclidean accumulation point of A_n. We wish to show that $B_{(x,y;1/n)} \subset f^{-1}[1/n, +\infty)$. To do this it is enough to show that the Euclidean interior of $B_{(x,y;1/n)}$ is in $f^{-1}[1/n, +\infty)$. For then the rest of $B_{(x,y;1/n)}$ is also in $f^{-1}[1/n, +\infty)$, by continuity of f on S^2.

So let (x', y') be in the Euclidean interior of $B_{(x,y;1/n)}$. There will be a Euclidean open square centered on (x', y') and contained in $B_{(x,y;1/n)}$. Say its sides are of length δ. Consider the open square of the same size centered on (x, y):



Take a point (x_m, y_m) of A_n which is contained in this square about (x, y). It is easy to see that $(x', y') \in B_{(x_n, y_n; 1/n)}$. Hence $(x', y') \in f^{-1}[1/n, +\infty)$, as was to be shown.

S. Mrówka has supplied an independent proof (see, [4]). It consists of showing that every bounded continuous function from S^2 to $\boldsymbol{R}$ is a pointwise limit of functions continuous from the Euclidean plane to $\boldsymbol{R}$, and then using a theorem to the effect that this property is equivalent to Theorem 3. An independent proof of the first result was supplied by W. G. Bade [1]. His proof generalizes to all finite powers S^n, as well as to $S^{\aleph_0}$. So does the proof we have just given. The generalization to S^n is obvious. For $S^{\aleph_0}$ we define, for each $p = (p(1), p(2), \ldots)$ in $S^{\aleph_0}$ the basic open sets

$$B_{(p,n)} = \{p'\colon\ p(k) \leqslant p'(k) < p(k)+1/n \ \text{ for } \ k = 1, \ldots, n\}\,.$$

Defining A_n as $\{p\colon B_{(p,n)} \subset f^{-1}[1/n, +\infty)\}$, we can show that $A = \bigcup_{n=1}^{\infty} A_n$ and that A_n is closed in the Euclidean topology.

For $m > \aleph_0$ the problem reduces to the case of $S^{\aleph_0}$. We have the following theorem of Gleason (quoted in [2], p. 130): *every continuous function from the product of arbitrarily many separable spaces into a first countable Hausdorff space can be factored through a countable subproduct.* So we express the appropriate projection of a cozero set as an Euclidean F_σ, and the cozero set itself will be the union of the preimages of the closed sets obtained.

4. We give two proofs of the main result, independently and almost simultaneously arrived at. The basic ideas in both are quite similar.

THEOREM. *The Sorgenfrey plane is strongly zero-dimensional.*

Proof A (P. Nyikos and P. Roy). Let A be a cozero set of S^2. We will show that A is an $(FG)_\sigma$.

As before, assume $A = f^{-1}(0, +\infty)$ where f is nonnegative, and let A_n be defined as above. Let

$$B_n = \bigcup \{B_{(x,y;1/n)}\colon\ (x, y) \in A_n\}\,.$$

Clearly, $A_n \subset B_n \subset A$. Next, let ∂B_n be the Euclidean boundary of B_n. We will exhibit a clopen set W_n containing $\partial B_n \cap A_n$ and contained in B_n. We will then have $A_n \subset W_n \cup B_n^0 \subset B_n \subset A$, where B_n^0 is the Euclidean interior of B_n. Since $W_n \cup B_n^0$ is an $(FG)_\sigma$ (W_n because it is clopen, B_n^0 by Lemma 2), it will follow that A is an $(FG)_\sigma$ as well.

Let $W_n = \bigcup \{B_{(x,y;1/n)}\colon\ (x, y) \in \partial B_n \cap A_n\}$. Clearly, $\partial B_n \cap A_n \subset W_n \subset B_n$, and W_n is S^2-open. We will now show W_n is S^2-closed.

Suppose $v = (x', y')$ is an S^2-limit point of W_n. Select a sequence $\{v_m\}$ of points of W_n converging to it in the Sorgenfrey topology. For each v_m let u_m be the corner point of the half-open square from which v_m is taken (if there is more than one candidate for u_m, pick one at random). By the Bolzano-Weierstrass theorem for the plane, the u_m have a subsequence which converges in the Euclidean topology. Let $u = (\bar{x}, \bar{y})$ be its limit point, and consider the clopen square $B_{(\bar{x}, \bar{y}; 1/n)}$ side $1/n$ with u in its lower left corner.

Since the u_n are in $A_n \cap \partial B_n$, so is u. Where is (x', y')? If it is in the square we are done, for it is then in W_n. Furthermore it must be on the Euclidean boundary of the square. By symmetry we may assume it is on the right-hand edge. We may also assume the point u to be the point $(0, 0)$.

We cannot have any u_m in the half-open square because that would put them in the Euclidean interior of B_n. Similarly, we cannot have any of them for a distance of less than $1/n$ below and to the left of u, for then u would be in the Euclidean interior of B_n, but we know that u is in $A_n \cap \partial B_n$. So we may assume all the u_m are in the second or fourth quadrants. But then (x', y') is not a limit point of the v_n in the Sorgenfrey topology. Hence W_n is S^2-closed, and we have

$$A = \bigcup_{n=1}^{\infty} (W_n \cup B_n^0)\,.$$

Proof B (R. Heath and D. Lutzer). Let f be as above, and let $Z = f^{-1}(0)$. Let F be the Euclidean closure of Z. For each n let $F_n = F \cap A_n$, where A_n is defined as above. Let

$$V_n = \bigcup \{B_{(x,y;1/n)}\colon\ (x, y) \in F_n\}\,.$$

We will show that V_n is S^2-clopen. Since the complement of F is an $(FG)_\sigma$ by Lemma 2, it will follow that the complement of Z is an $(FG)_\sigma$ also, making Z an $(FG)_\sigma$.

Clearly, V_n is S^2-open. Let (x', y') be an accumulation point of V_n in the Sorgenfrey topology. We will assume (as we may) that $(x', y') = (0, 0)$. There will be a sequence $\{v_m\}$ of points in $B_{(0,0;1/n)}$ contained in V_n and converging to the origin.

As in Proof A let u_m be the corner of a square in which v_m is found. The u_m are points of F_n used in defining V_n. If infinitely many u_m are in the first or third quadrants, then the origin is in V_n, either because it is a limit of points in F_n (a Euclidean closed set by the proof of Theorem 3) or because it is in one of the squares defined by the u_m.

Assume infinitely many u_m are in the second quadrant. The sequence $\{u_m\}$ has at least one accumulation point on the x-axis. If any limit point $(x, 0)$ is less than $1/n$ to the left of the origin, then the origin is in $B_{(x,0;1/n)}$ because $(x, 0)$ is in F_n. The other possibility is that $\{u_m\}$ converges to $(-1/n, 0)$. But this means that some point u_m is in the Euclidean interior of $B_{(x_k, y_k; 1/n)}$ for some other point $u_k = (x_k, y_k)$. And this contradicts the assumption that (x_m, y_m) is in F, the Euclidean closure of Z.

A similar argument holds if infinitely many points are in the fourth quadrant. Thus V_n is S^2-closed.

A third proof was supplied by S. Mrówka [4]. Neither his proof nor the two given above generalize to Sorgenfrey 3-space.

To see why the above proofs do not generalize, consider the line segment $\{(-x, x, x): 0 \leqslant x \leqslant 1\}$. Suppose this line segment were A_1 in the above notation. The set B_1 can be imagined as a parallelepiped with a square face resting on a table, an edge sloping up and to the left and away from the observer, the other edges parallel-plus a cube on top of the parallelepiped. Now if we try to define W_1 as above, we find that $W_1 = B_1$, and W_1 is not S^3-closed: the line segment $\{(-x, x+1, x): 0 \leqslant x \leqslant 1\}$ is in the S^3-closure of W_1 but is not in W_1 itself. Proof B does not generalize, either: if $A_1 = F_1$, then V_1 is the parallelepiped-plus-cube we have just constructed.

Another proof of Theorem 4 was given by W. G. Bade [1]. He proved first the following lemma: *Let K be compact in the Euclidean topology. Let $\sigma > 0$ and let*

$$W = \bigcup_{(x,y)\in K} B_{(x,y;\sigma)}.$$

If K does not meet the Euclidean interior of W, then W is clopen. (The proof of this lemma is similar to proofs A and B above.) Then he showed the following theorem: *Let V be any S^2-open set.* There exists *a sequence* $\{Q_n\}$ *of clopen sets such that*

$$V \subset \bigcup_{n=1}^{\infty} Q_n \subset \operatorname{cl} V.$$

(Here cl denotes the closure in the Sorgenfrey topology). Strong zero-dimensionality of S^2 then followed quickly. Unfortunately the lemma does not extend to S^3. We need only let K be the line segment A in the preceding discussion. Letting $\sigma = 1$, we see that W is not closed.

5. The anti-diagonal $D = \{(-x, x): x$ is a real number$\}$ is an important subset of S^2. It is closed, and it is discrete in the relative topology. Sorgenfrey's original proof that S^2 is not normal (see [7]) made use of the two disjoint closed sets $P = \{(-x, x): x$ is irrational$\}$ and $Q = \{(-x, x): x$ is rational$\}$. By using the Baire category theorem, Sorgenfrey showed that P and Q cannot be contained in disjoint open sets.

The following theorem gives additional information about zero-sets of D, relating Theorems 3 and 4.

THEOREM. *Let A be a subset of D. The following conditions are equivalent.*

1. *A is an $(FG)_\delta$ subset of S^2.*

2. *A is a zero-set of S^2.*
3. *A is a Euclidean G_δ set.*

Proof (Outline). We need only show that the third condition implies the first. For this it is enough to show the following result: *every Euclidean-closed subset F of D can be expressed as $C \cap D$, where C is a clopen subset of S^2.*

Consider the set F' gotten by rotating F through an angle of 45° counterclockwise. Consider now the graph G' of the distance function of F'. With each point on the x-axis we associate its distance to F'. The intersection of G' with the x-axis is just F' itself.

Now rotate G' 45° clockwise and consider the set C of all points above and to the right of the resulting set, G, including G itself. We have that $F = C \cap D$ and that C is S^2-clopen.

The rest is elementary. Any Euclidean G_δ subset $A = \bigcap_{n=1}^{\infty} U_n$ of D can be expressed as $(\bigcap_{n=1}^{\infty} C_n^c) \cap D$, where the C_n are S^2-clopen sets determined by the closed sets $F_n = U_n^c \cap D$. (Complementation is denoted by a superscripted c.) Since D is an $(FG)_\delta$ set, so is $A = (\bigcap_{n=1}^{\infty} C_n^c) \cap D$.

6. S. Mrówka has generalized the above result to arbitrary subsets of S^2:

THEOREM. *Let A be a subset of S^2. The following conditions on A are equivalent.*

1. *A is an $(FG)_\delta$ subset of S^2.*
2. *A is a zero-set of S^2.*
3. *A is a Euclidean G_δ, S^2-closed subset of S^2, and, for every Euclidean-closed $F \subset S^2$ with $F \cap A = \emptyset$, F and A are contained in disjoint S-open sets.*

Proof. See [4], Theorem 2.7.

Without the two additional conditions in 3, this theorem would be false. For example, any open circular disk is a Euclidean G_δ but is not S^2-closed and so cannot be a zero-set of S^2. Nor can we get by with just the first two conditions in 3, as the following example, discovered by P. Roy, shows.

7. EXAMPLE. We will use the notation of Theorem 5. Let A be the Cantor set on D. Let C be defined as before, and let $I_1, \ldots, I_n, \ldots$ be the sequence of removed intervals. Let C' be the subset of C obtained by deleting those points of G which are associated with points of I_n and are at a distance of $1/n$ or more from D. The set C' is an S^2-open, Euclidean F_σ (it is the union of the Cantor set, the Euclidean interior of C, and countably many line segments which are Euclidean F_σ's). But a standard argument, using the Baire Category Theorem applied to the Cantor set, shows that C' is not an $(FG)_\sigma$. Indeed, if $\bigcup_{n=1}^{\infty} C'_n = C'$ and all the C_n are clopen, define

$$A_{nm} = \{(x, y):\ B_{(x,y;1/n)} \subset C'_m\}$$

we have the Cantor set on D as a countable union of these (Euclidean-closed) sets and this is an impossibility.

A similar example was provided independently by S. Mrówka [4, Example 2.8].

Conclusion. By Theorem 4, we have eliminated S^2 as a possible counterexample to the conjectures cited in the introduction, although higher powers of S still pose a problem. Our result also leaves unsolved another problem which will now be discussed briefly. Additional information and references may be found in an earlier article [6].

An *N-compact space* is one which can be embedded as a closed subspace in a product of countable discrete spaces. It is known that every strongly 0-dimensional, realcompact space is N-compact. As for the converse, one readily obtains that every N-compact space is realcompact, but is every N-compact space strongly zero-dimensional? This question remains unanswered (1).

Now using Gleason's Theorem cited earlier, we can show that every product of countable discrete spaces is strongly zero-dimensional. Indeed, every *countable* product of countable discrete spaces is strongly zero-dimensional, because it is Lindelöf (being separable and metric) and has a base of clopen sets (cf. [2], 16.16). From this we can see that these two questions are equivalent:

1. Is every N-compact space strongly zero-dimensional?

2. Is every closed subspace of a strongly zero-dimensional realcompact space itself strongly zero-dimensional?

Since a product of N-compact spaces is itself N-compact, an affirmative answer to the above questions would imply the following: given a family $\{X_a\}_{a\in\mathcal{A}}$ of strongly zero-dimensional realcompact spaces, the product $\prod_{a\in\mathcal{A}} X_a$ is also strongly zero-dimensional. This is a special case of one conjecture mentioned in the introduction, and it too remains an open problem, even for finite products. In fact, since the Sorgenfrey line and all its powers are realcompact, they were candidates for a counterexample until recently.

(1) Added in proof. S. Mrówka has recently announced the existence of an N-compact space which is not strongly zero-dimensional.

References

[1] W. G. Bade, *Two properties of the Sorgenfrey line*, unpublished MS.
[2] L. Gillman and M. Jerison, *Rings of Continuous Functions*, Princeton 1960.
[3] J. Isbell, *Uniform Spaces*, Providence 1964.
[4] S. Mrówka, *Some problems related to N-compact spaces*, to appear.
[5] J. Nagata, *Dimension Theory*, New York 1965.
[6] P. Nyikos, *Not every 0-dimensional realcompact space is N-compact*, Bull. Amer. Math. Soc. 77 (1971), pp. 392–396.
[7] R. H. Sorgenfrey, *On the topological product of paracompact spaces*, Bull. Amer. Math. Soc. 53 (1947), pp. 613–632.
[8] T. Teresawa, *On the zero-dimensionality of some non-normal product spaces*, Sci. Rep. Tokyo Kyoiku Daigaku, Section A, 11 (1972), pp. 167–179.

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