$0 \leqslant r < k$) has diameter less than δ_1. Using the fact that the sequence $U_1, U_2, \ldots$ is null, choose a positive number s_{k+1}, $l < s_{k+1} < 2$, such that if $(\mathrm{Cl}\, U_j) \cap \{\mathrm{Bd}\, C_i \times (1, s_{k+1}]\} \neq \emptyset$, then $\mathrm{Cl}\, U_j \subset \mathrm{Bd}\, C_i \times (1, 2)$ and the projection π: $\mathrm{Bd}\, C_i \times (1, 2) \to \mathrm{Bd}\, C_i$ takes $\mathrm{Cl}\, U_j$ into some $V \in \mathfrak{V}$.

Choose $1 = s_0 < s_1 < \ldots < s_k < s_{k+1} < s_{k+2} = 2$ such that if $r \leqslant k$ and $\mathrm{Cl}\, U_j \cap \{\mathrm{Bd}\, C_i \times (s_0, s_r)\} \neq \emptyset$, then $\mathrm{Cl}\, C_j \subset \{\mathrm{Bd}\, C_i \times (s_0, s_{r+1})\}$. Let φ: $[1, 2] \to [1, 2]$ be the map which takes each segment $[s_r, s_{r+1}]$ linearly to $[t_r, t_{r+2}]$ $(0 \leqslant r < k+2)$ with $\varphi(s_r) = t_r$ $(0 \leqslant r \leqslant k+2)$. Define h_2: $S^n \to S^n$ by the formula

$$h_2(x) = \begin{cases} (c, \varphi(t)) & \text{if } x = (c, t),\ c \in \mathrm{Bd}\, C_1,\ 1 \leqslant t \leqslant 2\,, \\ x & \text{otherwise}\,. \end{cases}$$

Then $h = h_1 \circ h_2$: $S^n \to S^n$ satisfies the requirements of the lemma.

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UNIVERSITY OF WISCONSIN
Madison, Wis.
THE INSTITUTE FOR ADVANCED STUDY
Princeton, N.J.

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On decompositions of continua

by

J. J. Charatonik (Wrocław)

Abstract. The aim of the paper is to prove some theorems which generalize the author's earlier results concerning decompositions of λ-dendroids [4], and also some of Thomas' results on decompositions of irreducible continua [32]. An upper semi-continuous monotone decomposition of a metric continuum is called admissible if the layers of its irreducible subcontinua are contained in the elements of the decomposition. The main results of the paper say that the decomposition space of an admissible decomposition of a continuum is hereditarily arcwise connected and that every continuum has exactly one minimal admissible decomposition (called the canonical one). The structure of elements of the canonical decomposition of a continuum is shown. Some special continua are considered in the paper, e.g. continua X such that every upper semi-continuous monotone decomposition of X with a hereditarily arcwise connected decomposition space is admissible; monostratic continua; continua every element of the canonical decomposition of which has an empty interior. Some necessary and/or sufficient conditions are stated under which continua have the properties under consideration.

1. Introduction. Upper semi-continuous monotone decompositions of continua have been studied by a large number of authors. E.g. Z. Janiszewski in [18], B. Knaster in [19] and [20], K. Kuratowski in [21] and [22] and also W. A. Wilson in [34] investigated such decompositions for continua irreducible between two points. A continuation of this topic can be found in a sequence of papers. For example, E. S. Thomas, Jr., has given in [32] a large study of monotone decompositions of irreducible continua; E. Dyer [12], M. E. Hamstrom [16], W. S. Mahavier [26], H. C. Miller [28], E. E. Moise [29] and many others have discussed interesting particular problems concerning such decompositions. Some of these results, originally made for metric continua, have been extended to Hausdorff continua — see e.g. G. R. Gordh, Jr. [14] and W. S. Mahavier [27]. Besides decompositions of continua irreducible between two points, decompositions of some other spaces have been considered. In particular, R. W. FitzGerald and P. M. Swingle have studied in [13] decompositions of Hausdorff continua, especially those which have a semi-locally connected decomposition space. Some of Miller's results of [28] concerning decompositions of continua irreducible between two points were generalized by M. J. Russell in [31] to decompositions of continua irreducible about a finite set. Upper semi-continuous monotone decompositions of continua were investigated by G. R. Gordh, Jr. in [15] for smooth continua, and by the present author in [4] for λ-dendroids.

The aim of this paper is to extend and generalize some earlier results concerning upper semi-continuous monotone decompositions of continua having a special structure (e.g. of irreducible continua or of λ-dendroids) to those of arbitrary metric continua.

After the first and the second sections (Introduction and Preliminaries), in the third one a concept of an admissible decomposition of an arbitrary continuum is introduced and studied; the decomposition space of such a decomposition of a continuum is hereditarily arcwise connected.

The fourth section contains a proof of the uniqueness of the minimal admissible decomposition of a continuum. It is shown that for irreducible continua the minimal admissible decomposition coincides with the well-known classical decompositions of Kuratowski [21], [22], of Thomas [32] or of Wilson [35]; it is also shown that if the decomposed continuum is a λ-dendroid, then the decomposition in question gives the canonical decomposition described by the author in [4], and if the continuum is smooth, then its minimal admissible decomposition coincides with that given by Gordh in [15]. Finally the structure of elements of the minimal admissible decomposition is described. It is patterned after the author's ideas concerning strata of λ-dendroids in [4].

The fifth section contains investigations of some special kinds of mappings and of continua. In particular, it concerns continua with the property that every monotone upper semi-continuous decomposition having a hereditarily arcwise connected decomposition space is admissible. The concept of a monostratiform λ-dendroid ([5], p. 75) is generalized to that of a monostratic continuum. The class $\mathfrak{C}$ of mappings, originally — in [6] — studied for λ-dendroids only, is now extended to arbitrary continua. At the end of the section we introduce and study the class $\mathfrak{L}$ of continua which coincides (for irreducible continua) with the class of continua of type λ in Kuratowski's sense (see [22], p. 262 and [25], footnote on p. 197) or with the class of continua of type A' in Thomas' sense (see [32], p. 13).

2. Preliminaries. A continuum means a compact connected metric space. A mapping means a continuous function. A decomposition $\mathfrak{D}$ of a continuum X is a collection of closed subsets of X whose union is the whole X, such that $D_1 \in \mathfrak{D}$, $D_2 \in \mathfrak{D}$ implies $D_1 \cap D_2 = \emptyset$ or $D_1 = D_2$. A decomposition $\mathfrak{D}$ of a continuum X is called *upper semi-continuous* provided that if U is an open set in X which contains $D \in \mathfrak{D}$, then there exists an open subset V of U which contains D and is such that if D' is any set of $\mathfrak{D}$ intersecting V, then $D' \subset U$ (see [33], p. 122). In other words, $\mathfrak{D}$ is upper semi-continuous provided that if U is an open set in X and contains $D \in \mathfrak{D}$, then some open subset of U contains D and is the union of some elements of $\mathfrak{D}$ (see [32], p. 5; [24], § 19, II, Theorem 4, p. 185). The following is well known (see e.g. [24], § 19, II, Theorems 3 and 4, p. 185; [25], § 43, IV, Theorem 2, p. 66; [33], Chapter VII, Theorem (1.1), p. 122).

PROPOSITION 1. *Let X be a continuum and let $\mathfrak{D}$ be a decomposition of X. The following conditions are equivalent:*

(2.1) *$\mathfrak{D}$ is upper semi-continuous,*

(2.2) *for each closed set $A \subset X$ the union of all $D \in \mathfrak{D}$ which intersect A is closed,*

(2.3) *$D \cap \operatorname{Li} D_n \neq \emptyset$ implies $\operatorname{Ls} D_n \subset D$, where $D, D_n \in \mathfrak{D}$,*

(2.4) *if $D_1, D_2, \ldots$ is a convergent sequence of elements of $\mathfrak{D}$, then its limit is contained in a single element of $\mathfrak{D}$.*

A decomposition $\mathfrak{D}$ of a continuum X is said to be *monotone* if each element of $\mathfrak{D}$ is connected (in fact, it is a subcontinuum of X).

Let $\mathfrak{D}$ be a decomposition of a continuum X. The *quotient space* $X/\mathfrak{D}$ is a set whose points are elements of $\mathfrak{D}$ endowed with the *quotient topology*, i.e., such that a subset $A \subset X/\mathfrak{D}$ is open in $X/\mathfrak{D}$ provided that the union of all elements of $\mathfrak{D}$ which are points of A is open in X. The function $q\colon X \to X/\mathfrak{D}$ which assigns to a point of X the element of $\mathfrak{D}$ containing it is called the *quotient mapping*. It is well known that the quotient mapping is continuous (see e.g. [24], § 19, pp. 183–187). If the decomposition $\mathfrak{D}$ of a continuum X is upper semi-continuous, then the quotient space is a continuum ([33], Theorem (2.2), p. 123, and Corollary (3.11), p. 125). If, moreover, $\mathfrak{D}$ is monotone, then the quotient mapping q is monotone, and conversely ([33], Theorem (3.4), p. 126, and Theorem on p. 127).

A continuum I is said to be *irreducible between points a and b* (or shortly *irreducible*) if I contains a and b but no proper subcontinuum of I contains both of them. For every irreducible continuum I there exists a monotone mapping g of I into the closed interval $[0, 1]$ of reals (see [25], p. 199) the point-inverses of which, $g^{-1}(t)$, $0 \leqslant t \leqslant 1$, called *layers of I*, have the property that the decomposition of I into layers is the finest of all linear upper semi-continuous decompositions of I into continua (see [25], § 48, IV, Theorem 3, p. 200).

3. Admissible decompositions. Let X be a continuum. A decomposition $\mathfrak{D}$ of X is said to be *admissible* if

1° $\mathfrak{D}$ is upper semi-continuous,

2° $\mathfrak{D}$ is monotone,

3° for every irreducible continuum I in X, every layer T_t, $0 \leqslant t \leqslant 1$, of I is contained in some element of $\mathfrak{D}$.

It is immediately seen that every continuum X has an admissible decomposition, namely the trivial one, i.e., such that the whole X is the only element of the decomposition. The problem of characterizing continua X which have a non-trivial admissible decomposition is open.

PROPOSITION 2. *A non-trivial decomposition of an irreducible continuum is admissible if and only if it is admissible in the sense of Thomas,* [32], p. 8.

In fact, let a continuum X be irreducible between points a and b. Since conditions 1° and 2° are common for both definitions, it is enough to show that condition 3° holds if and only if every element of $\mathfrak{D}$ not containing a or b separates X. If condition 3° holds, and if an element D of $\mathfrak{D}$ does not contain a and b, then for a point $x \in D$ there is a layer T of X such that $x \in T \subset D$ by 3°. Since every layer not containing a and b separates X — as an element of the minimal decomposition of X admissible in the sense of Thomas — D separates X. The opposite way is obvious because every layer T_t of X, by the minimality of the decomposition of X into layers, is contained in some element of any decomposition of X admissible in the sense of Thomas, and condition 3° holds.

PROPOSITION 3. *If the continuum X is hereditarily arcwise connected, then each upper semi-continuous monotone decomposition of X is admissible.*

Indeed, in this case each irreducible subcontinuum of X is an arc, thus all layers of irreducible continua reduce to points, and condition 3° holds.

Let X be a continuum, $\mathfrak{D}$ — an admissible decomposition of X, and let $q\colon X \to X/\mathfrak{D}$ be the quotient mapping.

THEOREM 1. *If $\mathfrak{D}$ is an admissible decomposition of a continuum X, then the induced quotient space $X/\mathfrak{D}$ is hereditarily arcwise connected.*

Proof [1]. To show that $X/\mathfrak{D}$ is hereditarily arcwise connected, i.e. that every subcontinuum of $X/\mathfrak{D}$ is arcwise connected, it is sufficient to prove that every continuum $A \subset X/\mathfrak{D}$ which is irreducible between two of its points a and b, $a \neq b$, is an arc. Let $x \in q^{-1}(a)$ and $y \in q^{-1}(b)$, and let $C \subset q^{-1}(A) \subset X$ be a continuum irreducible between x and y. Since the decomposition $\mathfrak{D}$ is admissible, it satisfies 3° and thus inequality $a \neq b$ implies that the layer of the continuum C which contains the point x is different from the layer of C which contains the point y. It follows that there exists a mapping φ from C onto the unit segment I such that, for every $t \in I$, the inverse image $\varphi^{-1}(t)$ is a layer of C. It is easy to verify that the mapping $f = q \circ \varphi^{-1}\colon I \to X/\mathfrak{D}$ is well-defined (because the decomposition $\mathfrak{D}$ is admissible) and continuous. Since the continuum $f(I)$ is arcwise connected and $a, b \in f(I) \subset A$, and since the continuum A is irreducible between points a and b, it follows that A is an arc, and the proof is finished.

[1] The present form of this proof is due to J. Krasinkiewicz.

LEMMA 1. *Let M be a subcontinuum of a continuum X and let $\mathfrak{D}$ be an upper semi-continuous decomposition of X into closed sets D. Then the decomposition $\mathfrak{D}_1$ of M into non-empty sets $D \cap M$ is upper semi-continuous.*

In fact, for any closed set $A \subset X$ the union of all sets D with $D \cap A \neq \emptyset$ is a closed set by the definition of the upper semi-continuity of $\mathfrak{D}$. In particular, if A is a closed set in M, the union $\bigcup\{D\colon D \cap A \neq \emptyset\}$ is closed, whence $\bigcup\{D \cap M\colon D \cap A \neq \emptyset\} = (\bigcup\{D\colon D \cap A \neq \emptyset\}) \cap M$ is closed as the intersection of two closed sets. This shows that $\mathfrak{D}_1$ is upper semi-continuous.

LEMMA 2. *If $\mathfrak{D}$ is an upper semi-continuous decomposition of a continuum X into closed sets D, then the decomposition $\mathfrak{D}_2$ of X into components of sets D is upper semi-continuous.*

In fact, this is a consequence of a more general theorem (see [25], §47, VI, Theorem 6, p. 183; [17], Theorem 3.39, p. 137).

COROLLARY 1. *Let M be a subcontinuum of a continuum X, and let $\mathfrak{D}$ be an upper semi-continuous decomposition of X into closed sets D. Then the decomposition $\mathfrak{D}_3$ of M into components of the non-empty intersections $D \cap M$ is upper semi-continuous. If, moreover, $\mathfrak{D}$ is admissible, then $\mathfrak{D}_3$ is admissible.*

In fact, the first part of the corollary follows directly from Lemmas 1 and 2. So $\mathfrak{D}_3$ satisfies condition 1°. By definition $\mathfrak{D}_3$ is monotone, i.e., condition 2° holds. To prove that 3° is fulfilled observe that if a layer T of an irreducible continuum I in M has a point in common with a component D' of an intersection $D \cap M$, then it is contained in D by the admissibility of $\mathfrak{D}$, and in M by hypothesis; thus it is in $D \cap M$. Since T is a connected set having a non-empty intersection with the component D' of $D \cap M$, we have $T \subset D'$. Therefore $\mathfrak{D}_3$ is admissible.

4. The canonical decomposition. If $\mathfrak{D}$ and $\mathfrak{E}$ are upper semi-continuous monotone decompositions of a continuum X, then $\mathfrak{D} \leqslant \mathfrak{E}$ means that every element of $\mathfrak{D}$ is contained in some element of $\mathfrak{E}$, i.e., $\mathfrak{D}$ refines $\mathfrak{E}$. Clearly $\leqslant$ defines a partial ordering on the family of upper semi-continuous monotone decompositions of X.

THEOREM 2. *For every continuum X there exists an admissible decomposition of X which is minimal with respect to $\leqslant$.*

Proof (cf. [32], proof of Theorem 3, p. 8 and 9). Let $\{\mathfrak{D}_\alpha\colon \alpha \in A\}$ be a chain of admissible decompositions of X, and for $z \in X$ and $\alpha \in A$ let Z_α be an element of $\mathfrak{D}_\alpha$ containing z. For fixed $z \in X$, $\{Z_\alpha\colon \alpha \in A\}$ is a chain of continua and we denote by Z the intersection of this chain. Denoting by $\mathfrak{D}$ the collection $\{Z: z \in X\}$ we see that $\mathfrak{D}$ is a decomposition of X each of whose elements is a continuum. Let I be an irreducible continuum in X that contains z, and let T be the layer of I containing z. The decompositions $\mathfrak{D}_a$, $a \in A$, being admissible, we have $T \subset Z_a$ for each $a \in A$, whence $T \subset \bigcap \{Z_a: a \in A\} = Z$. Thus $\mathfrak{D}$ satisfies condition 3°. To prove the upper semi-continuity of $\mathfrak{D}$ suppose that U is open in X and contains $Z \in \mathfrak{D}$. For some $a \in A$ we have $Z_a \subset U$ and since $\mathfrak{D}_a$ is upper semi-continuous, some open subset V of U contains Z and is the union of elements of $\mathfrak{D}_a$. Thus V contains Z and is the union of elements of $\mathfrak{D}$. Therefore $\mathfrak{D}$ is upper-continuous, and thus admissible. Since $\mathfrak{D}$ refines each $\mathfrak{D}_a$, it is a lower bound for the chain. Applying the Kuratowski–Zorn lemma we complete the proof.

The following lemma results from Theorem 44 in [30], p. 299:

LEMMA 3. *Let $\mathfrak{D}$ and $\mathcal{E}$ be upper semi-continuous monotone decompositions of a continuum X. If there is an element of $\mathcal{E}$ that intersects two different elements of $\mathfrak{D}$, then the family $\mathcal{E}'$ of subsets of X whose elements are all components of the intersections $D \cap E$ for $D \in \mathfrak{D}$ and $E \in \mathcal{E}$ is an upper semi-continuous monotone decomposition of X that properly refines $\mathcal{E}$.*

COROLLARY 2. *Let $\mathfrak{D}$ and $\mathcal{E}$ be upper semi-continuous monotone decompositions of a continuum X. If there is an element of $\mathcal{E}$ that intersects two different elements of $\mathfrak{D}$, then $\mathcal{E}$ is not minimal.*

THEOREM 3. *A minimal admissible decomposition of a continuum is unique.*

Proof. Let $\mathfrak{D}$ and $\mathcal{E}$ be admissible decompositions of a continuum X, and suppose that some element of $\mathcal{E}$ meets two different elements of $\mathfrak{D}$. Further, let $\mathcal{E}'$ be the decomposition of X described in Lemma 3. We show that $\mathcal{E}'$ satisfies condition 3°. In fact, if T is a layer of an irreducible continuum I in X, then there are elements D and E of $\mathfrak{D}$ and $\mathcal{E}$ respectively such that $T \subset D \cap E$. Since the layer T is a continuum, and thus a connected set, it is contained in a component of the intersection $D \cap E$, i.e., in an element of $\mathcal{E}'$. Thus $\mathcal{E}'$ is admissible. Since it properly refines $\mathcal{E}$, $\mathcal{E}$ is not minimal. So we have proved that a minimal admissible decomposition of X refines every admissible decomposition of X, and thus the uniqueness is established.

To see the structure of elements of the minimal admissible decomposition of a continuum X, we can apply here the basic ideas used in [4] to describe elements of the canonical decomposition of a λ-dendroid (see [4], p. 25). Firstly we construct a particular admissible decomposition of X (called the canonical decomposition of X) whose elements will be called the strata of X, and next we show that it coincides with the minimal one.

Let X be a continuum and let x be a point of X. To describe the stratum $S(x)$ to which x belongs we shall define (by transfinite induction) an increasing sequence of continua $A_a(x)$ each of which contains the point x. With this in view let us consider in X all irreducible continua I that contain the point x and take in each of them the layer $T(x)$ to which x belongs. Put

$$(4.1) \qquad A_0(x) = \overline{\bigcup T(x)}\,,$$

where the union on the right side of the equality runs over all irreducible continua I such that $x \in I \subset X$. Now suppose that $A_\beta(x)$ are defined for $\beta < a$, and put

$$(4.2) \qquad A_a(x) = \begin{cases} \bigcup\{\operatorname{Ls}_{n\to\infty} A_\beta(x_n): \lim_{n\to\infty} x_n \in A_\beta(x)\}\,, & \text{if} \quad a = \beta+1\,, \\ \overline{\bigcup_{\beta<a} A_\beta(x)}\,, & \text{if} \quad a = \lim_{\beta<a} \beta\,, \end{cases}$$

where, in the case of $a = \beta+1$, the union is taken over all convergent sequences of points $x_n \in X$ with $\lim_{n\to\infty} x_n \in A_\beta(x)$.

So the sets $A_a(x)$ are well-defined for all $a < \Omega$. It follows from the definition — exactly as in [4], p. 19 — that the transfinite sequence $\{A_a(x)\}$ is increasing, i.e., that

$$(4.3) \qquad x \in A_0(x) \subset A_1(x) \subset \ldots \subset A_a(x) \subset \ldots$$

In the same manner as in [4], p. 19, Lemma 1, one can prove that

(4.4) The sets $A_a(x)$ are continua.

Thus $\{A_a(x)\}$ is an increasing sequence of continua. Since the space X is separable as a metric continuum, there exists a countable ordinal ξ such that

(4.5) If $\xi < \eta < \Omega$, then $A_\xi(x) = A_\eta(x)$.

Now we define the *stratum* $S(x)$ of X to which the point x belongs by

$$(4.6) \qquad S(x) = A_\xi(x)\,.$$

Thus we conclude from (4.3) and (4.5) that

$$(4.7) \qquad A_a(x) \subset S(x) \quad \text{for every ordinal } a < \Omega\,.$$

It follows immediately from the above definition of $S(x)$ that

(4.8) If X is an irreducible continuum, then $S(x) = T(x)$ for every $x \in X$, i.e., the notion of a stratum coincides with the notion of a layer.

(4.9) If the continuum X is hereditarily arcwise connected, then $S(x) = \{x\}$ for every $x \in X$.

(4.10) If X is a λ-dendroid, then the notion of a stratum defined above coincides with that defined in [4], p. 21.

Repeating letter by letter the proofs of Lemmas 2 and 3 and of Theorem 2 in [4], pp. 22–24 one can prove the following properties of strata of a continuum X:

(4.11) If $\lim_{n\to\infty} x_n = x$, then $\operatorname{Ls}_{n\to\infty} S(x_n) \subset S(x)$.

(4.12) If $S(x) \cap S(y) \neq \emptyset$, then $S(x) = S(y)$.

Therefore we see that for various x the strata $S(x)$ are either disjoint or identical by (4.12). Hence we can consider the decomposition of the continuum X into its strata $S(x)$. Call this decomposition *canonical*. Since the strata of X are continua by (4.6) and (4.4), the canonical decomposition is monotone. Exactly as in [4], Theorem 3, p. 25, we can show that

(4.13) The canonical decomposition of a continuum X is upper semi-continuous.

Since for each point $x \in X$ and for each irreducible continuum $I \subset X$ such that $x \in I$, the layer $T(x)$ of I to which x belongs is contained in the stratum $S(x)$ by (4.1) and (4.7), condition 3° holds true for the canonical decomposition. Hence

(4.14) The canonical decomposition of a continuum X is admissible.

We shall now prove the following

THEOREM 4. *The canonical decomposition of a continuum X coincides with the minimal admissible decomposition of X.*

Proof. Let $\mathfrak{D}$ be an arbitrary admissible decomposition of X and let D be an element of $\mathfrak{D}$. We claim that

(4.15) If $x \in D$, then $A_\alpha(x) \subset D$ for every $\alpha < \Omega$.

Apply transfinite induction. Firstly let $\alpha = 0$. Take a point $x \in X$, and let $T(x)$ denote the layer containing x of an irreducible continuum $I \subset X$. Since the decomposition $\mathfrak{D}$ is admissible, the condition $x \in D$ implies $T(x) \subset D$ by 3°. This leads to $\bigcup T(x) \subset D$, where the union is taken over all irreducible continua I containing x. The element D of $\mathfrak{D}$ being closed, we have $\overline{\bigcup T(x)} \subset D$, which means $A_0(x) \subset D$ by (4.1).

Secondly let $\alpha > 0$. Now the proof runs exactly as the corresponding part of the proof of Lemma 4 in [4], p. 28. So (4.15) follows.

Therefore, if $x \in D$, then — in particular — $A_\xi(x) \subset D$, where ξ is an ordinal for which (4.5) holds. According to definition (4.6) of the stratum we see that $x \in D$ implies $S(x) \subset D$ and the theorem is proved.

Theorem 4 can be reformulated in the following equivalent form:

COROLLARY 3. *If D is an element of an admissible decomposition of a continuum X, and if S is a stratum of X, then $S \cap D \neq \emptyset$ implies $S \subset D$.*

Since the decomposition of a hereditarily arcwise connected continuum into one-point sets is admissible and minimal, Theorem 4 implies

PROPOSITION 4. *Strata of a hereditarily arcwise connected continuum are one-point sets.*

Let $\mathfrak{D}$ be the canonical decomposition of a continuum X. The canonical mapping is defined to be the quotient mapping φ: $X \to X/\mathfrak{D}$. As the quotient mapping of an admissible decomposition, φ is onto, continuous and monotone (see e.g. [33], (4.1), Theorem, p. 127). In the particular case where X is a λ-dendroid the canonical mapping φ is considered in [4], p. 29. Corollary 1 and Theorem 4 imply

COROLLARY 4. *Let M be a subcontinuum of a continuum X, and let ψ: $M \to \psi(M)$ and φ: $X \to \varphi(X)$ be canonical mappings of M and of X respectively. Then for each point $t \in \psi(M)$ there is a point $s \in \varphi(X)$ such that $\psi^{-1}(t) \subset \varphi^{-1}(s)$.*

5. Special kinds of mappings and of continua. Let a mapping f of a continuum X be continuous and monotone. The mapping f is said to belong to the *class Φ* if for any point $y \in f(X)$, for any point $x \in X$ and for any irreducible continuum I in X it is true that if $x \in f^{-1}(y) \cap I$, then the layer $T(x)$ of x in I is contained in $f^{-1}(y)$. In other words, $f \in \Phi$ if, given a point $x \in X$ and an arbitrary irreducible continuum I that contains x, the whole layer $T(x)$ of x in I is mapped onto the point $f(x)$ under f.

The following two propositions can easily be deduced from the definitions.

PROPOSITION 5. *If the decomposition $\mathfrak{D}$ of a continuum X is admissible, then the quotient mapping q: $X \to X/\mathfrak{D}$ is in Φ.*

PROPOSITION 6. *If $f \in \Phi$, then the decomposition $\mathfrak{D}$ of X into continua $f^{-1}(y)$, $y \in f(X)$, is admissible.*

Thus Theorem 1 implies

COROLLARY 5. *If $f \in \Phi$, then $f(X)$ is hereditarily arcwise connected.*

Proposition 5 and Corollary 3 lead to

COROLLARY 6. *If $f \in \Phi$, then f takes every stratum of X into a point of $f(X)$.*

Proposition 7. *If a monotone mapping f of a hereditarily unicoherent continuum X is in Φ, then for every subcontinuum K of X the partial mapping $f|K$ also is in Φ.*

In fact, $f|K$ is monotone ([3], Lemma 1, p. 932), and if $x \in T(x) \subset I \subset K \subset X$, then $(f|K)\big(T(x)\big) = f\big(T(x)\big) = \{f(x)\}$.

It is not too difficult to show that the assumption of the hereditary unicoherence of X is essential in Proposition 7.

Proposition 8. *Every monotone mapping of a λ-dendroid onto an arcwise connected space is in Φ.*

In fact, if the mapping f of a λ-dendroid X is monotone, then $f(X)$ is a λ-dendroid (see [4], Theorem 4, p. 25). Being arcwise connected by hypothesis, $f(X)$ is a dendroid ([4], p. 15). Now the proposition follows from Corollary 3 in [4], p. 29 and from Proposition 5.

The projection f of a square onto its side shows that Proposition 8 fails to be true if X is an arbitrary continuum, even if $f(X)$ is hereditarily arcwise connected.

Proposition 9. *Every monotone mapping of a hereditarily arcwise connected continuum is in Φ.*

In fact, this follows from Propositions 3 and 5.

Corollary 7. *The hereditary arcwise connectedness of continua is an invariant under monotone mappings.*

In fact, this follows from Proposition 9 and Corollary 5.

Since a monotone image of an irreducible continuum is irreducible (see [25], § 48, I, Theorem 3, p. 192) we have

Proposition 10. *If a continuum X is irreducible and f is a monotone mapping of X onto an arcwise connected continuum, then $f \in \Phi$, and $f(X)$ is an arc or a point.*

Consider now the following example. Let x and y be rectangular coordinates in the plane. Put $A = \{(x, y)\colon 0 < x \leqslant 1$ and $y = \sin(1/x)\}$, $B = \{(x, y)\colon x = 0$ and $-1 \leqslant y \leqslant 1\}$, and let C be an arbitrary arc in the plane which joins points $(0, -1)$ and $(1, \sin 1)$ and which has only its end-points in common with $A \cup B$. Then $S = A \cup B \cup C$ is a continuum, namely the well-known $\sin 1/x$-circle. Let f be a monotone mapping of S onto the unit circle $\{(x, y)\colon x^2 + y^2 = 1\}$, the only non-degenerate point-inverse of which is the segment B. It can easily be observed that $f \in \Phi$. Moreover, it is not too difficult to conclude that if $g\colon S \to g(S)$ is a monotone mapping of S onto a hereditarily arcwise connected continuum $g(S)$, then $g \in \Phi$. But if $h\colon S \to h(S)$ is monotone and $h(S)$ is arcwise connected only, without being hereditarily arcwise connected, then h need not be in Φ, as an example of the identity mapping on S shows.

As we see from the example of the $\sin 1/x$-circle S and also from Propositions 8, 9 and 10, there are continua X with the property that every monotone mapping of X onto a hereditarily arcwise connected continuum is in Φ. Adopt the following definition. A continuum X belongs to the *class* $\mathfrak{K}$ if every monotone mapping of X onto a hereditarily arcwise connected continuum is in Φ. In other words, $X \in \mathfrak{K}$ if and only if every monotone upper semi-continuous decomposition with a hereditarily arcwise connected decomposition space is admissible (cf. Propositions 5 and 6). Propositions 8, 9 and 10 lead to

Corollary 8. *The class $\mathfrak{K}$ contains all λ-dendroids, all irreducible continua and also all hereditarily arcwise connected continua.*

The union of the square $\{(x, y)\colon 0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1\}$ and of the segment $\{(x, 0)\colon 1 \leqslant x \leqslant 2\}$ is an example of a continuum which has a non-trivial admissible decomposition but is not in $\mathfrak{K}$.

It is known that if f is an arbitrary monotone mapping of a λ-dendroid X onto a dendroid (thus onto a hereditarily arcwise connected continuum) and if φ is the canonical mapping of X, then $\varphi^{-1}\big(\varphi(x)\big) \subset f^{-1}\big(f(x)\big)$ for every $x \in X$ (see [4], Corollary 3, p. 29). This result cannot be generalized to arbitrary continua X, as can be seen by the example of the unit square I^2, where φ is the trivial map of I^2 onto a point, and f the projection of I^2 onto its side I. But if we assume that $X \in \mathfrak{K}$, then the following theorem shows that this is the case.

Theorem 5. *A continuum X is in $\mathfrak{K}$ if and only if for every monotone mapping f of X onto a hereditarily arcwise connected continuum and for every point x in X we have*

$$\varphi^{-1}\big(\varphi(x)\big) \subset f^{-1}\big(f(x)\big)\,, \tag{5.1}$$

where φ denotes the canonical mapping of X.

Proof. Given a continuum X in $\mathfrak{K}$, the decomposition $\mathfrak{D}$ of X into continua $f^{-1}(y)$, $y \in f(X)$, is admissible by the definition of $\mathfrak{K}$ and by Proposition 6. Since $\varphi^{-1}\big(\varphi(x)\big)$ is a stratum of X, inclusion (5.1) follows from Corollary 3.

Conversely, assume that inclusion (5.1) holds for every $x \in X$ and for every monotone mapping f of X onto a hereditarily arcwise connected continuum. Let $T(x)$ be the layer of the point x in an irreducible continuum $I \subset X$. Since $\varphi^{-1}\big(\varphi(x)\big)$ is an element of the canonical (and thus admissible) decomposition of X, we have $T(x) \subset \varphi^{-1}\big(\varphi(x)\big)$ by condition 3°, and (5.1) leads to $T(x) \subset f^{-1}(y)$ for each $y \in f(X)$. Therefore the decomposition of X into continua $f^{-1}(y)$ is shown to satisfy condition 3°. It obviously satisfies conditions 1° and 2°, and so the decomposition in question is admissible. Thus $f \in \Phi$ by Proposition 5, whence $X \in \mathfrak{K}$.

COROLLARY 9. *A continuum X is in $\mathcal{K}$ if and only if for every monotone mapping f of X onto a hereditarily arcwise connected continuum there exists one and only one mapping g of $\varphi(X)$ onto $f(X)$ such that the diagram*

$$\begin{tikzcd} X \arrow[rr, "\varphi"] \arrow[dr, "f"'] & & \varphi(X) \arrow[dl, "g"] \\ & f(X) & \end{tikzcd} \tag{5.2}$$

commutes, and g is monotone (here φ denotes the canonical mapping of X).

In fact, the corollary follows from Theorem 6, and the proof is quite similar to that of Theorem 7 in [4], p. 29 and 30.

COROLLARY 10. *The decomposition space of an admissible decomposition of a continuum X belonging to $\mathcal{K}$ is a monotone image of $\varphi(X)$.*

A continuum X is said to be *monostratic* if it consists of only one stratum, i.e. if the canonical mapping φ is the trivial one of X into a point. It is easy to verify that each indecomposable continuum is monostratic. As an example of a monostratic continuum which is hereditarily decomposable one can take a monostratiform λ-dendroid. A non-trivial example of such a λ-dendroid can be found in [5]. In general, for λ-dendroids, the concept of monostratiformity defined in [5], p. 75 and investigated in [5], [3], [6] and [7], p. 58 coincides with the concept of monostraticity. Both the above examples (an indecomposable continuum and a monostratiform λ-dendroid) are in the class $\mathcal{K}$. An n-dimensional cube, where $n > 1$, is a monostratic continuum which does not belong to $\mathcal{K}$.

The next proposition follows from Corollary 9.

PROPOSITION 11. *A continuum $X \in \mathcal{K}$ is monostratic if and only if every monotone mapping of X onto a hereditarily arcwise connected continuum is trivial.*

It is easy to verify the following

PROPOSITION 12. *If a continuum X is the union of a family of monostratic continua such that for every two points of X there is a chain of members of the family joining them, then X is monostratic.*

THEOREM 6. *Let M be a monostratic subcontinuum of a continuum X and let $\mathfrak{D}$ be an admissible decomposition of X. Then M is contained in some element of $\mathfrak{D}$.*

Proof. Suppose on the contrary that there are two different elements D' and D'' of $\mathfrak{D}$ such that $D' \cap M \neq \emptyset \neq D'' \cap M$. The decomposition of M into components of the intersections $D \cap M$, where $D \in \mathfrak{D}$, is admissible according to Corollary 1, and is not trivial, which contradicts the assumption that M is monostratic.

If we take the canonical decomposition as $\mathfrak{D}$ in Theorem 6, we have

COROLLARY 11. *Monostratic subcontinua of a continuum X are contained in strata of X.*

Generalizing the concept of the class $\mathcal{C}$ of mappings introduced in [6], p. 337, let us adopt the following definition. A mapping f of a continuum X into a continuum Y is said to belong to the *class* $\mathcal{C}$ if f takes every stratum of X into a stratum of Y. In other words, if $\varphi\colon X \to \varphi(X)$ and $\psi\colon Y \to \psi(Y)$ are canonical mappings of continua X and Y respectively, then the mapping $f\colon X \to Y$ is defined to be in $\mathcal{C}$ provided that for every point $s \in \varphi(X)$ there is a point $t \in \psi(Y)$ such that $f(\varphi^{-1}(s)) \subset \psi^{-1}(t)$. The following proposition can immediately be seen from the definition.

PROPOSITION 13. *If f_1 and f_2 are mappings of a continuum X into Y and of Y into Z respectively, f_1 and f_2 both being in $\mathcal{C}$, then the mapping $f_2 f_1\colon X \to Z$ is in $\mathcal{C}$.*

As in [6], p. 338 one can observe that the canonical decomposition $\mathfrak{D}$ of a continuum X into its strata, or the canonical mapping $\varphi\colon X \to \varphi(X)$, defines an equivalence relation on X: two points of X are in the relation if and only if they belong to the same stratum of X, or — in the other words — if they are mapped on the same point of $\varphi(X)$ under φ. Thus it follows from Theorem 7.7 in [11], p. 17, and from the definition of the class $\mathcal{C}$ of mappings that for every mapping $f\colon X \to Y$ in $\mathcal{C}$ there exists one and only one mapping g (called the mapping induced by f) of $\varphi(X)$ into $\psi(Y)$ (where ψ is the canonical mapping on Y) such that

$$g(\varphi(x)) = \psi(f(x)) \quad \text{for} \quad x \in X, \tag{5.3}$$

i.e. that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & Y \arrow[d, "\psi"] \\ \varphi(X) \arrow[r, "g"'] & \psi(Y) \end{tikzcd} \tag{5.4}$$

Exactly as in [6], Property 2, p. 338, one can prove

PROPOSITION 14. *If X and Y are continua and if $f\colon X \to Y$ belongs to $\mathcal{C}$, then the induced mapping g of $\varphi(X)$ into $\psi(Y)$ is continuous. Conversely, if f is continuous and if there exists a mapping g such that diagram (5.4) commutes, then f belongs to $\mathcal{C}$ and g is the mapping induced by f.*

Since strata of hereditarily arcwise connected continua are points, Corollaries 5 and 6 imply

PROPOSITION 15. *If $f \in \Phi$, then $f \in \mathcal{C}$.*

PROPOSITION 16. *If a continuum X belongs to $\mathcal{K}$ and a mapping $f\colon X \to Y$ is monotone and onto, then $f \in \mathcal{C}$.*

Indeed, the superposition ψf in (5.4) is a monotone mapping of X onto $\psi(Y)$ which is hereditarily arcwise connected; so Corollary 9 is applicable. Thus there exists exactly one mapping g of $\varphi(X)$ onto $\psi f(X) = \psi(Y)$ which satisfies (5.3). Therefore $f \in \mathcal{C}$ by Proposition 14.

The following two propositions are easy consequences of the definitions.

PROPOSITION 17. *The class* $\mathcal{C}$ *contains all mappings of continua into monostratic continua.*

PROPOSITION 18. *Monostraticity of continua is an invariant under mappings belonging to* $\mathcal{C}$.

Propositions 18 and 16 imply

PROPOSITION 19. *Monostraticity of continua belonging to* $\mathcal{K}$ *is an invariant under monotone mappings.*

Recall that a mapping f of a continuum X onto Y is called *confluent* if for every continuum Q in Y each component of $f^{-1}(Q)$ is mapped onto Q under f (see [1], p. 213). The mapping f is said to be *open* (or *interior*) if the image under f of every open set in X is open in Y. Each monotone and each open mapping is confluent (see [1], V and VI, p. 214). Following J. B. Fugate (see [6], p. 340) one can ask the following question: is monostraticity of continua belonging to $\mathcal{K}$ an invariant under confluent or open mappings? The question is unanswered.

An irreducible continuum is said to be of *type* λ (see [25], p. 197, footnote) or of *type* A' (see [32], p. 13 and [14]) if it has a non-trivial admissible decomposition whose elements (i.e. the layers of the continuum) have empty interiors. Extending this concept to arbitrary continua, we define a class $\mathcal{L}$ of continua in the following way. A continuum X is said to be in the *class* $\mathcal{L}$ if and only if it admits a non-trivial admissible decomposition each of whose elements has an empty interior. Theorem 4 implies immediately

PROPOSITION 20. *A continuum* X *belongs to the class* $\mathcal{L}$ *if and only if each stratum of the canonical decomposition of* X *has an empty interior.*

Since for irreducible continua layers coincide with strata, we have

PROPOSITION 21. *Each irreducible of type* λ *continuum belongs to the class* $\mathcal{L}$.

Since the strata of a hereditarily arcwise connected continuum are one-point sets (see Proposition 4), we have

PROPOSITION 22. *Each hereditarily arcwise connected continuum belongs to the class* $\mathcal{L}$.

A continuum X is said to be *hereditarily unicoherent at a point* p if the intersection of any two subcontinua each of which contains p is connected (see [15]). It is easy to observe that a continuum X is hereditarily unicoherent at p if and only if, given any point $x \in X$, there exists a unique subcontinuum $I(p, x)$ which is irreducible between p and x

(see [15], Theorem 1.3). A continuum X is said to be *smooth at a point* p if X is hereditarily unicoherent at p and for each convergent net of points $\{a_n: n \in D\}$ in X the condition $\lim a_n = a$ implies that the net $\{I(p, a_n): n \in D\}$ is convergent, and $\operatorname{Lim} I(p, a_n) = I(p, a)$ (here D denotes a directed set). A continuum X is said to be *smooth* if there exists a point p such that X is smooth at p (see [15]). Theorem 5.2 in [15] implies the following

PROPOSITION 23. *Each smooth continuum is in* $\mathcal{L}$.

To compare the classes $\mathcal{K}$ and $\mathcal{L}$ of continua let us observe first that there are continua belonging to both of them (cf. e.g. Corollary 8 and Propositions 21 and 22). To see the difference between the classes considered, we describe two examples of continua which show that neither $\mathcal{K} \setminus \mathcal{L}$ nor $\mathcal{L} \setminus \mathcal{K}$ is empty.

Firstly let A be an arc with end-points a and b, and let B be an indecomposable continuum such that the common part $A \cap B$ reduces to the point b. Let C denote the composant of B containing b. Thus $A \cup B$ is a continuum irreducible between a and each point of $B \setminus C$. As an irreducible continuum, $A \cup B$ is in $\mathcal{K}$ by Corollary 8, and it is not in $\mathcal{L}$ since its stratum B does not have an empty interior.

Secondly let (x, y, z) be a point of the Euclidean 3-dimensional space endowed with a rectangular coordinate system $Oxyz$. Put

$$R = \{(0, y, z):\ -1 \leqslant y \leqslant 1 \text{ and } 0 \leqslant z \leqslant 1\}$$

and

$$S_{nm} = \left\{(x, y, z):\ 0 < x \leqslant 2^{-n},\ y = \sin\frac{1}{x} \text{ and } z = \frac{2m-1}{2^n}\right\},$$

where $n = 1, 2, \ldots$ and $m = 1, 2, \ldots, 2^{n-1}$. Defining

$$J = R \cup \bigcup_{n=1}^{\infty} \Big(\bigcup_{m=1}^{2^{n-1}} S_{nm}\Big)$$

we see that J is a continuum for which R is the only non-degenerate stratum. Since the interior of R is empty, we have J to be in $\mathcal{L}$. On the other hand take the projection f of J into the plane $y = 0$, i.e. the mapping f of J that maps each point (x, y, z) of J to $(x, 0, z)$. Obviously f is continuous with the hereditarily arcwise connected image of J:

$$f(J) = \{(0, 0, z):\ 0 \leqslant z \leqslant 1\} \cup$$
$$\cup \bigcup_{n=1}^{\infty}\left(\bigcup_{m=1}^{2^{n-1}}\left\{(x, 0, z):\ 0 < x \leqslant 2^{-n} \text{ and } z = \frac{2m-1}{2^n}\right\}\right).$$

Further, f is not in Φ, because $f(R) = \{(0, 0, z):\ 0 \leqslant z \leqslant 1\}$ is not a point. Consequently J is not in $\mathcal{K}$.

Corollary 11 and Proposition 20 imply

Proposition 24. *If a continuum X is in the class $\mathfrak{L}$, then every monostratic subcontinuum of X has an empty interior.*

Kuratowski ([25], § 48, VII, Theorem 3, p. 216) and Thomas ([32], Theorem 10, p. 15) have characterized irreducible continua of type λ (i.e. of type A' in the sense of Thomas, [32], p. 13) as irreducible continua I with the property that each indecomposable subcontinuum of I has an empty interior. Since each indecomposable continuum is monostratic, one can believe that monostratic continua play a similar role in the theory of decompositions of arbitrary continua as indecomposable continua do in the theory of decompositions of irreducible ones, and that — consequently — continua belonging to the class $\mathfrak{L}$ can be characterized as those which have the property that each monostratic subcontinuum has an empty interior. The following example shows that this is not the case, i.e. that the inverse to Proposition 24 does not hold.

Let Oxy be the rectangular coordinate system in the plane. Let N_V be the irreducible continuum lying in the unit square with the opposite vertices $(0,0)$ and $(1,1)$, described by Knaster in [20], section 2, p. 570 (see also [25], § 48, I, Example 5, p. 191). Put

$$I = \{(x, 0)\colon\ 0 \leqslant x \leqslant 1\}$$

and

$$S = \{(x, y)\colon\ x = |\sin(1/y)| \text{ and } -1 \leqslant y < 0\}$$

and let $K = N_V \cup I \cup S$. We see that $I \cup S$ is an irreducible continuum between the point $(\sin 1, -1)$ and every point of I and that the layer I of $I \cup S$ intersects every layer of N_V. Further, K is a continuum each monostratic subcontinuum of which is a single point. Continua having this property are called hereditarily stratified (see [3], p. 933; cf. also [6], p. 343 and [7], p. 58). The strata of K are single points of S and the continuum $N_V \cup I$ — as the only non-trivial stratum. Since the stratum $N_V \cup I$ of K has a non-empty interior, K is not in $\mathfrak{L}$; however, K contains no non-trivial monostratic subcontinuum.

It is easy to observe that the example of the continuum K is a particular case of a more general situation described below in Proposition 25, the proof of which is immediate.

Proposition 25. *If a stratum of a continuum X intersects each stratum of a continuum Y, then Y is contained in a stratum of $X \cup Y$. If, moreover, Y has common points with only one stratum of X, then the decomposition spaces of X and of $X \cup Y$ are homeomorphic.*

The example given above of the hereditarily stratified continuum K which does not belong to the class $\mathfrak{L}$ is not a λ-dendroid. The following problem is open. Suppose X is a λ-dendroid. Does it follow that, if every monostratic subcontinuum of X has an empty interior, then X is in $\mathfrak{L}$?

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INSTITUTE OF MATHEMATICS OF THE WROCŁAW UNIVERSITY
(INSTYTUT MATEMATYCZNY UNIWERSYTETU WROCŁAWSKIEGO)

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The Sorgenfrey plane in dimension theory

by

Peter Nyikos (Pittsburgh, Penn.)

Abstract. The Sorgenfrey plane S^2 is shown in several different ways to be strongly zero-dimensional. This eliminates it as a possible counterexample to the conjecture that the product of strongly zero-dimensional spaces is itself strongly zero-dimensional. Related properties of S^2 are demonstrated, and several allied conjectures in dimension theory are discussed.

One of the drawbacks of ordinary Lebesque covering dimension is that it does not behave well under products. One would like to have the inequality $\dim(X\times Y)\leqslant \dim X+\dim Y$ hold for arbitrary spaces X and Y. But, as Nagata points out in his text [5, p. 196], it does not even hold for the case where X and Y are paracompact spaces of dimension 0, and X is homeomorphic to Y. The counterexample there given by Nagata is the Sorgenfrey line S, which is the real line with upper half-open intervals $[a, b)$ as a base for the topology [7]. We have $\dim S = 0$ because S is Lindelöf and the base given consists of clopen [closed-and-open] sets [cf. 2, 16.16]. But $\dim S\times S > 0$ because a space of covering dimension zero is automatically normal [5, p. 196] and the Sorgenfrey plane, $S\times S$, is not normal [7].

A definition of covering dimension has been adopted by some authors [2, Chapter 16], [3, p. 97] which agrees with Lebesgue dimension for normal spaces and in some respects is more satisfactory for non-normal spaces. One simply replaces open sets by cozero sets at one point in the definition of Lebesgue dimension: $\dim X\leqslant n$ if every finite cover of X by cozero sets can be refined to an open cover of order n. (A cover $\mathfrak{U}$ of a space X is of *order* m if every point of X belongs to at most $m+1$ members of $\mathfrak{U}$.)

For $\dim X = 0$ this is equivalent to the following simpler condition [2, 16.17]: given two disjoint zero-sets Z_1 and Z_2 of X, there is a clopen set C such that $Z_1\subset C$, $Z_2\cap C=\emptyset$. A Tychonoff space satisfying this property will be called a *strongly zero-dimensional space*.

In this paper, we show that the Sorgenfrey plane is strongly zero-dimensional. This keeps alive the conjecture that $\dim X\times Y\leqslant \dim X+\dim Y$ holds for arbitrary spaces for this kind of covering dimension. The