All $\aleph_1$-dense sets of reals can be isomorphic

by

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Abstract. A non-empty set A of real numbers without endpoints is $\aleph_1$-dense if between every two members of A there are exactly $\aleph_1$ members of A. The proposition that all $\aleph_1$-dense sets of reals are order-isomorphic is shown to be relatively consistent with $ZFC+2^{\aleph_0}=\aleph_2$.

0. Introduction. If $\varkappa$ is an infinite cardinal, let us call a set A of real numbers without end points *$\varkappa$-dense* if A has power $\varkappa$ and between every two distinct members of A there are exactly $\varkappa$ members of A.

Consider the following assertion:

(*) All $\aleph_1$-dense sets of reals are order-isomorphic.

Proposition (*) can be regarded as an extension of Cantor's famous theorem on the rationals, which asserts in particular that all $\aleph_0$-dense sets of reals are isomorphic.

Using a diagonalization argument (see [9]), Sierpiński showed that there exists a collection of $2^{\aleph_0}$ mutually non-isomorphic sets of reals, each of which has power $2^{\aleph_0}$. It follows from this that (*) fails very badly if the continuum hypothesis (CH) is true. Essentially the same argument can be used to produce a model of ZFC (Zermelo–Fraenkel set theory plus the axiom of choice) in which CH is false and (*) still fails very badly. (One starts with a model of ZFC+CH and blows up the continuum with Cohen reals. See [3] for a description of this method of forcing.)

Nevertheless (*) is relatively consistent with ZFC, and it is the purpose of this paper to give the proof.

The results in this paper were announced in [2].

1. Notation and terminology. Our set-theoretical usage is standard. If x is a set then $|x|$ is the cardinality of x. Since we assume the axiom of choice throughout, cardinals may be identified with initial ordinals. Therefore we may make statements like "$\alpha<2^{\aleph_0}$", where α is an ordinal.

If P is a partial ordering and $p,q\in P$, then p and q are *compatible* if there exists $r\in P$ with $p,q\leqslant r$; otherwise p and q are *incompatible*. P has the *countable chain condition* if every set of pairwise incompatible elements of P is countable.

Let A and B be ordered sets with order types φ and ψ respectively. We write $\varphi \leqslant \psi$ to mean that A is order-isomorphic to a subset of B. The order type of the set of all real numbers is denoted by λ.

A class C of order types is *well quasi-ordered* by $\leqslant$ if for any sequence φ_n, $n < \omega$, of types in C there exist i and j such that $i < j$ and $\varphi_i \leqslant \varphi_j$.

2. Statement of results. We prove the relative consistency of (*) in two steps:

THEOREM 1. *Assume* CH. *Suppose A and B are $2^{\aleph_0}$-dense sets of reals. Then there exist sequences $\langle A_a: a < 2^{\aleph_0}\rangle$ and $\langle B_a: a < 2^{\aleph_0}\rangle$ satisfying*

(i) $A = \bigcup \{A_a: a < 2^{\aleph_0}\}$ *and* $B = \bigcup \{B_a: a < 2^{\aleph_0}\}$.

(ii) $A_a \cap A_\beta = B_a \cap B_\beta = 0$ *if* $a \neq \beta$.

(iii) A_a *and* B_a *are countable and dense in* A *and* B *respectively.*

(iv) *Let P be the set of all finite, (one-one) order-preserving functions p mapping A into B which satisfy the restriction that $x \in A_a$ iff $p(x) \in B_a$ for all $x \in$ domain p. Let P be partially ordered by inclusion. Then P has the countable chain condition.*

THEOREM 2. *It is relatively consistent with* ZFC *that* (*) *holds and* $2^{\aleph_0} = \aleph_2$.

Theorem 1 is proved by combining a diagonalization argument similar to Sierpiński's with a back-and-forth argument similar to Cantor's. No knowledge of the theory of forcing and Boolean-valued models for set theory is presupposed. Such knowledge is, however, presupposed for the proof of Theorem 2.

It is easy to see that if (*) holds, then there is an order type φ (namely the type of the $\aleph_1$-dense set) such that for all uncountable types ψ, if $\psi \leqslant \lambda$ then $\varphi \leqslant \psi$. In the terminology of [1], the set $\{\varphi\}$ is a basis for $\{\psi: \psi \leqslant \lambda$ and ψ is uncountable$\}$. Thus Theorem 2 yields also the relative consistency of the assertion that $\{\psi: \psi \leqslant \lambda$ and ψ is uncountable$\}$ has a finite basis.

We also wish to mention an unpublished result obtained recently by Laver. Let M be the class of order types obtained by closing the set $\{0, 1\}$ under the following operations:

(1) well-ordered sums,

(2) conversely well-ordered sums,

(3) arbitrary countable sums.

Laver proved in [5] that the class M is well quasi-ordered by $\leqslant$. Now let M^+ be the class of types obtained by closing $\{0, 1\}$ under the operations (1), (2), (3) and

(4) $\aleph_1$-powered real sums.

By combining the methods of [5] with the methods of this paper, Laver has proved the relative consistency of the assertion that M^+ is well quasi-ordered by $\leqslant$.

We conclude this section with two open questions.

1. Is it consistent to have all $\aleph_2$-dense sets of reals isomorphic? The answer would be yes if Theorem 1 is true without the assumption of CH.

2. The proof of Theorem 2 can easily be extended to show the consistency of Martin's axiom (see [6] and [11]) together with (*). Does (*) follow from Martin's axiom $+ 2^{\aleph_0} > \aleph_1$?

3. Proof of Theorem 1. By replacing A and B with isomorphic images, if necessary, we may assume that both A and B are dense in the reals. Let $I = \{r \times s: r$ and s are rational intervals (i.e., closed intervals with rational end points)$\}$. We say that $r \times s$ and $r' \times s'$ are *separated* if $r \cap r' = s \cap s' = 0$. Let S be the set of all finite, pairwise separated subsets of I. Let P' be the set of all finite, order-preserving functions mapping A into B. Given $x \in S$, let $P(x) = \{p \in P': |p| = |x|$ and $|p \cap i| = 1$ for every $i \in x\}$. Clearly for every $p \in P'$ there are many $x \in S$ such that $p \in P(x)$.

Let $\langle d_a: a < 2^{\aleph_0}\rangle$ enumerate $\{d$: for some $x \in S$, d is a countable subset of $P(x)\}$. For each a, let $x_a \in S$ be such that $d_a \subseteq P(x_a)$ and let $c_a = \{p \in P': \forall x \in S$ if $p \in P(x)$ then $P(x) \cap d_a \neq 0\}$. Intuitively, c_a is the closure of d_a. Note that if U is an uncountable subset of $P(x)$ for some $x \in S$, then there exists $a < 2^{\aleph_0}$ such that $d_a \subseteq U \subseteq c_a$. Let $\langle a_a: a < 2^{\aleph_0}\rangle$ and $\langle b_a: a < 2^{\aleph_0}\rangle$ enumerate A and B respectively, and let $\langle r_n: n < \omega\rangle$ enumerate the set of rational intervals.

The construction of the A_a and B_a takes place by transfinite recursion. Let us assume that we have constructed A_β and B_β for all $\beta < a$. Suppose a is even (if a is odd we reverse the roles of A and B). We will define sequences $\langle k_{an}: n < \omega\rangle$ and $\langle l_{an}: n < \omega\rangle$ of members of A and B respectively such that for all n, $k_{an+1} \in A \cap r_n$ and $l_{an} \in B \cap r_n$. The definition will be made in the order $k_{a0}, l_{a0}, k_{a_1}, l_{a_1}$, etc. At the end we will let $A_a = \{k_{an}: n < \omega\}$ and $B_a = \{l_{an}: n < \omega\}$. Let k_{a0} be the least member of the sequence $\langle a_a: a < 2^{\aleph_0}\rangle$ not yet in some A_β. Now suppose that we have already defined k_{an}. We will show how to define l_{an}. (The passage from l_{an} to k_{an+1} is similar.) For each $m \leqslant n$, each $\beta < a$, each $i \in x_\beta$ and each p in the already-constructed part of P, let $X_{mp\beta i} = \{l \in B: p \cup \{(k_{am}, l)\} \in c_\beta$ and $(k_{am}, l) \in i\}$ *if this set is countable*; otherwise let $X_{mp\beta i} = 0$. Let l_{an} be any element of

$$(B \cap r_n) - (\bigcup_{\beta < a} B_\beta \cup \bigcup_{m,p,\beta,i} X_{mp\beta i}).$$

This completes the construction.

Assertions (i), (ii) and (iii) of Theorem 1 are clear. We must check assertion (iv). Suppose U is an uncountable collection of pairwise incompatible members of P. We may assume that $|p| = n$ for all $p \in U$ and that n is the smallest possible number for which there exists such a collection all of whose members have power n. An easy argument using the fact that each A_a and B_a is countable shows that we may also assume that for all $p, q \in U$, if $p \neq q$ then domain $p \cap$ domain $q =$ range $p \cap$ range $q = 0$. Since S is countable we may assume that $U \subseteq P(x)$ for some $x \in S$. Finally, we may assume that $a < 2^{\aleph_0}$ and $i_0 \in x_a$ are such that $d_a \subseteq U \subseteq c_a$ and for all $p \in U$ the last-constructed member of p is in i_0, and the some coordinate of that member is last constructed, say the second coordinate. Let $Q = \{p - i_0 \colon p \in U\}$. Let $A' = A - \bigcup_{p \in d_a} \text{domain } p$.

LEMMA (a) *Assume $n = 1$. Then for all $a \in A'$ $\{b \in B \colon \{(a, b)\} \in c_a\}$ is countable.*

(b) *Assume $n > 1$. Then there are only countably many $q \in Q$ with the property that for some $a \in A'$, $\{b \in B \colon q \cup \{(a, b)\} \in c_a\}$ is uncountable.*

Proof. We define sets $K_1, K_2 \subseteq c_a$ as follows. Suppose $p \in c_a$ and $p \cap i_0 = \{(a, b)\}$. We put $p \in K_1$ (K_2) if and only if for all $x \in S$, if $p \in P(x)$ then there exist $p' \in d_a \cap P(x)$, $a' \in A$ and $b' \in B$ such that $p' \cap i_0 = \{(a', b')\}$ and $a' < a$ $(a' > a)$. Clearly if $p \in c_a$, $p \cap i_0 = \{(a, b)\}$ and $a \in A'$ then either $p \in K_1$ or $p \in K_2$ (or both).

(a) If part (a) is false, then there exist $a \in A'$ and $b_1, b_2 \in B$ such that $\{(a, b_1)\}$, $\{(a, b_2)\} \in K_1$ or $\{(a, b_1)\}$, $\{(a, b_2)\} \in K_2$. Assume for concreteness that $\{(a, b_1)\}$, $\{(a, b_2)\} \in K_1$ and $b_1 < b_2$. Let s_1 and s_2 be rational intervals such that $s_1 \cap s_2 = 0$, $b_1 \in s_1$ and $b_2 \in s_2$. Let r be a rational interval containing a. Since $\{(a, b_1)\} \in K_1$ we can find $\{(a_1', b_1')\} \in d_a \cap (r \times s_1)$ such that $a_1' < a$. Now let r' be a rational interval containing a but not a_1'. Since $\{(a, b_2)\} \in K_1$ there exists $\{(a_2', b_2')\} \in d_a \cap (r' \times s_2)$ such that $a_2' < a$. But then $a_1' < a_2'$ and $b_1' < b_2'$. Hence $\{(a_1', b_1')\}$ and $\{(a_2', b_2')\}$ are compatible, contradicting the fact that both are members of U. This establishes part (a).

(b) Suppose $i_0 = r_0 \times s_0$. If part (b) is false, then there exist disjoint rational intervals $s, t \subseteq s_0$, $j \in \{1, 2\}$ and uncountable $Q' \subseteq Q$ such that for each $q \in Q'$ there are $a_q \in A'$ and $b_1, b_2 \in B$ such that $b_1 < b_2$, $b_1 \in s$, $b_2 \in t$ and $q \cup \{(a_q, b_1)\}$, $q \cup \{(a_q, b_2)\} \in K_j$. We assume for concreteness that $j = 1$. Since Q' is an uncountable subset of P' all of whose members have power $n-1$, it follows from the minimality of n that there exist $q_1, q_2 \in Q'$ such that $q_1 \neq q_2$ and q_1 and q_2 are compatible. Since by assumption on U domain $q_1 \cap$ domain $q_2 =$ range $q_1 \cap$ range $q_2 = 0$, we can find $x_1, x_2 \in S$ such that $q_1 \in P(x_1)$, $q_2 \in P(x_2)$, every member of x_1 is separated from every member of x_2 and for each $i \in x_1 \cup x_2$ there exists $i' \in x_a$ such that $i \subseteq i'$. Then any member of $P(x_1)$ is compatible with any member of $P(x_2)$. We may assume that $a_{q_1} \leqslant a_{q_2}$. Of course $r_0 \times s \subseteq i_0$ so $x_1' = x_1 \cup \{r_0 \times s\} \in S$. Since for some $b \in B$, $q_1 \cup \{(a_{q_1}, b)\} \in P(x_1')$ there is some $p_1 \in d_a \cap P(x_1')$, some $a_1' \in A$ and some $b_1' \in B$ such that $p_1 \cap i_0 = \{(a_1', b_1')\}$ and $a_1' < a_{q_1}$. Now let r be a rational interval around a_{q_2} small enough so that $a_1' \notin r$ and $r \times t \subseteq i_0$. Let $x_2' = x_2 \cup \{r \times t\}$. Since for some $b \in B$, $q_2 \cup \{(a_{q_2}, b)\} \in P(x_2')$ there exist $p_2 \in d_a \cap P(x_2')$, $a_2' \in A$ and $b_2' \in B$ so that $p_2 \cap i_0 = \{(a_2', b_2')\}$ and $a_2' < a_{q_2}$. But then $a_1' < a_2'$ and $b_1' < b_2'$. Since every member of $P(x_1)$ is compatible with every member of $P(x_2)$, it follows that p_1 is compatible with p_2, a contradiction since $p_1, p_2 \in U$. This establishes part (b), and completes the proof of the lemma.

It follows from the lemma, and from the fact that domain $p_1 \cap$ domain $p_2 =$ range $p_1 \cap$ range $p_2 = 0$ for all $p_1, p_2 \in U$, that there exist $q \in Q$, $a \in A'$ and $b \in B$ such that $q \cup \{(a, b)\} \in U$, a and b are constructed after stage a (*), and for all $a' \in A'$ $\{b' \in B \colon q \cup \{(a', b')\} \in c_a\}$ is countable (of course if $n = 1$ then $q = 0$). Say $a = k_{\beta m}$ and $b = l_{\beta n}$ for β even (the other case is similar) and greater than a. Of course $m \leqslant n$ since by assumption b is the last-constructed coordinate of (a, b). But then $b = l_{\beta n} \in X_{mqai_0}$, which is countable, and this contradicts the definition of $l_{\beta n}$. Hence P satisfies the countable chain condition and the proof is complete.

4. Proof of Theorem 2. We assume that the reader is familiar with the theory of forcing and generic sets, the theory of Boolean-valued models for set theory, and the relationship between them. Suitable references are [10], [7] and [8].

Assume that $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$ in V, the universe of all sets. Suppose A and B are $\aleph_1$-dense sets of reals. If we let $\mathcal{B}_P$ be the complete Boolean algebra associated with the partial ordering P of Theorem 1, then it follows immediately from Theorem 1 that $\mathcal{B}_P$ has the countable chain condition (i.e., every chain is countable) and that in $V^{\mathcal{B}_P}$ the statement "A and B are isomorphic" is Boolean valid. Furthermore, since $|P| = \aleph_1$ we have $|\mathcal{B}_P| = \aleph_1$, so "$2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$" is also Boolean valid in $V^{\mathcal{B}_P}$.

Of course, the above argument could be repeated inside $V^{\mathcal{B}_P}$ for any A' and B' for which the statement "A' and B' are $\aleph_1$-dense sets of reals" is Boolean valid. Using the methods of Solovay and Tennenbaum [11], this process can be iterated $\aleph_2$ times to obtain a Boolean universe $V^{\mathcal{B}}$ in which the statement "$2^{\aleph_0} = \aleph_2$ and (*)" is Boolean valid. Hence if ZFC$+ 2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} = \aleph_2$ is consistent, then so is ZFC$+ 2^{\aleph_0} = \aleph_2 + (*)$. By results of Gödel [4], if ZF is consistent, so is ZFC$+ 2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} = \aleph_2$. This completes the proof.

(*) This is the only place in the proof where the continuum hypothesis is required.

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A positional characterization of the $(n-1)$-dimensional Sierpiński curve in S^n $(n \neq 4)$

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Abstract. Let X be a compact metric continuum which can be embedded in the n-sphere S^n, say by a map $h\colon X \to S^n$, in such a manner that the components of $S^n - h(X)$ form a null sequence $U_1, U_2, \ldots$ satisfying the following conditions: (1) $S^n - U_i$ is an n-cell for each i, (2) $\mathrm{Cl}\,U_i \cap \mathrm{Cl}\,U_j = \emptyset$ if $i \neq j$ (Cl denotes closure), and (3) $\mathrm{Cl}(\bigcup U_i) = S^n$. Then X is called an $(n-1)$-*dimensional Sierpiński curve*. A beautiful theorem of G. T. Whyburn [11] states that, for $n = 2$, there is precisely one $(n-1)$-dimensional Sierpiński curve X up to homeomorphism and that properties (1), (2), and (3) are satisfied for each embedding $h\colon X \to S^2$. We observe in this note that recent developments in the topology of manifolds allow one to extend Whyburn's result directly to higher dimensions $(n \neq 4)$.

Conventions. In all proofs we shall assume that $n = 3$ or $n \geqslant 5$. Our manifolds will have no boundary. If X is an $(n-1)$-dimensional Sierpiński curve and $h\colon X \to S^n$ an embedding of the type ensured by that fact, then $h(X)$ will be called an $\mathcal{S}$-*curve*; i.e., an $\mathcal{S}$-curve is a nicely embedded Sierpiński curve. We assume the reader is thoroughly familiar with [11] and simply indicate the alterations necessary in higher dimensions.

The recent developments alluded to in the first paragraph are the following.

Annulus Theorem [7]. *Let U be a connected open subset of a topological n-manifold M $(n \neq 4)$ and let B and B' be two locally flat n-cells in U. Then there is a homeomorphism $h\colon M \to M$, fixed outside U, such that $h(B) = B'$.*

Approximation Theorem for Cellular Maps [2] [10]. *Let $f\colon M \to N$ denote a proper cellular map of n-manifolds $(n \neq 4)$ and $\{U_a\}$ an f-saturated open covering of M (i.e., $f^{-1}f(U_a) = U_a$ for each index a). Then there is a homeomorphism $g\colon N \to M$ such that $g \circ f = \text{identity} \mod \{U_a\}$ (i.e., for each $p \in M$, there is an index a such that $\{p, g \circ f(p)\} \subset U_a$).*

Corollary. *Suppose K is a compact subset of M such that $f^{-1}f(p) = p$ for each $p \in K$. Then g may be chosen so that $g \circ f|K = \text{identity}$. Hence,*

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