Factors of E^{n+3} with infinitely many bad points

by

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Abstract. In this paper it is shown that: if G_1 is an upper semi-continuous decomposition of E^n such that the set of non-degenerate elements is a countable set of arcs and G_2 is an upper semi-continuous decomposition of E^m such that the set of non-degenerate elements is a set of "nearly" straight arcs, then $E^n/G_1 \times E^m/G_2 \cong E^{n+m}$.

1. Introduction. K. W. Kwun [6] shows that for $n \geqslant 6$ n-dimensional Euclidean E^n space can be represented as the product of two spaces, neither of which is a manifold. In particular, he shows that if α is an arc in E^n and β is an arc in E^m then $E^n/\alpha \times E^m/\beta = E^{m+n}$. Each of these factors can fail to be a manifold at only one point. In [5] it was shown that E^6 can be written as the product of two spaces each of which fails to be a manifold at an uncountable number of points.

In this paper we show that for $n \geqslant 6$, E^n can be written as the product of two spaces each of which has an infinite number of points which fail to have Euclidean neighborhoods.

2. A class of factors of E^n. Suppose α is an arc in E^n (i.e., α is any homeomorphic image of the unit interval) such that $P = \Pi_1/\alpha$ is an injection, where Π_1 is the projection of E^n onto the 1st coordinate. Then α is said to have *property QS*.

Let $A_1, A_2, \ldots$ be a sequence of compact n-manifolds (not necessarily connected) in E^n satisfying

i. $A_{i+1} \subset \operatorname{int} A_i$ for all $i = 1, 2, \ldots$

ii. Each component of $A_\infty = \bigcap_i A_i$ is an arc with property QS.

Let Γ_n be the upper semi-continuous decomposition of E^n into the arcs of A_∞ and the points of $E^n - A_\infty$. Further, let Γ_n be the associated decomposition space. Note that Γ_n might be any of the following decompositions of E^3, (a) "dogbone space", (b) "unused example", (c) "segment space" [4], and (d) "straight arcs space" [2].

THEOREM 1. *Let α be an arc in E^m and let Γ_n be as defined above. Then $E^m/\alpha \times \Gamma_n$ is topologically E^{m+n}.*

3. Shrinking the disks of $a\times A_\infty$. As a first step in proving Theorem 1 we shall show that we can shrink the components of $a\times A_\infty\subset E^m\times E^n$ to points with a pseudo-isotopy (see [6] and [5]). Let K' be the upper semi-continuous decomposition of $E^m\times E^n$ which consists of the components of $a\times A_\infty$, which are disks, and the points of $(E^m\times E^n)-(a\times A_\infty)$. Further, let K be the decomposition space associated with K'.

LEMMA 1. *Let A be a component of A_r and U_a a neighborhood of a in E^m. For every $\varepsilon>0$ there exists an integer $N>r$ and a homeomorphism $h\colon E^m\times E^n\to E^m\times E^n$ such that*

i. $h=\mathrm{id}$ *on* $(E^m\times E^n)-(U_a\times A)$,

ii. $\operatorname{diam}\big(h(a\times A')\big)<\varepsilon$ *for each component A' of $A\cap A_N$.*

In order to prove Lemma 1 we need the following two lemmas from [5].

LEMMA A (Lemma 3 of [5]). *Suppose $\varepsilon>0$ and $A_1, A_2, \ldots$ are as defined above, then there exists a finite collection of n-cells C_i, $i=1,2,\ldots,p$ satisfying*:

1. *For each C_i there exists an arc $a_i\in A_\infty\cap\operatorname{int}C_i$ such that the distance from x to boundary of C_i is less than ε for each $x\in a_i$.*

2. *There exists an integer m such that if A is a component of A_m then $A\subset\operatorname{int}C_i$ for some $i=1,2,\ldots,p$.*

LEMMA B (Lemma 2.2 of [5]). *Suppose $B\neq\emptyset$ is a compact subset of the interior of an n-cell, I^n, and C is a compact subset of I^n disjoint from B. Similarly, suppose $D\neq\emptyset$ is a compact subset of interior of an m-cell, I^m, and E is a compact subset of I^m disjoint from D. Then there exists an $(n+m)$-cell G with the following properties*:

1. $B\times D\subset\operatorname{int}G\subset G\subset\operatorname{int}I^n\times\operatorname{int}I^m$,

2. $G\cap[(B\times E)\cup(C\times D)\cup(C\times E)]=\emptyset$.

Proof of Lemma 1. Let Q be an m-cell in E^m containing the arc a. Let $\delta=\min(\varepsilon$, distance from a to boundary of $U_a)$. By Lemma A there exists a finite collection of m-cells $C_1, C_2, \ldots, C_p$ in A and an integer N such that $C_i\subset\operatorname{int}A$ for each $i=1,2,\ldots,p$ and each component of $A\cap A_N$ is in the interior of some C_i. Let W_i be the union of components of $A\cap A_N$ contained in C_i but not in C_j for any $j<i$. Without loss of generality we shall assume that each W_i, $i=1,2,\ldots,p$ is non-empty.

Let $U_\delta=\{x\in E^m|\ \operatorname{dist}(x,a)<\delta\}$. By Lemma B there exists a collection of $(n+m)$-cells $G_1, G_2, \ldots, G_p$ in $E^m\times E^n$ with the properties

i. $G_i\subset\operatorname{int}Q\times\operatorname{int}C_i$ for $i=1,2,\ldots,p$,

ii. $G_i\cap\big((Q-U_\delta)\times W_j\big)=\emptyset$ for $i=j$,

iii. $G_i\cap(a\times W_j)=\emptyset$ if $j\neq i$,

iv. $G_i\cap\big((Q-U_\delta)\times W_i\big)=\emptyset$ for all $i=1,2,\ldots,p$,

v. $a\times W_i\subset\operatorname{int}G_i$ for all $i=1,2,\ldots,p$.

For each $i=1,2,\ldots,p$ let $H_i\subset G_i$ be an $(n+m)$-cell such that $\operatorname{diam}(H_i)<\delta$ and $H_i\subset G_i-G_j$ for $j\neq i$. Clearly such cells exist. Let h_i be a homeomorphism of G_i onto itself such that $h_i(W_i)\subset H_i$ and $h_i=\mathrm{id}$ on boundary of G_i. Clearly each h_i has an extension h_i^* of E^{m+n} onto itself. $h=h_1^*\circ h_2^*\circ\ldots\circ h_p^*$ is the desired homeomorphism.

LEMMA 2. *Let $A_1, A_2, \ldots$ be defined as above. Then there exists a pseudo-isotopy $H\colon E^{m+n}\times I\to E^{m+n}$ such that*

1. $H/E^{m+n}\times\{0\}=\mathrm{id}$.

2. *If $H_t(x)=H(x,t)$ then for all $t<1$, H_t is a homeomorphism of E^{m+n} onto itself which is the identity on the complement of a compact set.*

3. *H_1 maps E^{m+n} onto itself and maps each component of $a\times A_\infty$ onto a distinct point.*

4. *If $x\in E^{m+n}-(a\times A_\infty)$ then $H_1^{-1}\big(H_1(x)\big)=x$.*

Proof. Let $\varepsilon_i=(\tfrac{1}{2})^i$ and let $N_i=\{x\in E^m|\ \operatorname{dist}(x,a)<1/i\}$. We define a sequence of homeomorphisms h_i, positive integers $M(i)$, and positive numbers δ_i for $i=1,2,\ldots$ inductively as follows. First, let $M(1)=1$. By Lemma 1 we get a homeomorphism $h\colon E^{m+n}\to E^{m+n}$ and an integer $M(2)$ such that

$$h_1(x)=x \quad\text{for}\quad x\in E^{m+n}-(N_1\times A_1)\,, \tag{1}$$

$$\operatorname{diam}h_1(a\times A_{M(2)})<\delta_1=\varepsilon_1\,. \tag{2}$$

Assume that $h_k\colon E^{m+n}\to E^{m+n}$ has been defined so that

$$h_k(x)=x \quad\text{for}\quad x\in E^{m+n}-(N_k\times A_{M(k)})\,,$$

$$\operatorname{diam}\big(h_k(a\times A)\big)<\delta_k \quad\text{for each component } A\in A_{M(k+1)}$$

where $\delta_k>0$ and if $W\subset E^{m+n}$ with $\operatorname{diam}(W)<\delta_k$ then $\operatorname{diam}\big(h(w)\big)<\varepsilon_k$. Since h_k is uniformly continuous there exists a δ_{k+1} such that if $W\subset E^{m+n}$ and $\operatorname{diam}(W)<\delta_{k+1}$ then $\operatorname{diam}\big(h_k(W)\big)<\varepsilon_{k+1}$. By Lemma 1, there exists a homeomorphism h_{k+1} mapping E^{m+n} onto E^{m+n} and an integer $N_{N(k+2)}$ satisfying

$$h_{k+1}(x)=x \quad\text{if}\quad x\in E^{m+n}-(N_{k+1}\times A_{M(k+1)})\,,$$

$$\operatorname{diam}\big(h_{k+1}(a\times A)\big)<\delta_{k+1} \quad\text{for each component } A\subset A_{M(k+2)}\,.$$

Define a sequence $f_i=h_1\circ h_2\circ\ldots\circ h_i$ for $i=1,2,\ldots$ Since each h_i is isotopic to the identity we have that for each $i=1,2,\ldots$, h_i is isotopic to h_{i+1}. Moreover, the isotopy between h_i and h_{i+1} moves no point more than ε_{i-1} for $i=2,3,\ldots$ If we define $f(x)=\lim f_i(x)$ for $x\in E^{m+n}$ then f is well defined and continuous. Thus we can use the f_i along with connecting isotopies to construct the desired pseudo-isotopy.

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LEMMA 3. *Let P: $E^{m+n} \to K$ be the projection map onto the decomposition space K defined above. Then K is homeomorphic to E^{m+n} and K has a metric with respect to which P is uniformly continuous.*

4. Shrinking the arcs. In order to complete the proof of Theorem 1 we must shrink each of the arcs of

1. $P(\alpha \times w)$ for each $w \in E^n - A_\infty$ and

2. $P(z \times \beta)$ for each $z \in E^m - \alpha$ and β a component of A_∞.

To accomplish this shrinking we shall use the method of [6]. So let $X_1 = \{P(\alpha \times w) |\ w \in E^n - A_\infty\}$ and $X_2 = \{P(z \times \beta) |\ z \in E^m - \alpha$ and β is a component of $A_\infty\}$.

Let $T_1 \supset T_2 \supset \ldots$ be the sequence of compact neighborhoods of α in E^m used in the proof of Lemma 1. Now define the following open sets:

$$U_1 = \bigcup P\big(\operatorname{int} T_1 \times (E^n - A_i)\big),$$

$$U_2 = \bigcup P\big((E^m - T_i) \times \operatorname{int} A_1\big).$$

Note that $X_i \subset U_i$ for $i = 1, 2$ and $U_1 \cap U_2 = \emptyset$.

By the Lemma of [6] there exists a pseudo-isotopy of E^{m+n} which shrink the arcs $P(\alpha \times w)$ to points and is fixed on $E^{m+n} - U_1$. To complete the proof we need a pseudo-isotopy which shrinks the arcs $P(z \times \beta)$ to points and is fixed on $E^{m+n} - U_2$. The existence of such a pseudo-isotopy is established if we amend the construction of the pseudo-isotopy for U_1 given in [6] as follows.

1. Replace T_i^1 by T_i and T_i^2 by A_i.

2. In the proof of the Lemma use Theorem 3.6 of [5] in place of Theorem 1 of [6].

5. E^{m+n} mod an infinite set of disks. Let G be an upper semi-continuous decomposition of E^m such that the set H of non-degenerate elements of G is a compact set each of whose components is an arc.

In light of Lemma 2 and the proof of Theorem 1 of [3], we have the following result.

Let L' be the upper semi-continuous decomposition of E^{m+n} whose only non-degenerate elements are the disks $\alpha \times \beta$ where $\alpha \in H$ and $\beta \in A_\infty$. Let L be the decomposition space associated with L'.

LEMMA 4. *L is homeomorphic to E^{m+n} and L has a metric with respect to which the projection map P: $E^{m+n} \to K$ is uniformly continuous.*

Now by Theorem 3 of [7], Theorem 3.6 of [5], and the proof of Theorem 1 above, we have the following result.

THEOREM 2. *Let G^* be the decomposition space associated with the decomposition G and let Γ_n be the space defined above. Then $G^* \times \Gamma_n$ is topologically E^{m+n}.*

References

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