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Strongly additive functions on lattices

by

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Abstract. Let $\mathfrak{L}$ be a lattice of subsets of a set T, with $\phi \in \mathfrak{L}$, let G be a complete metric abelian group, and let $\lambda\colon \mathfrak{L} \to G$ be a strongly additive, (σ, δ)-additive function. It is shown that monotone convergence, together with a regularity condition, are necessary and sufficient conditions for the unique extension of λ to a strongly additive, (σ, δ)-additive function on the generated (σ, δ)-lattice. This generalizes a previous special case, in which $\mathfrak{L}$ is an algebra and G is a Banach space.

1. Introduction. The term *lattice* will refer to a lattice $\mathfrak{L}$ of subsets of a fixed set T, such that $\phi \in \mathfrak{L}$. Let λ map $\mathfrak{L}$ into an Abelian group G. As was remarked by Smiley ([6], p. 239), *additivity*, of ring context, generalizes, in lattice context, to *strong additivity*:

$$\lambda(A \cup B) = \lambda(A) + \lambda(B) - \lambda(A \cap B)\,, \qquad \lambda(\phi) = 0\,.$$

The following result ([5], p. 189; [3], p. 327) will serve as the basic lemma:

1.1. THEOREM. *Let $\mathfrak{L}$ be a lattice, $\mathcal{R}(\mathfrak{L})$ the ring generated by $\mathfrak{L}$, and G an Abelian group. A strongly additive function $\lambda\colon \mathfrak{L} \to G$ extends uniquely to an additive function $\bar{\lambda}\colon \mathcal{R}(\mathfrak{L}) \to G$.*

When μ denotes a strongly additive function on a lattice $\mathfrak{L}$ into an Abelian group G, $\bar{\mu}$ denotes its additive extension to $\mathcal{R}(\mathfrak{L})$, according to 1.1.

If G is a complete metric Abelian group we may suppose its metric ϱ invariant ([4], p. 487). Then $|x| = \varrho(0, x)$ is a non-negative function on G with the properties: $|x| = 0 \Leftrightarrow x = 0$, $|x+y| \leqslant |x| + |y|$, $|-x| = |x|$. Henceforth, a *function* maps a lattice into a fixed complete metric Abelian group G.

A function λ on $\mathfrak{L}$ is σ-*additive* (δ-*additive*) if, for every increasing (decreasing) sequence (L_n) in $\mathfrak{L}$, with $\lim_n L_n \in \mathfrak{L}$, we have $\lambda(L_n) \to \lambda(\lim_n L_n)$. If λ is σ-additive and δ-additive we say that λ is (σ, δ)-*additive*. For strongly additive functions, (σ, δ)-additivity is the natural generalization of σ-additivity in ring context. The σ-ring generalizes to (σ, δ)-*lattice*, that is, lattice closed under countable unions and countable intersections. Let $\lambda\colon \mathfrak{L} \to G$ be strongly additive and (σ, δ)-additive. The purpose of the paper is to establish necessary and sufficient conditions for the unique extension of λ to a strongly additive (σ, δ)-additive function on the (σ, δ)-lattice generated by $\mathfrak{L}$. When $\mathfrak{L}$ is a ring the conditions reduce to *monotone convergence*:

For every monotone sequence (L_n) in $\mathfrak{L}$, $(\lambda(L_n))$ converges. Accordingly, the result will contain the extension theorem for an *algebra* given in ([2], p. 1254).

2. Extension theorems. Let $\mathfrak{L}$ be a lattice and let $L \subseteq T$. If $L \in \mathfrak{L}$ we say that L is an $\mathfrak{L}$-*set*; a sequence of $\mathfrak{L}$-sets will be called an $\mathfrak{L}$-*sequence*. We denote an increasing (decreasing) $\mathfrak{L}$-sequence by $L_n\uparrow$, $L_n \in \mathfrak{L}$ $(L_n\downarrow, L_n \in \mathfrak{L})$, and, if E is the limit set, we write $L_n\uparrow E$, $L_n \in \mathfrak{L}$ $(L_n\downarrow E, L_n \in \mathfrak{L})$. The domain of a function λ (which is always a lattice) is denoted $\mathfrak{D}(\lambda)$. Let λ be a function and let $E \subset T$. The class $\{L\colon L \in \mathfrak{D}(\lambda), L \subseteq E\}$, directed by $\supseteq$, defines the Moore-Smith sequence $(\lambda(L))_{L\subseteq E, L\in\mathfrak{D}(\lambda)}$. Similarly the class $\{L\colon L \in \mathfrak{D}(\lambda), L \supseteq E\}$, if non-empty, directed by $\subseteq$, defines the Moore-Smith sequence $(\lambda(L))_{L\supseteq E, L\in\mathfrak{D}(\lambda)}$. The lattice of countable unions (countable intersections) of $\mathfrak{L}$-sets is denoted $\mathfrak{L}_\sigma(\mathfrak{L}_\delta)$. All sets, other than classes of sets, are subsets of T.

2.1. Definition. Let λ, μ be functions:

(a) λ is μ-*lower regular* if, for every $E \in \mathfrak{D}(\lambda)$, $\lim\limits_{F\subseteq E, F\in\mathfrak{D}(\mu)} \mu(F) = \lambda(E)$.

(b) λ is μ-*upper regular* if, for every $E \in \mathfrak{D}(\lambda)$, the class $\{F\colon F \in \mathfrak{D}(\mu), F \supseteq E\}$ is non-empty, and $\lim\limits_{F\supseteq E, F\in\mathfrak{D}(\mu)} \mu(F) = \lambda(E)$.

2.2. Lemma. *Let λ be a function and let μ be a strongly additive function. Let $E \in \mathfrak{D}(\lambda)$ and let $\varepsilon > 0$ be arbitrary:*

(a) *If λ is μ-lower regular and F is a $\mathfrak{D}(\mu)$-set contained in E such that $F \subseteq G \subseteq E$ and $G \in \mathfrak{D}(\mu)$ implies $|\mu(G)-\mu(F)| < \varepsilon$, then $A, B \in \mathfrak{D}(\mu)$, $A, B \subseteq E$ and $A-B \subseteq E-F$ implies $|\bar{\mu}(A-B)| < 2\varepsilon$.*

(b) *If λ is μ-upper regular and F is a $\mathfrak{D}(\mu)$-set containing E such that $F \supseteq G \supseteq E$ and $G \in \mathfrak{D}(\mu)$ implies $|\mu(G)-\mu(F)| < \varepsilon$, then $A, B \in \mathfrak{D}(\mu)$, $A, B \supseteq E$ and $A-B \subseteq F-E$ implies $|\bar{\mu}(A-B)| < 2\varepsilon$.*

Proof. Under the hypotheses of (a), (b) we have, respectively,

$$|\bar{\mu}(A-B)| = |\bar{\mu}((A\cup B\cup F)-(B\cup F))| = |\mu(A\cup B\cup F)-\mu(B\cup F)| < 2\varepsilon,$$
$$|\bar{\mu}(A-B)| = |\bar{\mu}((A\cap F)-(A\cap B\cap F))| = |\mu(A\cap F)-\mu(A\cap B\cap F)| < 2\varepsilon.$$

2.3. Lemma. *Let λ be a function, let μ be a strongly additive function, and let $\varepsilon > 0$ be arbitrary. If λ is μ-lower regular (μ-upper regular) and (E_n) is a decreasing (increasing) $\mathfrak{D}(\lambda)$-sequence, there exists a decreasing (increasing) $\mathfrak{D}(\mu)$-sequence (F_n) such that $F_n \subseteq E_n$ $(F_n \supseteq E_n)$ and $|\mu(F_n)-\lambda(E_n)| < \varepsilon$. Further, given a $\mathfrak{D}(\mu)$-set A contained in (containing) $\lim\limits_n E_n$, we may choose the sequence (F_n) so as to satisfy the additional condition*

$$A \subseteq \lim_n F_n \subseteq \lim_n E_n \quad (A \supseteq \lim_n F_n \supseteq \lim_n E_n).$$

Proof. In the first case, choose $\mathfrak{D}(\mu)$-sets B_n contained in E_n such that $B_n \subseteq C \subseteq E_n$ and $C \in \mathfrak{D}(\mu)$ implies $|\mu(C)-\lambda(E_n)| < 2^{-(n+2)}\varepsilon$, $|\mu(C)-\mu(B_n)| < 2^{-(n+2)}\varepsilon$. Let $F_n = \bigcap\limits_1^n B_i$, so that $F_n \in \mathfrak{D}(\mu)$, $F_n \subseteq E_n$, $F_n\downarrow$. For $n > 1$, $B_n - F_n$ is expressible as the disjoint union

$$[B_n-(B_n\cap B_1)] \cup [(B_n\cap B_1)-(B_n\cap B_1\cap B_2)] \cup \ldots$$
$$\ldots \cup [(B_n\cap B_1\cap\ldots\cap B_{n-2})-(B_n\cap B_1\cap\ldots\cap B_{n-1})]$$

of which the jth term is contained in $E_j - B_j$. By 2.2(a),

$$|\mu(B_n)-\mu(F_n)| = |\bar{\mu}(B_n-F_n)| < \sum_1^{n-1} 2^{-j-1}\varepsilon < 2^{-1}\varepsilon,$$

so that $|\lambda(E_n)-\mu(F_n)| < \varepsilon$.

In the second case, choose $\mathfrak{D}(\mu)$-sets B_n containing E_n such that $B_n \supseteq C \supseteq E_n$ and $C \in \mathfrak{D}(\mu)$ implies $|\mu(C)-\lambda(E_n)|$, $|\mu(C)-\mu(B_n)| < 2^{-(n+2)}\varepsilon$. Let $F_n = \bigcup\limits_1^n B_i$, then it suffices to observe that, for $n > 1$, $F_n - B_n$ is expressible as the disjoint union

$$[(B_1\cup B_2\cup\ldots\cup B_n)-(B_2\cup\ldots\cup B_n)] \cup [(B_2\cup\ldots\cup B_n)-$$
$$-(B_3\cup\ldots\cup B_n)] \cup \ldots \cup [(B_{n-1}\cup B_n)-B_n],$$

of which the jth term is contained in $B_j - E_j$.

If, in the respective cases, A is a $\mathfrak{D}(\mu)$-set contained in (containing) $\lim\limits_n E_n$, we may choose the B_n so as to satisfy the additional condition

$$A \subseteq B_n \subseteq E_n \quad (A \supseteq B_n \supseteq E_n).$$

Henceforth λ is a fixed function of domain $\mathfrak{L}$.

2.4. Lemma. *Let λ be strongly additive and σ-additive. Then the following conditions are equivalent:*

(a) *For every set E, $\lim\limits_{L\subseteq E, L\in\mathfrak{L}} \lambda(L)$ exists.*

(b) *For every increasing $\mathfrak{L}$-sequence (L_n), $\lim\limits_n \lambda(L_n)$ exists. Moreover, if (a) or (b) holds, then for every increasing $\mathfrak{L}$-sequence (L_n) with limit E, we have $\lim\limits_{L\subseteq E, L\in\mathfrak{L}} \lambda(L) = \lim\limits_n \lambda(L_n)$.*

Proof. (a) $\Rightarrow$ (b): Let $E = \bigcup\limits_1^\infty L_n$ so that $\lim\limits_{L\subseteq E, L\in\mathfrak{L}} \lambda(L) = g$ exists, by (a). It will be shown that $\lim\limits_n \lambda(L_n) = g$. Given $\varepsilon > 0$, there exists an

$\mathfrak{L}$-set L contained in E such that $L \subseteq L' \subseteq E$, $L' \in \mathfrak{L}$ implies $|\lambda(L) - \lambda(L')|$, $|\lambda(L') - g| < \varepsilon$. Then

$$|\lambda(L_n) - g| \leqslant |\lambda(L_n) - \lambda(L)| + |\lambda(L) - g|$$
$$< |\bar{\lambda}(L_n - L)| + |\bar{\lambda}(L - L_n)| + \varepsilon$$
$$= |\lambda(L_n \cup L) - \lambda(L)| + |\lambda(L) - \lambda(L \cap L_n)| + \varepsilon .$$

Finally, $|\lambda(L_n \cup L) - \lambda(L)| < \varepsilon$ and $\lambda(L) - \lambda(L \cap L_n) \to 0$ (σ-additivity).

(b) $\Rightarrow$ (a): If $\lim\limits_{L \subseteq E, L \in \mathfrak{L}} \lambda(L)$ does not exist, for some $\varepsilon > 0$, and every $\mathfrak{L}$-set contained in E, there exists an $\mathfrak{L}$-set L' such that $L \subseteq L' \subseteq E$ and $|\lambda(L) - \lambda(L')| \geqslant \varepsilon$. We may construct inductively an increasing $\mathfrak{L}$-sequence (L_n) such that $|\lambda(L_n) - \lambda(L_{n+1})| \geqslant \varepsilon$, contrary to (b).

Similarly we prove the dual lemma:

2.5. Lemma. *Let λ be strongly additive and δ-additive. Then the following conditions are equivalent*:

(a) *For every set E, contained in at least one $\mathfrak{L}$-set,* $\lim\limits_{L \supseteq E, L \in \mathfrak{L}} \lambda(L)$ *exists.*

(b) *For every decreasing $\mathfrak{L}$-sequence (L_n), $\lim\limits_n \lambda(L_n)$ exists. Moreover, if* (a) *or* (b) *holds, then for every decreasing $\mathfrak{L}$-sequence (L_n) with limit E,* $\lim\limits_{L \supseteq E, L \in \mathfrak{L}} \lambda(L) = \lim\limits_n \lambda(L_n)$.

Henceforth, up to the statement of Theorem 2.10, λ is assumed to be strongly additive, (σ, δ)-additive and monotonely convergent. By Lemma 2.4, λ extends to the function λ_σ on $\mathfrak{L}_\sigma$: $\lambda_\sigma(E) = \lim\limits_{L \subseteq E, L \in \mathfrak{L}} \lambda(L)$; and, by Lemma 2.5, λ extends to the function λ_δ on $\mathfrak{L}_\delta$: $\lambda_\delta(E) = \lim\limits_{L \supseteq E, L \in \mathfrak{L}} \lambda(L)$. Accordingly, λ_σ (λ_δ) is λ-lower regular (λ-upper regular).

2.6. Lemma. *The extension λ_σ (λ_δ) is strongly additive, monotonely convergent and σ-additive (δ-additive).*

Proof. We treat λ_σ, the proof for λ_δ being similar. The strong additivity is clear, so it remains to show that

(a) If $E_n \uparrow E$, $E_n \in \mathfrak{L}_\sigma$ then $\lambda_\sigma(E_n) \to \lambda_\sigma(E)$.

(b) If $E_n \downarrow$, $E_n \in \mathfrak{L}_\sigma$ then $(\lambda_\sigma(E_n))$ converges.

To prove (a), for each n choose an $\mathfrak{L}$-set L_n, $L_n \subseteq E_n$, such that $L_n \subseteq L \subseteq E_n$ and $L \in \mathfrak{L}$ implies $|\lambda(L) - \lambda_\sigma(E_n)| < n^{-1}$. For each n there exists an increasing $\mathfrak{L}$-sequence $(L_{nm})_{m=1,2,\ldots}$ converging to E_n. Write $K_m = \bigcup\limits_{i=1}^{m} (L_{im} \cup L_i)$, so that $L_m \subseteq K_m \subseteq E_m$, $K_m \in \mathfrak{L}$, and $K_m \uparrow E$. Then $|\lambda(K_m) - \lambda_\sigma(E_m)| < m^{-1}$ and $\lambda(K_m) \to \lambda_\sigma(E)$ (2.4), so that $\lambda_\sigma(E_m) \to \lambda_\sigma(E)$.

To prove (b), let $\varepsilon > 0$ be arbitrary. By 2.3 there exists a decreasing $\mathfrak{L}$-sequence (L_n) such that $|\lambda(L_n) - \lambda_\sigma(E_n)| < \varepsilon$, hence $(\lambda_\sigma(E_n))$ is Cauchy.

2.7. Lemma. (a) *If λ_σ is λ_δ-lower regular it is δ-additive.*

(b) *If λ_δ is λ_σ-upper regular it is σ-additive.*

Proof. We prove (a), (b) being similar:

Let $E_n \downarrow E$, E_n, $E \in \mathfrak{L}_\sigma$, and let $\varepsilon > 0$ be arbitrary. There exists an $\mathfrak{L}_\delta$-set F contained in E such that $F \subseteq F' \subseteq E$ and $F' \in \mathfrak{L}_\delta$ implies $|\lambda_\delta(F') - \lambda_\sigma(E)| < \varepsilon$. By 2.3 there exists a decreasing $\mathfrak{L}_\delta$-sequence (F_n) such that $|\lambda_\delta(F_n) - \lambda_\sigma(E_n)| < \varepsilon$ and $F \subseteq \lim\limits_n F_n \subseteq E$. By 2.6, $(\lambda_\sigma(E_n))$ converges. We then have

$$|\lim_n \lambda_\sigma(E_n) - \lambda_\sigma(E)| \leqslant |\lim_n \lambda_\delta(F_n) - \lambda_\sigma(E)| + \varepsilon < 2\varepsilon .$$

2.8. Lemma. *λ_σ is λ_δ-lower regular if and only if λ_δ is λ_σ-upper regular.*

Proof. It will be shown that if λ_σ is λ_δ-lower regular then λ_δ is λ_σ-upper regular. The proof of the converse is analogous. Let $E \in \mathfrak{L}_\delta$. Since λ_σ is strongly additive and monotonely convergent (2.6), and also δ-additive (hypotheses and 2.7), $\lim\limits_{F \supseteq E, F \in \mathfrak{L}_\sigma} \lambda_\sigma(F) = \mu(E)$ exists (2.5). Let $\varepsilon > 0$ be arbitrary. There exists an $\mathfrak{L}_\sigma$-set F containing E such that $F \supseteq F' \supseteq E$ and $F' \in \mathfrak{L}_\sigma$ implies $|\lambda_\sigma(F') - \mu(E)|$, $|\lambda_\sigma(F') - \lambda_\sigma(F)| < \varepsilon$. Let $L_n \downarrow E$, $L_n \in \mathfrak{L}$. We have

$$|\lambda(L_n) - \mu(E)| \leqslant |\lambda(L_n) - \lambda_\sigma(F)| + |\lambda_\sigma(F) - \mu(E)|$$
$$< |\lambda_\sigma(L_n \cup F) - \lambda_\sigma(F)| + |\lambda_\sigma(F) - \lambda_\sigma(L_n \cap F)| + \varepsilon .$$

Since $|\lambda_\sigma(F) - \lambda_\sigma(F \cap L_n)| < \varepsilon$ and $\lambda_\sigma(L_n \cup F) \to \lambda_\sigma(F)$ (2.7) we have $\lambda(L_n) \to \mu(E)$. On the other hand, $\lambda(L_n) \to \lambda_\delta(E)$ (2.5), therefore $\mu(E) = \lambda_\delta(E)$.

All of the proceeding lemmas (except 2.2) are implicitely involved in the statement or in the proof of 2.9, which is the basic lemma for the proof of Theorem 2.10. To reduce the terminology, we say that a function is *λ-regular* if it is λ_δ-lower regular and λ_σ-upper regular.

2.9. Lemma. *Let λ_σ be λ_δ-lower regular, or (equivalently) let λ_δ be λ_σ-upper regular. Let μ be a strongly additive, (σ, δ)-additive, monotonely convergent, λ-regular function.*

(a) *If μ extends λ_δ then μ_σ is λ-regular and δ-additive.*

(b) *If μ extends λ_σ then μ_δ is λ-regular and σ-additive.*

Proof. We prove (a), (b) being similar.

Let $E \in (\mathfrak{D}(\mu))_\sigma$, let $\varepsilon > 0$ be arbitrary, and let $\nu_1(E) = \lim\limits_{F \subseteq E, F \in \mathfrak{L}_\delta} \lambda_\delta(F)$. Then E contains an $\mathfrak{L}_\delta$-set F such that $F \subseteq F' \subseteq E$ and $F' \in \mathfrak{L}_\delta$ implies $|\lambda_\delta(F') - \nu_1(E)| < \varepsilon$. By the definition of μ_σ, E contains a $\mathfrak{D}(\mu)$-set K such that $|\mu(K) - \mu_\sigma(E)| < \varepsilon$, and, because $\mathfrak{L}_\delta \subseteq \mathfrak{D}(\mu)$, we may suppose that $F\subseteq K$. Because μ is λ_δ-lower regular, there exists an $\mathfrak{L}_\delta$-set H such that $F\subseteq H\subseteq K$ and $|\lambda_\delta(H)-\mu(K)|<\varepsilon$. Then

$$|\nu_1(E)-\mu_\sigma(E)|\leqslant|\nu_1(E)-\lambda_\delta(H)|+|\lambda_\delta(H)-\mu(K)|+|\mu(K)-\mu_\sigma(E)|<3\varepsilon.$$

We conclude that $\nu_1(E)=\mu_\sigma(E)$, proving the λ_δ-lower regularity of μ_σ. To prove the λ_σ-upper regularity of μ_σ, we note that E is contained in some $\mathfrak{L}_\sigma$-set (because E is a countable union of $\mathfrak{D}(\mu)$-sets and, since μ is λ_σ-upper regular, every $\mathfrak{D}(\mu)$-set is contained in some $\mathfrak{L}_\sigma$-set), therefore $\lim_{F\supseteq E, F\in\mathfrak{L}_\sigma}\lambda_\sigma(F)=\nu_2(E)$ exists. There exists an $\mathfrak{L}_\sigma$-set F containing E such that $F\supseteq F'\supseteq E$ and $F'\in\mathfrak{L}_\sigma$ implies $|\lambda_\sigma(F')-\nu_2(E)|<\varepsilon$. Let $E_n\uparrow E$, $E_n\in\mathfrak{D}(\mu)$. There exists an increasing $\mathfrak{L}_\sigma$-sequence (F_n) such that $|\lambda_\sigma(F_n)-\mu(E_n)|<\varepsilon$ and $F\supseteq\lim_n F_n\supseteq E$. We have

$$|\nu_2(E)-\mu_\sigma(E)|\leqslant|\nu_2(E)-\lambda_\sigma(\lim_n F_n)|+|\lambda_\sigma(\lim_n F_n)-\mu_\sigma(E)|$$
$$<\varepsilon+|\lim_n\lambda_\sigma(F_n)-\lim_n\mu(E_n)|<2\varepsilon,$$

and therefore $\nu_2(E)=\mu_\sigma(E)$. To establish the δ-additivity of μ_σ, let $A_n\downarrow A$, $A_n, A\in(\mathfrak{D}(\mu_\sigma))$. Since μ_σ is λ_δ-lower regular, A contains an $\mathfrak{L}_\delta$-set F such that $F\subseteq F'\subseteq A$ and $F'\in\mathfrak{L}_\delta$ implies $|\lambda_\delta(F')-\mu_\sigma(A)|<\varepsilon$; and there exists a decreasing $\mathfrak{L}_\delta$-sequence (F_n) such that $|\lambda_\delta(F_n)-\mu_\sigma(A_n)|<\varepsilon$ and $F\subseteq\lim_n F_n\subseteq E$. We then have

$$|\lim_n\mu_\sigma(A_n)-\mu_\sigma(A)|\leqslant|\lim_n\mu_\sigma(A_n)-\lim_n\lambda_\delta(F_n)|+|\lim_n\lambda_\delta(F_n)-\mu_\sigma(A)|$$
$$\leqslant\varepsilon+|\lambda_\delta(\lim_n F_n)-\mu_\sigma(A)|<2\varepsilon,$$

completing the proof.

Applying 2.9 to $\mu=\lambda_\delta$, we have the

COROLLARY. *Let λ_σ be λ_δ-lower regular, or (equivalently) let λ_δ be λ_σ-upper regular. Then $\lambda_{\delta\sigma}$ is λ-regular and δ-additive.*

2.10. THEOREM. *Let λ be a strongly additive, (σ,δ)-additive function on a lattice $\mathfrak{L}$ of subsets of a set T, with values in a complete metric Abelian group G. Then λ is uniquely extendable to a strongly additive, (σ,δ)-additive function λ' on the (σ,δ)-lattice $\mathfrak{L}'$ generated by $\mathfrak{L}$, if and only if the following two conditions are satisfied:*

(a) *λ is monotonely convergent.*

(b) *λ_σ is λ_δ-lower regular, or (equivalently) λ_δ is λ_σ-upper regular.*

Proof of necessity. Since λ' is (σ,δ)-additive on the (σ,δ)-lattice $\mathfrak{L}'$, λ' is monotonely convergent, and so is its restriction λ. Let $E\in\mathfrak{L}_\sigma$ and let $\lim_{F\subseteq E, F\in\mathfrak{L}_\delta}\lambda_\delta(F)=\mu(E)$ (which exists because λ_δ is a restriction of λ'). Let $\varepsilon>0$ be arbitrary. There exists an $\mathfrak{L}_\delta$-set K contained in E such that $K\subseteq H\subseteq E$ and $H\in\mathfrak{L}_\delta$ implies $|\lambda_\delta(H)-\mu(E)|$, $|\lambda_\delta(H)-\lambda_\delta(K)|<\varepsilon$. Let $L_n\uparrow E$, $L_n\in\mathfrak{L}$. Then

$$|\lambda(L_n)-\mu(E)|\leqslant|\lambda(L_n)-\lambda_\delta(K)|+|\lambda_\delta(K)-\mu(E)|$$
$$<|\lambda_\delta(L_n\cup K)-\lambda_\delta(K)|+|\lambda_\delta(K)-\lambda_\delta(L_n\cap K)|+\varepsilon.$$

But $|\lambda_\delta(L_n\cup K)-\lambda_\delta(K)|<\varepsilon$ and $\lambda_\delta(K)-\lambda_\delta(L_n\cap K)=\lambda'(K)-\lambda'(L_n\cap K)\to 0$, and therefore $\lambda(L_n)\to\mu(E)$, so that $\mu(E)=\lambda_\sigma(E)$.

Proof of sufficiency. Let Π be the set of all pairs $(\mathfrak{K},\mu)$, where $\mathfrak{K}$ is a lattice, $\mathfrak{L}_{\delta\sigma}\subseteq\mathfrak{K}\subseteq\mathfrak{L}'$, and $\mu\colon\mathfrak{K}\to G$ is an extension of $\lambda_{\delta\sigma}$ with the properties:

(i) μ is strongly additive, (σ,δ)-additive, monotonely convergent and λ-regular.

(ii) μ is the only strongly additive, (σ,δ)-additive extension of $\lambda_{\delta\sigma}$ on $\mathfrak{K}$.

We partially order Π by the usual formula:

$$(\mathfrak{K}_2,\mu_2)\geqslant(\mathfrak{K}_1,\mu_1)\Leftrightarrow\mathfrak{K}_2\supseteq\mathfrak{K}_1\quad\text{and}\quad\mu_2\text{ extends }\mu_1.$$

By 2.6, the hypothesis and the corollary of 2.9, $(\mathfrak{L}_{\delta\sigma},\lambda_{\delta\sigma})\in\Pi$. Let Λ be any non-empty linearly ordered subset of Π. Then $\mathfrak{K}_0=\bigcup\{\mathfrak{K}\colon(\mathfrak{K},\mu)\in\Lambda\}$ is a lattice such that $\mathfrak{L}_{\delta\sigma}\subseteq\mathfrak{K}_0\subseteq\mathfrak{L}'$. The function $\mu_0\colon\mathfrak{K}_0\to G$ is well defined if we write $\mu_0(E)=\mu(E)$, where $(\mathfrak{K},\mu)$ is any element of Λ such that $E\in\mathfrak{K}$. We will verify (i) and (ii) for μ_0. It is clear that μ_0 is strongly additive and λ-regular. Let (E_n) be a decreasing $\mathfrak{K}_0$-sequence. Let $\varepsilon>0$ be arbitrary. Because μ_0 is λ_δ-lower regular, Lemma 2.3 implies the existence of a decreasing $\mathfrak{L}_\delta$-sequence (F_n) such that $|\lambda_\delta(F_n)-\mu_0(E_n)|<\varepsilon$, and it follows that $(\mu_0(E_n))$ is Cauchy. If, further, $E_n\downarrow E$, $E\in\mathfrak{K}_0$, then E contains an $\mathfrak{L}_\delta$-set K such that $K\subseteq K'\subseteq E$ and $K'\in\mathfrak{L}_\delta$ implies $|\lambda_\delta(K')-\mu_0(E)|<\varepsilon$. According to 2.3 we may suppose that $K\subseteq\lim_n F_n\subseteq E$, and we conclude that $|\lim_n\mu_0(E_n)-\mu_0(E)|<2\varepsilon$. Treating increasing $\mathfrak{K}_0$-sequences similarly, we conclude that μ_0 is monotonely convergent and (σ,δ)-additive. To verify (ii) for μ_0, we note that every strongly additive, (σ,δ)-additive extension of $\lambda_{\delta\sigma}$ on $\mathfrak{K}_0$ coincides with the restriction of μ_0 to $\mathfrak{K}$, for every $(\mathfrak{K},\mu)\in\Lambda$. We may now apply Zorn's lemma to assert the existence of a maximal element $(\mathfrak{K}',\mu')$ of Π.

By 2.6, μ'_σ is strongly additive, σ-additive and monotonely convergent. By the hypothesis and 2.9, μ'_σ is λ-regular and δ-additive. By 2.4, μ'_σ is the only strongly additive, (σ,δ)-additive extension of μ' on $\mathfrak{K}'_\sigma$. We have shown that $(\mathfrak{K}'_\sigma,\mu'_\sigma)\in\Pi$, so, by the maximality, $\mathfrak{K}'_\sigma=\mathfrak{K}'$. Similarly $\mathfrak{K}'_\delta=\mathfrak{K}'$, and therefore $\mathfrak{K}'=\mathfrak{L}'$. Then $\lambda'=\mu'$ is the required extension, and the proof is complete.

If the domain of a strongly additive function is a ring, then the condition of (σ, δ)-additivity reduces to σ-additivity.

2.11. THEOREM. *Let λ be a σ-additive function on a ring $\mathfrak{R}$ of subsets of a set T, with values in a complete metric Abelian group G. Then λ is uniquely extendable to a σ-additive function λ' on the σ-ring $\mathfrak{R}'$ generated by $\mathfrak{R}$ if only if λ is monotonely convergent.*

Proof. Let $\mathfrak{L}'$ $(\mathfrak{R}')$ be the (σ, δ)-lattice (σ-ring) generated by $\mathfrak{R}$. We note that $\mathfrak{R}'$ is the monotone class generated by $\mathfrak{R}$ ([1], p. 12). Since $\mathfrak{L}'$ is a monotone class containing $\mathfrak{R}$, $\mathfrak{L}' \supseteq \mathfrak{R}'$. On the other hand, $\mathfrak{R}'$ is a (σ, δ)-lattice containing $\mathfrak{R}$, hence also $\mathfrak{L}'$, so we have $\mathfrak{R}' = \mathfrak{L}'$. Now the theorem will follow, as a corollary of 2.10, if we show that the monotone convergence of λ implies the λ_δ-lower regularity of λ_σ. Let $E \in \mathfrak{R}_\sigma$. Since λ_δ is monotonely convergent (hypothesis and 2.6), the argument (b) $\Rightarrow$ (a) of the proof of 2.4 shows that $\lim\limits_{F \subseteq E, F \in \mathfrak{R}_\delta} \lambda_\delta(F) = \mu(E)$ exists. Let $\varepsilon > 0$ be arbitrary. There exists an $\mathfrak{R}_\delta$-set K contained in E such that $K \subseteq K' \subseteq E$ and $K' \in \mathfrak{R}_\delta$ implies $|\lambda_\delta(K') - \mu(E)|$, $|\lambda_\delta(K')| - \lambda_\delta(K)| < \varepsilon$. Let $R_n \uparrow R$, $R_n \in \mathfrak{R}$. Then

$$|\lambda(R_n) - \mu(E)| \leqslant |\lambda(R_n) - \lambda_\delta(K)| + |\lambda_\delta(K) - \mu(E)|$$
$$< |\lambda_\delta(R_n \cup K) - \lambda_\delta(K)| + |\bar{\lambda}_\delta(K - R_n)| + \varepsilon$$

and $|\lambda_\delta(R_n \cup K) - \lambda_\delta(K)| < \varepsilon$. Because $\mathfrak{R}$ is a ring, $K - R_n \in \mathfrak{R}_\delta$, so $\bar{\lambda}_\delta(K - R_n) = \lambda_\delta(K - R_n) \to 0$ (2.6). Therefore $\lambda(R_n) \to \mu(E)$. On the other hand, $\lambda(R_n) \to \lambda_\sigma(E)$, therefore $\mu(E) = \lambda_\sigma(E)$.

References

[1] N. Dinculeanu, *Vector Measures*, 1967.
[2] G. Fox, *Inductive extension of a vector measure under a convergence condition*, Can. J. Math. 20 (1968), pp. 1246–1255.
[3] J. Kisyński, *Remark on strongly additive set functions*, Fund. Math. 63 (1968), pp. 327–332.
[4] V. Klee, *Invariant metrics in groups (solution of a problem of Banach)*, Proc. Amer. Math. Soc. 3 (1952), pp. 484–487.
[5] B. J. Pettis, *On the extension of measures*, Ann. of Math. 54 (1951), pp. 186–197.
[6] M. Smiley, *A note on measure functions in a lattice*, Bull. Amer. Math. Soc. 46 (1940), pp. 239–241.

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Various approaches to the fundamental groups

by

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Abstract. The notion of the fundamental group introduced by K. Borsuk in [1] is useful in the Borsuk approach to the theory of shapes. However, if one is concerned with the Mardešić and Segal approach (see [3], [4]), then some other notions seem to be more convenient. One of them, the notion of the limit homotopy group, has been defined by the author in [7]. Another one, the notion of the shape group, is defined here in § 4. As regards compact metric spaces, these three approaches turn out to be in some sense equivalent (§ 6). To explain these connections we start with preliminaries concerning the category theory (§ 1).

For the convenience of the reader, definitions of the two categories $\mathfrak{R}^*, \hat{\mathfrak{R}}^*_{\sim}$ (of ANR($\mathfrak{R}$)-systems) and of the categories $\mathfrak{G}^*, \hat{\mathfrak{G}}^*$ (of inverse systems of groups) are recalled in the Appendix.

1. Isomorphism and quasi-isomorphism of functors. One of the basic concepts in category theory is the notion of natural transformation and of natural equivalence of functors (see [6], p. 59). The notion of natural equivalence enables us to identify two functors $\Pi, \Pi' \colon \mathfrak{K} \to \mathfrak{L}$, which, from the intuitive point of view, coincide. Here, the natural transformations are treated as morphisms in some category of functors; then the natural equivalence is simply an isomorphism in this category. In turn, this notion of isomorphism of functors from $\mathfrak{K}$ to $\mathfrak{L}$, where the categories $\mathfrak{K}, \mathfrak{L}$ are both fixed, is extended to the notion of quasi-isomorphism of functors. It enables us to study the connection between two functors $\Pi\colon \mathfrak{K} \to \mathfrak{L}$ and $\Pi'\colon \mathfrak{K}' \to \mathfrak{L}$.

Given two categories $\mathfrak{K}, \mathfrak{L}$, we are concerned with covariant functors from $\mathfrak{K}$ to $\mathfrak{L}$. Let us consider the category $\mathcal{M}^{\mathfrak{K}\mathfrak{L}}$ [1] with all those functors as objects and with morphisms defined as follows:

for $\Pi, \Pi' \in \mathrm{Ob}_{\mathcal{M}}$

$$\Lambda \in \mathrm{Mor}_{\mathcal{M}}(\Pi, \Pi') \quad \text{whenever } \Lambda = \{\lambda_X\}_{X \in \mathrm{Ob}_{\mathfrak{K}}}$$

where

$$\lambda_X \in \mathrm{Mor}_{\mathfrak{L}}\big(\Pi(X), \Pi'(X)\big)$$

(1) We shall often write $\mathcal{M}$ instead of $\mathcal{M}^{\mathfrak{K}\mathfrak{L}}$.