Covering dimension modulo a class of spaces

by

J. M. Aarts (Delft) and **T. Nishiura** (Detroit, Mich.) (1)

Abstract. All spaces are to be metrizable. The paper is motivated by the following extension problem: Let $\mathcal{T}$ be a topologically closed class of spaces and n an integer. Characterize those spaces X which have an extension Y such that $Y \in \mathcal{T}$ and the dimension of the remainder $Y \setminus X$ does not exceed n. Covering dimension applied to the remainder $Y \setminus X$ leads naturally to the concept of a $\mathcal{T}$-border cover and the order of a $\mathcal{T}$-border cover. Analogous to dimension theory, $\mathcal{T}$-dim and $\mathcal{T}$-Dim are defined, the covering dimensions modulo a class $\mathcal{T}$, according to whether finite or arbitrary $\mathcal{T}$-border covers are refined.

A solution to the above characterization problem is given in terms of $\mathcal{T}$-Dim under reasonably weak restrictions on $\mathcal{T}$. The following analogues of theorems of dimension theory are proved under mild assumptions on $\mathcal{T}$: 1. The characterization of $\mathcal{T}$-dim by mappings into spheres. 2. Sum theorems for $\mathcal{T}$-dim, $\mathcal{T}$-Dim and $\mathcal{T}$-Ind, the strong inductive dimension modulo the class $\mathcal{T}$. 3. Dowker's theorem: $\mathcal{T}\text{-dim} = \mathcal{T}\text{-Dim}$. 4. $\mathcal{T}\text{-dim} \leqslant \mathcal{T}\text{-Ind}$.

With the aid of the so-called ambiguous spaces, the notions of extensions and kernels (introduced in [3]) are shown to be complementary. A unified theory of kernels and extensions results.

1. Introduction. All spaces are understood to be metrizable.

The present paper is an investigation of the following problem of extensions of spaces (cf. [5]):

Let $\mathcal{T}$ be a topologically closed class of spaces and n be an integer. Find a characterization of those spaces X which possess an extension $Y \in \mathcal{T}$ whose remainder $Y \setminus X$ has dimension not exceeding n.

This problem has been succesfully resolved for the class of compact spaces by Smirnov [9]. The solution uses certain sequences of so called border covers (see Section 2 for a definition) with a uniformly bounded order. Also, Aarts [1] has shown that, for the class of topologically complete spaces, consideration of the order of border covers yields a solution to the extension problem. In this case, no special sequence of border covers is required. The present work gives a reasonable solution to the

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general extension problem via a covering dimension theory using border covers. The solution includes the results of [1]. It also becomes evident from examples that, for the class of compact spaces, a solution to the extension problem will require more structure such as those assumed by Smirnov [9].

Covering dimension theory using border covers also yields a nice way to investigate kernels which were studied in [3]. (For the definition of a kernel, see Section 2.) In [3], we have shown by examples that extensions and kernels are divergent concepts. None the less, we show in the present paper that these two notions are complementary and that covering dimension theory using border covers results in a unified theory of extensions and kernels.

The paper begins with notations and basic definitions in Section 2. The standard procedures for defining covering dimensions are followed to give the two covering dimensions modulo a class of spaces $\mathcal{P}$ by means of border covers; the smaller, $\mathcal{P}$-dim, being the result of finite border covers and the larger, $\mathcal{P}$-Dim, the result of arbitrary border covers. As the development proceeds, the conditions on $\mathcal{P}$ will, in general, become more restrictive. With minimal conditions, mappings into spheres are discussed in Section 3. This leads to a characterization of $\mathcal{P}$-dim in Section 4. Relationships between $\mathcal{P}$-Dim and kernels are derived in Section 5. The solution of the extension problem via $\mathcal{P}$-Dim is discussed in Section 6. A unified theory of extensions and kernels is presented in Section 7. Section 8 concerns finite sum theorems for the two covering dimensions. Countable sum theorems for the covering dimensions and for $\mathcal{P}$-Ind, the strong inductive dimension which neglects a class $\mathcal{P}$ (see 2.4), are discussed in Section 9. Section 10 develops the analogue of Dowker's theorem: $\mathcal{P}\text{-dim} = \mathcal{P}\text{-Dim}$, and also develops the inequality $\mathcal{P}\text{-Ind} \geqslant \mathcal{P}\text{-dim}$. All these partial results are put together in Section 11.

2. Definitions and notations. The nature of the problem under consideration requires a substantial number of definitions. To ease the exposition and for the convenience of the reader, we collect most of them in this section.

2.1. Let $\mathcal{M}$ denote the class of metrizable spaces. A subclass $\mathcal{P}$ of $\mathcal{M}$ is said to be *topologically closed* if $X \in \mathcal{P}$ whenever X is homeomorphic to Y and $Y \in \mathcal{P}$. *Throughout this paper all classes considered are assumed to be topologically closed. We also assume throughout that* $\emptyset \in \mathcal{P}$.

2.2. Definition. Let $\mathcal{P}$ be a class of spaces. A *$\mathcal{P}$-border cover* of a space X is an open collection $\mathcal{V}$ such that $(X \setminus \bigcup \mathcal{V}) \in \mathcal{P}$. The set $X \setminus \bigcup \mathcal{V}$ is called the *enclosure of* $\mathcal{V}$. The *order of a $\mathcal{P}$-border cover* (*at a point*) is defined as usual (e.g. [8] Definition I.3). Whenever no confusion is likely to arise, we shall use "border cover" rather than "$\mathcal{P}$-border cover".

In dealing with border covers we adopt the following convention. If $\mathcal{V} = \{V_\gamma \mid \gamma \in \Gamma\}$ is a collection in X and $Y \subset X$, then $\mathcal{V}|Y$ — the restriction of $\mathcal{V}$ to Y — is the collection $\{V_\gamma \cap Y \mid \gamma \in \Gamma\}$. We also adopt the standard conventions for collections ([8], Section I.1).

2.3. We now define the *small* and *large covering dimensions modulo a class* $\mathcal{P}$, $\mathcal{P}$-dim and $\mathcal{P}$-Dim.

Definition. Let $\mathcal{P}$ be a class of spaces and X be a space. $\mathcal{P}\text{-dim}\, X \leqslant n$ if for any *finite* $\mathcal{P}$-border cover $\mathcal{U}$ of X there exists a $\mathcal{P}$-border cover $\mathcal{V}$ such that $\mathcal{V} < \mathcal{U}$ and order $\mathcal{V} \leqslant n+1$. $\mathcal{P}\text{-Dim}\, X \leqslant n$ if for any $\mathcal{P}$-border cover $\mathcal{U}$ of X there exists a $\mathcal{P}$-border cover $\mathcal{V}$ such that $\mathcal{V} < \mathcal{U}$ and order $\mathcal{V} \leqslant n+1$. It will be agreed that $\mathcal{P}\text{-dim}\, X = -1$ ($\mathcal{P}\text{-Dim}\, X = -1$) if and only if $X \in \mathcal{P}$. When $\mathcal{P} = \{\emptyset\}$, we drop the prefix $\mathcal{P}$ and simply write dim. (That Dim = dim on $\mathcal{M}$ is a well-known theorem of Dowker [8], Theorem II.6). We shall show that $\mathcal{P}\text{-Dim} = \mathcal{P}\text{-dim}$ under rather weak assumptions (Section 11). It is unknown whether or not $\mathcal{P}\text{-dim} = \mathcal{P}\text{-Dim}$ in general.

2.4. The following definitions were given in [3]. The notion of $\mathcal{P}$-deficiency is motivated by the extension problem. One of the goals of the present paper is to find relationships between $\mathcal{P}$-dim, $\mathcal{P}$-Dim, $\mathcal{P}$-deficiency, $\mathcal{P}$-surplus and strong inductive dimension which neglects $\mathcal{P}$.

Definitions. Let $\mathcal{P}$ be a class of spaces and X be a space. A *$\mathcal{P}$-kernel* (*$\mathcal{P}$-hull*) of X is a space $Y \in \mathcal{P}$ with $Y \subset X$ ($Y \supset X$). (Thus, the enclosure of a $\mathcal{P}$-border cover of X is a closed $\mathcal{P}$-kernel of X.)

$\mathcal{P}\text{-Sur}\, X = \min\{\operatorname{Ind} X \setminus Y \mid Y \text{ is a } \mathcal{P}\text{-kernel of } X\}$ and

$\mathcal{P}\text{-Def}\, X = \inf\{\operatorname{Ind} Y \setminus X \mid Y \text{ is a } \mathcal{P}\text{-hull of } X\}$.

(Here, Ind is the strong inductive dimension. Ind = dim on $\mathcal{M}$.)

$\mathcal{P}$-Sur and $\mathcal{P}$-Def are, respectively, the *strong surplus* and *strong deficiency with respect to* $\mathcal{P}$.

The *strong inductive dimension which neglects* $\mathcal{P}$, of a space X, $\mathcal{P}\text{-Ind}\, X$, is defined in an inductive manner similar to that of strong inductive dimension, [8] Definition 1.5. The induction is started by taking $\mathcal{P}\text{-Ind}\, X = -1$ if and only if $X \in \mathcal{P}$. For peculiarities of this definition see [3].

2.5. Monotonicity of a class. In our discussions we will need to assume certain monotonicity conditions on a class of spaces $\mathcal{P}$. That is, if Y is a certain type of subset of X and $X \in \mathcal{P}$, then $Y \in \mathcal{P}$. The types we will consider are closed, open, F_σ and G_δ. We will use these modifiers to express the type of monotonicity we wish to use. For example, closed monotone class $\mathcal{P}$. The relationships between the various types of monotone classes are obvious.

2.6. Additivity of a class. We will use two basic kinds of additivity conditions on a class $\mathcal{P}$ of spaces.

DEFINITION 1. Let $\mathcal{P}$ be a class of spaces. If $Z \in \mathcal{P}$ whenever $Z = X \cup Y$ with X closed, $X \in \mathcal{P}$ and $Y \in \mathcal{P}$, then we say $\mathcal{P}$ is *weakly additive*.

DEFINITION 2. Let $\mathcal{P}$ be a class of spaces. Suppose $X \in \mathcal{P}$ whenever there is a closed cover $\mathcal{F}$ of a certain type with $\mathcal{F} \subset \mathcal{P}$. Then we say $\mathcal{P}$ is *closed additive of that type*. There are three types we will consider: finite, countable and locally finite. We will use the expressions *finitely closed additive*, *countably closed additive* and *locally finitely closed additive*.

2.7. MULTIPLICATIVITY OF A CLASS. The multiplicativity condition, which will be used, will be defined in Section 6.2.

2.8. CONVENTIONS. If A is a subset of X, then $B_X(A)\big(\mathrm{cl}_X(A)\big)$ denotes the boundary (closure) of A in X. If the space X need not be emphasized the subscripts will be dropped.

Let F be a subset of a space X. Then we agree to the following:

M1. $A \subset B \bmod F$ means $A \backslash F \subset B \backslash F$.

M2. order $\mathfrak{V} \leqslant n \bmod F$ means order $\mathfrak{V}|\,(X\backslash F) \leqslant n$.

M3. $A \cap B = \emptyset \bmod F$ means $(A \cap B)\backslash F = \emptyset$.

These notations should be read as *A is contained in B modulo F*, etc. They will also be used in composite formulas. In connection with this convention the following propositions are very useful and will be tacitly used in the sequel. The easy proofs are omitted.

PROPOSITION 1. *If F is a closed subset of X, then* $\mathrm{cl}_{X\backslash F}(A\backslash F) = (\mathrm{cl}_X A)\backslash F$ *for every subset A of X.*

PROPOSITION 2. *If U is open in X, then $B_{X\backslash F}(U\backslash F) \subset B_X(U)\backslash F$ for every subset F of X. Furthermore, when F is closed, $B_{X\backslash F}(U\backslash F) = B_X(U)\backslash F$.*

PROPOSITION 3. *If $F_1 \subset F_2$, then relationship $A \subset B \bmod F_1$ implies $A \subset B \bmod F_2$. A similar statement holds for order modulo F_1 and F_2.*

3. Mappings in spheres. The propositions of this section will assume only the minimal conditions listed in Section 2.1 for the class $\mathcal{P}$. That is, *$\mathcal{P}$ is topologically closed and $\emptyset \in \mathcal{P}$*. After the presentation of some simple consequences of the definitions, we will deduce a theorem which relates the covering dimensions modulo $\mathcal{P}$ and the extendability of partial mappings into spheres under the added condition that $\mathcal{P}$ be closed monotone.

3.1. PROPOSITION 1. *$\mathcal{P}$-dim and $\mathcal{P}$-Dim are topological invariants.*

PROPOSITION 2. $\mathcal{P}$-dim $\leqslant$ $\mathcal{P}$-Dim $\leqslant$ dim.

Observe that $\mathcal{P}\text{-}\dim X = -1$ if and only if $\mathcal{P}\text{-}\mathrm{Dim}\, X = -1$ if and only if $\mathcal{P}\text{-}\mathrm{Ind}\, X = -1$ if and only if $X \in \mathcal{P}$.

3.2. EXAMPLE 1. Let $\mathcal{S} = \{X|\ X \text{ is } \sigma\text{-compact}\}$. Let $Z = B \times I^n$, where I denotes the unit interval and $B = \{t|\ t \in I \text{ and } t \text{ is irrational}\}$.

We shall show $\mathcal{S}\text{-}\dim Z = \mathcal{S}\text{-}\mathrm{Dim}\, Z = n$. In view of Proposition 2 above we have only to show $\mathcal{S}\text{-}\dim Z \geqslant n$. Since $\dim I^n = n$, there is an open cover $\mathfrak{U} = \{U_\gamma|\ \gamma \in \Gamma\}$ of I^n such that for every open cover $\mathfrak{V}$ of I^n with $\mathfrak{V} < \mathfrak{U}$ we have order $\mathfrak{V} \geqslant n+1$. $\mathfrak{U}^* = \{B \times U_\gamma|\ \gamma \in \Gamma\}$ is a cover of $B \times I^n$. Let $\mathfrak{W}$ be a border cover of Z with enclosure C such that $\mathfrak{W} < \mathfrak{U}^*$. Since B is not σ-compact, there is a $q \in B$ with $\{q\} \times I^n \cap C = \emptyset$. Therefore, $n+1 \leqslant$ order $\mathfrak{W}|\ \{q\} \times I^n \leqslant$ order $\mathfrak{W}$. Consequently, $\mathcal{S}\text{-}\dim Z \geqslant n$.

EXAMPLE 2. Let $\mathcal{K} = \{X|\ X \text{ is compact}\}$ and consider the space Z of Example 1. In exactly the same way it can be proved that $\mathcal{K}\text{-}\dim Z = \mathcal{K}\text{-}\mathrm{Dim}\, Z = n$.

3.3. The following proposition will be very important later on.

PROPOSITION. *Suppose $\mathcal{P}\text{-}\mathrm{Dim}\, X \leqslant n$. Let $\{U_\gamma|\ \gamma \in \Gamma\}$ be a locally finite open collection and let $\{F_\gamma|\ \gamma \in \Gamma\}$ be a closed collection such that $F_\gamma \subset U_\gamma$, $\gamma \in \Gamma$. Then there exist open collections $\{V_\gamma|\ \gamma \in \Gamma\}$ and $\{W_\gamma|\ \gamma \in \Gamma\}$ and a closed $\mathcal{P}$-kernel G of X such that*

$$F_\gamma \subset V_\gamma \subset \mathrm{cl}_X V_\gamma \subset W_\gamma \subset U_\gamma \bmod G, \quad \gamma \in \Gamma,$$

and

$$\textit{order } \{W_\gamma \backslash \mathrm{cl}_X V_\gamma|\ \gamma \in \Gamma\} \leqslant n \bmod G.$$

The corresponding result holds for $\mathcal{P}\text{-}\dim X \leqslant n$ and Γ finite.

Proof. Let $\mathfrak{U}_\gamma = \{U_\gamma, X\backslash F_\gamma\}$, $\gamma \in \Gamma$. Then $\bigwedge\{\mathfrak{U}_\gamma|\ \gamma \in \Gamma\}$ is a locally finite open covering. Since $\mathcal{P}\text{-}\mathrm{Dim}\, X \leqslant n$, there exists a $\mathcal{P}$-border cover $\mathcal{N} = \{N_\delta|\ \delta \in \Delta\}$ such that $\mathcal{N} < \bigwedge\{\mathfrak{U}_\gamma|\ \gamma \in \Gamma\}$, order $\mathcal{N} \leqslant n+1$ and such that every N_δ intersects at most finitely many members of $\{F_\gamma|\ \gamma \in \Gamma\}$.

Let G be the enclosure of $\mathcal{N}$. Restrict $\{U_\gamma|\ \gamma \in \Gamma\}$ and $\{F_\gamma|\ \gamma \in \Gamma\}$ to $X\backslash G$ and proceed as in the proof of [8], II.5.B ("covering" in [8] should be read as "covering of $X\backslash G$"). Then in $X\backslash G$ we get open collections $\{V_\gamma|\ \gamma \in \Gamma\}$ and $\{W_\gamma|\ \gamma \in \Gamma\}$ (observe that open in $X\backslash G$ is the same as open in X for subsets of $X\backslash G$) such that

$$F_\gamma\backslash G \subset V_\gamma \subset \mathrm{cl}_{X\backslash G} V_\gamma \subset W_\gamma \subset U_\gamma\backslash G, \quad \gamma \in \Gamma,$$

and

$$\text{order } \{W_\gamma \backslash \mathrm{cl}_{X\backslash G} V_\gamma|\ \gamma \in \Gamma\} \leqslant n.$$

In view of Proposition 1 in 2.8 this is just what we wanted to show.

The proof for $\mathcal{P}\text{-}\dim X \leqslant n$ is similar. Note that the condition that every N_δ intersects at most finitely many members of $\{F_\gamma|\ \gamma \in \Gamma\}$ is trivially satisfied, because Γ is finite.

3.4. THEOREM. *Suppose $\mathcal{P}$ is closed monotone. Then, for each closed subset F of X, $\mathcal{P}\text{-}\dim F \leqslant \mathcal{P}\text{-}\dim X$ and $\mathcal{P}\text{-}\mathrm{Dim}\, F \leqslant \mathcal{P}\text{-}\mathrm{Dim}\, X$.*

Proof. Let $\{U_\gamma|\ \gamma \in \Gamma\}$ be a finite $\mathcal{F}$-border cover of F with enclosure G. Let U_γ^* be an open set in X with $U_\gamma^* \cap F = U_\gamma$.

$$\mathcal{U} = \{U_\gamma^*|\ \gamma \in \Gamma\} \cup \{X\backslash F\}$$

is a finite border cover of X. Let $\mathcal{V}$ be a border cover of X with enclosure H such that $\mathcal{V} < \mathcal{U}$ and order $\mathcal{V} \leqslant \mathcal{F}\text{-}\dim X+1$. $\mathcal{V}|\ F$ is a $\mathcal{F}$-border cover of F with enclosure $H \cap F$ of order $\leqslant \mathcal{F}\text{-}\dim X+1$. It follows that $\mathcal{F}\text{-}\dim F \leqslant \mathcal{F}\text{-}\dim X$. The proof for $\mathcal{F}$-Dim is similar.

3.5. The following theorem is highly useful in the computation of lower bounds for $\mathcal{F}$-dim. It also will lead to a characterization of $\mathcal{F}$-dim in Section 4.4.

THEOREM ($n \geqslant 0$). *Suppose $\mathcal{F}$ is closed monotone and $\mathcal{F}\text{-}\dim X \leqslant n$, Then for every closed subset C of X and every continuous mapping f of C into the n-sphere S^n there exists a closed $\mathcal{F}$-kernel F of X with $F \subset X\backslash C$ such that f has a continuous extension over $X\backslash F$.*

Proof. The proof essentially follows the "only if" part of the proof of [8] Theorem III.2. We may assume that f is defined in a closed neighborhood W of C, since S^n is an absolute neighborhood retract. Let Y be the complement of the interior of W. Theorem 3.4 gives $\mathcal{F}\text{-}\dim Y \leqslant n$. Let $B = W \cap Y$. Then f is a continuous map of B into S^n. We will regard S^n to be boundary of I^{n+1}. With the aid of Proposition 3.3, the proofs [8] III.1.A and III.2.A give the existence of a closed $\mathcal{F}$-kernel F in Y and a continuous mapping g of $Y\backslash F$ into I^{n+1} such that the origin is not an element of $g(Y\backslash F)$, and g and f agree on $B\backslash F$. Consequently, g and f agree on the set $B\backslash F$ which is the intersection of the closed subsets $W\backslash F$ and $Y\backslash F$ of $X\backslash F$. It is clear that F is a closed $\mathcal{F}$-kernel of X. Thus we have an extension of f: $C \to S^n \subset I^{n+1}$ to a continuous mapping h: $X\backslash F \to I^{n+1}$ such that $h(X\backslash F)$ does not contain the origin of I^{n+1}. The theorem now follows easily.

Remark. A similar theorem for $\mathcal{F}$-Ind has been proved in [2] with $\mathcal{F}$ closed monotone and X hereditarily normal, not just metrizable.

3.6. EXAMPLE. As before, $\mathcal{K}$ denotes the class of compact spaces. Let E^n denote Euclidean n-space ($n \geqslant 1$). We shall compute $\mathcal{K}\text{-}\dim E^n = \mathcal{K}\text{-}\mathrm{Dim}\, E^n = n$. In view of 3.1 Proposition 2 it is sufficient to prove $\mathcal{K}\text{-}\dim E^n \geqslant n$. We shall derive a contradiction from the assumption $\mathcal{K}\text{-}\dim E^n \leqslant n-1$. Let $C_m = \{x|\ \varrho(x, p_m) = 1\}$ where p_m is a point the distance of which to the origin q is $3m$, $m = 1, 2, \ldots$ Let

$$C = \bigcup \{C_m|\ m = 1, 2, \ldots\}.$$

A continuous map f of C onto the standard sphere S^{n-1} is defined by sending each C_m isometrically onto S^{n-1}. If $\mathcal{K}\text{-}\dim E^n \leqslant n-1$ then, by virtue of Theorem 3.5, there exists a compact set F in E^n with $F \subset E^n\backslash C$ such that f has a continuous extension over $X\backslash F$. Since F is bounded it follows that for some k the partial map $f|\ C_k$ can be continuously extended over $\{x|\ \varrho(x, p_k) \leqslant 1\}$. That is, the n-cell can be retracted to its boundary. This is a contradiction.

4. Characterizations of $\mathcal{F}$-dim. The classes in this section are assumed to be *weakly additive* and *open monotone* except in the last theorem where closed monotone will be added. Three characterization theorems for $\mathcal{F}$-dim will be given.

4.1. THEOREM. *If F is a closed $\mathcal{F}$-kernel of X, then*

$$\mathcal{F}\text{-}\dim X = \mathcal{F}\text{-}\dim X\backslash F \quad \textit{and} \quad \mathcal{F}\text{-}\mathrm{Dim}\, X = \mathcal{F}\text{-}\mathrm{Dim}\, X\backslash F.$$

This theorem justifies the name "covering dimension modulo $\mathcal{F}$". Notice the equality! A similar result for $\mathcal{F}$-Sur follows easily from the definition.

Proof. (i) $\mathcal{F}\text{-}\dim X \leqslant \mathcal{F}\text{-}\dim X\backslash F$. Let $\mathcal{U}$ be a finite border cover of X with enclosure G. Since F is closed and $\mathcal{F}$ is open monotone, $\mathcal{U}|\ X\backslash F$ is a border cover of $X\backslash F$ with enclosure $G\backslash F$. Let $\mathcal{V}$ be a border cover of $X\backslash F$ which refines $\mathcal{U}|\ X\backslash F$, has order $\mathcal{V} \leqslant \mathcal{F}\text{-}\dim X\backslash F+1$ and has enclosure H. Since H is closed in $X\backslash F$, $F \cup H$ is closed in X. $\mathcal{F}$ being weakly additive, $\mathcal{V}$ is a border cover of X with enclosure $F \cup H$.

(ii) $\mathcal{F}\text{-}\dim X\backslash F \leqslant \mathcal{F}\text{-}\dim X$. Let $\mathcal{V}$ be a finite border cover of $X\backslash F$ with enclosure G. $G \cup F$ is a closed $\mathcal{F}$-kernel of X and hence $\mathcal{V}$ is a border cover of X with enclosure $G \cup F$. Let $\mathcal{U}$ be a border cover of X which refines $\mathcal{V}$, has order $\mathcal{U} \leqslant \mathcal{F}\text{-}\dim X+1$ and has enclosure H. Open monotonicity implies $\mathcal{U}|\ X\backslash F$ is a border cover of $X\backslash F$ with enclosure $H\backslash F$ of order $\leqslant \mathcal{F}\text{-}\dim X+1$. The proof for $\mathcal{F}$-Dim is similar.

4.2. THEOREM. *$\mathcal{F}\text{-}\dim X \leqslant n$ if and only if for every closed $\mathcal{F}$-kernel F of X in the subspace $X\backslash F$ the following holds: for every open collection $\{U_i|\ i = 1, \ldots, k\}$ and closed collection $\{F_i|\ i = 1, \ldots, k\}$ with $F_i \subset U_i$ there exist a closed $\mathcal{F}$-kernel G of $X\backslash F$ and an open collection $\{V_i|\ i = 1, \ldots, k\}$ such that (in X)*

$$F_i \subset V_i \subset \mathrm{cl}_X V_i \subset U_i \bmod G \cup F, \quad i = 1, \ldots, k,$$

and

$$\textit{order}\ \{B_X(V_i)|\ i = 1, \ldots, k\} \leqslant n \bmod G \cup F.$$

Proof. Necessity follows from Proposition 3.3, Theorem 4.1 and the propositions in 2.8. We prove the sufficiency. Let $\{U_i|\ i = 1, \ldots, k\}$ be a finite border cover with enclosure F. In the subspace $X\backslash F$ there exists a closed covering $\{F_i|\ i = 1, \ldots, k\}$ with $F_i \subset U_i$. Let G be a closed $\mathcal{F}$-kernel of $X\backslash F$ and let $\{Y_i|\ i = 1, \ldots, k\}$ be an open collection such that

$$F_i \subset Y_i \subset \mathrm{cl}_X Y_i \subset U_i \bmod G \cup F, \quad i = 1, \ldots, k,$$

and

$$\operatorname{order}\{B_X(Y_i)|\ i=1,\dots,k\}\leqslant n \bmod G\cup F.$$

In view of the weak additivity of $\mathcal{F}$ we have $H=F\cup G\in\mathcal{F}$. Observe that H is closed in X. Let $V_i=Y_i\backslash H$. Then $\{V_i|\ i=1,\dots,k\}$ is a $\mathcal{F}$-border cover of X with enclosure H. Also, it is evident that

$$F_i\subset V_i\subset \operatorname{cl}_X V_i\subset U_i \bmod H,\quad i=1,\dots,k,$$

and

$$\operatorname{order}\{B_X(V_i)|\ i=1,\dots,k\}\leqslant n \bmod H.$$

Now we restrict our attention to the subspace $X\backslash H$ and proceed as in the proof of [8], Theorem II.8. We result in an open cover $\mathcal{W}$ of $X\backslash H$ which refines $\{U_i\backslash H|\ i=1,\dots,k\}$ and order $\mathcal{W}\leqslant n+1$. Since $\mathcal{W}$ is a cover of $X\backslash H$, $\mathcal{W}$ is a $\mathcal{F}$-border cover of X. The proof is now completed.

4.3. THEOREM. *$\mathcal{F}$-$\dim X\leqslant n$ if and only if for every closed $\mathcal{F}$-kernel F of X in the subspace $X\backslash F$ the following holds:*

for every open collection $\{U_i|\ i=1,\dots,n+1\}$ and closed collection $\{F_i|\ i=1,\dots,n+1\}$ with $F_i\subset U_i$, there exist a closed $\mathcal{F}$-kernel G in $X\backslash F$ and an open collection $\{V_i|\ i=1,\dots,n+1\}$ such that (in X)

$$F_i\subset V_i\subset \operatorname{cl}_X V_i\subset U_i \bmod G\cup F,\quad i=1,\dots,n+1$$

and

$$\bigcap\{B_X(V_i)|\ i=1,\dots,n+1\}=\emptyset \bmod G\cup F.$$

Proof. The necessity follows from the preceding theorem. We prove the sufficiency by modifying slightly the proof of the corollary to Theorem II.8 in [8]. Let, in the subspace $X\backslash F$, $\{U_i|\ i=1,\dots,k\}$ be a given finite open collection and $\{F_i|\ i=1,\dots,k\}$ be a closed collection such that $F_i\subset U_i$. As in the above mentioned proof, we number all the combinations $C=\{i_1,\dots,i_{n+1}\}$ of $n+1$ numbers from $\{1,\dots,k\}$ as $C_1,\dots,C_m$. In $X\backslash F$ we can find a closed $\mathcal{F}$-kernel H_1 and an open collection $\{V_i^1|\ i\in C_1\}$ such that

$$F_i\subset V_i^1\subset \operatorname{cl}_X V_i^1\subset U_i \bmod H_1\cup F,\quad i\in C_1,$$

and

$$\bigcap\{B_X(V_i^1)|\ i\in C_1\}=\emptyset \bmod H_1\cup F.$$

Weak additivity gives $G_1=F\cup H_1$ is a closed $\mathcal{F}$-kernel of X. Let A_i^1 be open subsets of $X\backslash G_1$ such that

$$B_X(V_i^1)\subset A_i^1\subset U_i \bmod G_1\ (i\in C_1)\quad\text{and}\quad \bigcap\{A_i^1|\ i\in C_1\}=\emptyset.$$

Let $W_i^1=V_i^1\cup A_i^1$ $(i\in C_1)$. Then

$$F_i\subset V_i^1\subset \operatorname{cl}_X V_i^1\subset W_i^1\subset U_i \bmod G_1,\quad i\in C_1$$

and

$$\bigcap\{W_i^1\backslash \operatorname{cl}_X V_i^1|\ i\in C_1\}=\emptyset \bmod G_1.$$

Now we can proceed in a similar way as in the proof of the above mentioned corollary and define, for each $l=1,2,\dots,m$, closed $\mathcal{F}$-kernels G_l of X and open subsets V_i^l and W_i^l of $X\backslash G_l$, $i\in C_l$, such that

1) $F\subset G_1\subset G_2\subset\dots\subset G_m$;

2) if $i\in C_l\backslash\bigcup\{C_j|\ j<l\}$, then

$$F_i\subset V_i^l\subset \operatorname{cl}_X V_i^l\subset W_i^l\subset U_i \bmod G_l;$$

3) if $i\in\bigcup\{C_j|\ j<l\}$, then

$$F_i\subset V_i^j\subset \operatorname{cl}_X V_i^j\subset V_i^l\subset \operatorname{cl}_X V_i^l\subset W_i^l\subset W_i^j\subset U_i \bmod G_l$$

for every j for which $1\leqslant j\leqslant l-1$, $i\in C_j$;

4) $\bigcap\{W_i^l\backslash \operatorname{cl}_X V_i^l|\ i\in C_l\}=\emptyset \bmod G_l$.

Clearly 2, 3 and 4 remain true if we replace G_l by $\bigcup\{G_j|\ j=1,\dots,m\}$. Also, $\bigcup\{G_j|\ j=1,\dots,m\}$ is a closed $\mathcal{F}$-kernel of X. The computations in the last few lines of [8] p. 31 can be carried out in $X\backslash\bigcup\{G_j|\ j=1,\dots,m\}$. Let $G=\bigcup\{G_j|\ j=1,\dots,m\}\backslash F$. G is closed in $X\backslash F$.

The result is that for any combination C_l of $n+1$ numbers from $\{1,\dots,k\}$ we have $\bigcap\{B_X(V_i)|\ i\in C_l\}=\emptyset \bmod G\cup F$. This means order $\{B_X(V_i)|\ i=1,\dots,k\}\leqslant n \bmod G\cup F$. Therefore, by the preceding theorem we conclude $\mathcal{F}$-$\dim X\leqslant n$.

4.4. We now give a characterization for $\mathcal{F}$-dim using extendability of mappings in the n-sphere S^n.

DEFINITION. Y is said to be an *extensor of* $X \bmod \mathcal{F}$ if for every closed $\mathcal{F}$-kernel F of X and every closed subset C of $X\backslash F$ and every continuous map $f\colon C\to Y$ there exists a closed $\mathcal{F}$-kernel G of $X\backslash F$ such that $G\subset X\backslash C$ and f can be extended over $X\backslash(F\cup G)$.

PROPOSITION. *If S^n is an extensor of $X \bmod \mathcal{F}$, then $\mathcal{F}$-$\dim X\leqslant n$.*

Proof. Let F be a closed $\mathcal{F}$-kernel of X and let $\{U_i|\ i=1,\dots,n+1\}$ be an open collection and $\{F_i|\ i=1,\dots,n+1\}$ be a closed collection in $X\backslash F$ with $F_i\subset U_i$. We will consider S^n to be the boundary of I^{n+1} where $I=[-1,1]$. For each i, let $f_i\colon X\backslash F\to I$ be continuous with $f_i(F_i)=-1$ and $f_i\big(X\backslash(F\cup U_i)\big)=1$. Then $f=(f_1,\dots,f_{n+1})$ is a continuous map of $X\backslash F$ into I^{n+1}.

Let $C=f^{-1}(S^n)$. Since S^n is an extensor $\bmod\mathcal{F}$, there exists a closed $\mathcal{F}$-kernel G of $X\backslash F$ such that $G\subset X\backslash C$ and $f|C$ has a continuous extension g over $X\backslash(F\cup G)$.

Recalling our agreement that S^n is the boundary of I^{n+1} and letting $g=(g_1,\dots,g_{n+1})$ we define

$$V_i=\{x|\ x\in X\backslash(F\cup G)\ \text{and}\ g_i(x)<0\}.$$

In view of Propositions 1 and 2 in 2.8 it follows that

$$F_i\subset V_i\subset \mathrm{cl}_X V_i\subset U_i \bmod G\cup F,\qquad i=1,\dots,n+1,$$

and

$$\bigcap\{B_X(V_i)|\ i=1,\dots,n+1\}=\emptyset \bmod G\cup F.$$

Then $\mathcal{S}$-$\dim X\leqslant n$ by Theorem 4.3.

Our final theorem of this section needs the added assumption of closed monotonicity as well as those assumed throughout this section.

THEOREM. *Suppose $\mathcal{S}$ is closed monotone. Then $\mathcal{S}$-$\dim X\leqslant n$ if and only if S^n is an extensor of $X \bmod \mathcal{S}$.*

Proof. Sufficiency has been proved in the previous proposition. Necessity: Let F be a closed $\mathcal{S}$-kernel in X. Then, by virtue of Theorem 4.1, $\mathcal{S}$-$\dim X\backslash F\leqslant n$. Let C be a closed subset of $X\backslash F$ and $f\colon C\to S^n$. Theorem 3.5 now completes the proof.

5. The excision theorem. In this section we will assume the classes to be *closed monotone* and *weakly additive*. In [3] Theorem 4.5 it has been proved that $\mathcal{S}$-Ind $\leqslant$ $\mathcal{S}$-Sur under the sole assumption that $\mathcal{S}$ is closed monotone. The assumptions of this section yield the corresponding relations for covering dimensions modulo $\mathcal{S}$. The reverse inequality for the large covering dimension modulo $\mathcal{S}$ is then obtained under the additional assumption that $\mathcal{S}$ be countably closed additive.

5.1. LEMMA. *Let Y be a subspace of X. For every open collection $\{U_\gamma|\ \gamma\in\Gamma\}$ of Y there exists an open collection $\{V_\gamma|\ \gamma\in\Gamma\}$ of X such that $U_\gamma=V_\gamma\cap Y$ and order $\{U_\gamma|\ \gamma\in\Gamma\}=$ order $\{V_\gamma|\ \gamma\in\Gamma\}$.*

Proof. See [6] § 15, XIII.

5.2. THEOREM. $\mathcal{S}$-dim $\leqslant$ $\mathcal{S}$-Dim $\leqslant$ $\mathcal{S}$-Sur.

Proof. Suppose $\mathcal{S}$-Sur $X\leqslant n$. Let $\{U_\gamma|\ \gamma\in\Gamma\}$ be a border cover of X with enclosure F. Using the definition of the strong surplus of X with respect to $\mathcal{S}$, we choose a $\mathcal{S}$-kernel G of X with $\mathrm{Ind}\,X\backslash G\leqslant n$. $\{U_\gamma\backslash G|\ \gamma\in\Gamma\}$ is an open cover of the space $X\backslash(G\cup F)$. Since $\mathrm{Ind}\,X\backslash(G\cup F)\leqslant n$, there exists an open cover $\{V_\delta|\ \delta\in\Delta\}$ which is a refinement of $\{U_\gamma\backslash G|\ \gamma\in\Gamma\}$ and has order $\leqslant n+1$ at each point of $X\backslash(G\cup F)$. By using Lemma 5.1 above we may assume $\{V_\delta|\ \delta\in\Delta\}$ is an open collection in X, refines $\{U_\gamma|\ \gamma\in\Gamma\}$ and has order $\leqslant n+1$ at each point of X. $X\backslash\bigcup\{V_\delta|\ \delta\in\Delta\}$ is a closed subset of $G\cup F$. The weak addivity of $\mathcal{S}$ gives $G\cup F\in\mathcal{S}$. Hence $\{V_\delta|\ \delta\in\Delta\}$ is a border cover of X. Thus $\mathcal{S}$-Dim $X\leqslant n$. The theorem is completed by referring to Proposition 2 in 3.1.

5.3. THE EXCISION THEOREM. *Suppose $\mathcal{S}$ is countably closed additive. If $\mathcal{S}$-Dim $X\leqslant n$, then there exists an F_σ $\mathcal{S}$-kernel G of X such that $\mathrm{Ind}\,X\backslash G\leqslant n$.*

Proof. Let $\{U_\gamma|\ \gamma\in\Gamma_i\}$ be a locally finite open cover of X with mesh $<1/i$ and let $\{F_\gamma|\ \gamma\in\Gamma_i\}$ be a closed cover of X, $i=1,2,\dots$, such that $F_\gamma\subset U_\gamma$ for $\gamma\in\bigcup\{\Gamma_i|\ i=1,2,\dots\}$.

By use of Proposition 3.3 we define open sets in X

$$V_\gamma^1,\ W_\gamma^1 \text{ for every } \gamma\in\Gamma_1,$$
$$V_\gamma^2,\ W_\gamma^2 \text{ for every } \gamma\in\Gamma_1\cup\Gamma_2,$$
$$\dots\dots\dots\dots$$
$$V_\gamma^j,\ W_\gamma^j \text{ for every } \gamma\in\Gamma_1\cup\dots\cup\Gamma_j,$$
$$\dots\dots\dots\dots$$

and closed $\mathcal{S}$-kernels G_j such that $G_j\subset G_{j+1}$, $j=1,2,\dots$, in such a way that

$$F_\gamma\subset V_\gamma^j\subset \mathrm{cl}_X V_\gamma^j\subset W_\gamma^j\subset U_\gamma \bmod G_j,\qquad j\geqslant 1,$$
$$\mathrm{cl}_X V_\gamma^{j-1}\subset V_\gamma^j,\quad \mathrm{cl}_X W_\gamma^j\subset W_\gamma^{j-1} \bmod G_j,\qquad j\geqslant 2,$$
$$\text{order}\,\{W_\gamma^j\backslash \mathrm{cl}_X V_\gamma^j|\ \gamma\in\Gamma_1\cup\dots\cup\Gamma_j\}\leqslant n \bmod G_j,\qquad j\geqslant 1.$$

This construction can be done, because G_j is a closed $\mathcal{S}$-kernel in X and $\mathcal{S}$-Dim $X=\mathcal{S}$-Dim $X\backslash G_j$ by Theorem 4.1. (Observe that $\mathcal{S}$ is open monotone, since $\mathcal{S}$ is countably closed additive). Finally let

$$V_\gamma=\bigcup\{V_\gamma^j|\ j=1,2,\dots\}\quad \text{for}\quad \gamma\in\bigcup\{\Gamma_i|\ i=1,2,\dots\}$$

and

$$G=\bigcup\{G_j|\ j=1,2,\dots\}.$$

One readily computes

$$F_\gamma\subset V_\gamma\subset U_\gamma \bmod G.$$

Also,

$$(\mathrm{cl}_{X\backslash G}(V_\gamma\backslash G))\backslash(V_\gamma\backslash G)\subset(\mathrm{cl}_{X\backslash G}(W_\gamma^{j+1}\backslash G))\backslash(V_\gamma^{j+1}\backslash G)$$
$$\subset((\mathrm{cl}_X W_\gamma^{j+1})\backslash V_\gamma^{j+1})\backslash G\subset(W_\gamma^j\backslash \mathrm{cl}_X V_\gamma^j)\backslash G\quad \text{for}\quad j\geqslant 1.$$

Consequently, we have

$$\text{order}\,\{B_{X\backslash G}(V_\gamma\backslash G)|\ \gamma\in\bigcup\{\Gamma_i|\ i=1,2,\dots\}\}\leqslant n.$$

Since $\{V_\gamma\backslash G|\ \gamma\in\bigcup\{\Gamma_i|\ i=1,2,\dots\}\}$ is a σ-locally finite open basis for the space $X\backslash G$, it follows from [8] Theorem II.9 that $\mathrm{Ind}\,X\backslash G\leqslant n$. Since G is the countable closed union of members of $\mathcal{S}$ we have G is an F_σ $\mathcal{S}$-kernel of X.

Remark. It should be observed that the condition of countable closed additivity implies that of weak additivity.

5.4. As a result of Theorems 5.2 and 5.3 we have the following theorem, the analogue of which for small covering dimension modulo $\mathcal{P}$ is discussed in Section 11.

THEOREM. *Suppose $\mathcal{P}$ is countably closed additive. Then* $\mathcal{P}$-Sur $= \mathcal{P}$-Dim.

EXAMPLES. The conditions of the preceding theorem are satisfied by several classes of spaces. We mention only:

the class of all countable spaces;

the class $\mathcal{S}$ of all σ-compact spaces;

the class $\mathcal{A}(\alpha)$ of all absolute Borel sets of additive class α, $1 \leqslant \alpha < \Omega$, which will be discussed in 7.3.

In case $\mathcal{P} = \{\emptyset\}$, we have $\mathcal{P}$-Sur $=$ Ind and $\mathcal{P}$-Dim $=$ dim.

COUNTEREXAMPLE. Let $\mathcal{C}$ be the class of topologically complete spaces. Here we present an example of a space X with $\mathcal{C}$-Dim$X = 0$ and $\mathcal{C}$-Sur$X = 1$.

X is the subspace of $I \times I$ defined by $X = I \times Q \cup Q \times I$, where I is the unit interval and Q is the space of rationals in I.

$\mathcal{C}$-Dim$X = 0$ follows from the results of the next section.

$\mathcal{C}$-Sur$X = 1$ has been computed in [3], Section 5.1, Example 3. So the complement of any $\mathcal{C}$-kernel in X has dimension one. None the less, every $\mathcal{C}$-border cover of X has a border cover refinement of order one! It is evident that $\mathcal{C}$ is not countably closed additive.

6. Extension theorems. Except in the definitions below, all classes in this section will be *open monotone* and *weakly additive*.

6.1. In [3] Theorem 4.7 it has been proved that $\mathcal{P}$-Ind $\leqslant \mathcal{P}$-Def under the assumption $\mathcal{P}$ is closed and open monotone. Similar inequalities hold for $\mathcal{P}$-dim and $\mathcal{P}$-Dim.

THEOREM. *Suppose $\mathcal{P}$ is closed monotone. Then* $\mathcal{P}$-dim $\leqslant \mathcal{P}$-Dim $\leqslant \mathcal{P}$-Def.

Proof. In view of Proposition 2 of Section 3.1 we need only prove $\mathcal{P}$-Dim $\leqslant \mathcal{P}$-Def. Suppose $\mathcal{P}$-Def$X \leqslant n$ and F is a $\mathcal{P}$-hull of X with Ind$F \setminus X \leqslant n$. Let $\{U_\gamma|\ \gamma \in \Gamma\}$ be a border cover of X with enclosure G. For each γ, let U_γ^* be an open subset of F with $U_\gamma^* \cap X = U_\gamma$. $W = \bigcup\{U_\gamma^*|\ \gamma \in \Gamma\}$ is an open subset of F and hence $W \in \mathcal{P}$. Since $G \cap W = \emptyset$, weak additivity gives $Y = G \cup W$ is a $\mathcal{P}$-hull of X. Also, Ind$Y \setminus X \leqslant n$. As in the proof of Theorem 5.2, let $\mathcal{V}$ be an open collection in Y such that order $\mathcal{V} \leqslant n+1$, $\mathcal{V}|X < \{U_\gamma|\ \gamma \in \Gamma\}$ and $\mathcal{V}$ is a cover of $Y \setminus X$. $Y \setminus \bigcup \mathcal{V}$ is a closed subset of Y contained in X. It follows that $\mathcal{V}|X$ is a border cover of X of order $\leqslant n+1$. Hence $\mathcal{P}$-Dim$X \leqslant n$.

6.2. DEFINITION. A class $\mathcal{P}$ is said to be *countably open multiplicative* if for each $Y \in \mathcal{P}$ and each nonempty countable collection $\{X_i|\ i = 1, 2, \ldots\}$ of $\mathcal{P}$-kernels of Y the intersection $X = \bigcap\{X_i|\ i = 1, 2, \ldots\}$ belongs to $\mathcal{P}$, whenever $X_j \setminus X$ is open in $Y \setminus X$ for $j = 1, 2, \ldots$ It is clear that if $\mathcal{P}$ is open monotone and countably open multiplicative, then $\mathcal{P}$ is closed monotone.

6.3. LEMMA. *Suppose $\mathcal{P}$ is countably open multiplicative. Then* $\mathcal{P}$-Dim$X = \mathcal{P}$-Def$X$ *provided X has a $\mathcal{P}$-hull.*

Proof. In view of Theorem 6.1 we need only prove $\mathcal{P}$-Def$X \leqslant \mathcal{P}$-Dim$X$. Let Y be a $\mathcal{P}$-hull of X. We may assume X is dense in Y. Let $\mathcal{U}_1 = \{S_1(x)|\ x \in Y \setminus X\}$, where $S_1(x) = \{y \in Y|\ \varrho(x, y) < 1\}$. Let $\mathcal{U}_1' = \mathcal{U}_1|X$. $\mathcal{U}_1'$ is a border cover of X with enclosure $Y \setminus \bigcup \mathcal{U}_1$. Let $\mathcal{V}_1$ be a border cover of X with enclosure F_1 such that $\mathcal{V}_1 < \mathcal{U}_1'$ and order $\mathcal{V}_1 \leqslant n+1$, where $n = \mathcal{P}$-Dim$X$. Let $\mathcal{W}_1$ be an open collection in Y such that $\mathcal{W}_1|X = \mathcal{V}_1$. Since X is dense in Y, we have order $\mathcal{W}_1 \leqslant n+1$. Suppose $\mathcal{W}_1, \ldots, \mathcal{W}_{k-1}$ have been defined. Let $\mathcal{U}_k = \{S_{1/k}(x)|\ x \in Y \setminus X\}$. Let $\mathcal{V}_k$ be a border cover of X with enclosure F_k which refines $(\mathcal{U}_k \wedge \mathcal{W}_{k-1})|X$ and has order $\leqslant n+1$. Let $\mathcal{W}_k$ be an open collection such that $\mathcal{W}_k|X = \mathcal{V}_k$, $\mathcal{W}_k < \mathcal{W}_{k-1}$ and order $\mathcal{W}_k \leqslant n+1$. Weak additivity implies $X_k = F_k \cup (\bigcup \mathcal{W}_k) \in \mathcal{P}$.

Denote by Z the set $\bigcap\{X_k|\ k = 1, 2, \ldots\}$. Since $F_k \subset X \subset X_k$ for each k, $Z \supset X$ and $X_k \setminus Z = (\bigcup \mathcal{W}_k) \setminus Z$. Consequently, countable open multiplicativity of $\mathcal{P}$ implies Z is a $\mathcal{P}$-hull of X. We shall show Ind$Z \setminus X \leqslant n$. Let $\mathcal{W}_k' = \mathcal{W}_k|Z \setminus X$. Then $\{\mathcal{W}_k'|\ k = 1, 2, \ldots\}$ is a sequence of open coverings of $Z \setminus X$ such that

1) $\mathcal{W}_{k+1}' < \mathcal{W}_k'$, $k = 1, 2, \ldots$;

2) order $\mathcal{W}_k' \leqslant n+1$, $k = 1, 2, \ldots$;

3) mesh $\mathcal{W}_k' \leqslant$ mesh $\mathcal{W}_k =$ mesh $\mathcal{V}_k \leqslant$ mesh $\mathcal{U}_k \leqslant 2/k$.

By [8] Theorem V.1 it follows that Ind$Z \setminus X \leqslant n$. Hence $\mathcal{P}$-Def$X \leqslant n = \mathcal{P}$-Dim$X$.

6.4. DEFINITION. A class $\mathcal{P}$ is said to be *universal* if every space X has a $\mathcal{P}$-hull.

A G_δ monotone class $\mathcal{P}$ is universal if and only if $\mathcal{P}$ contains the class $\mathcal{C}$ of topologically complete spaces.

From Lemma 6.3 we get the following theorem.

THE EXTENSION THEOREM. *Suppose $\mathcal{P}$ is countably open multiplicative and universal. Then* $\mathcal{P}$-Def $= \mathcal{P}$-Dim.

EXAMPLES. The conditions of the preceding theorem are satisfied by the class $\mathcal{C}$ of all topologically complete spaces and by the class $\mathcal{M}(\alpha)$ of all absolute Borel sets of multiplicative class α, $2 \leqslant \alpha < \Omega$, which will be discussed in 7.3.

Observe that the class $\mathcal{C}$ is the smallest class which satisfies the conditions of the preceding theorem. In view of Example 3.6 the covering dimension turns out not to be the right way to characterize compactness deficiency. It is also clear why for the characterization of compactness deficiency — $\mathcal{K}$-Def — in [9] a special system of border covers with compact enclosures is employed.

COUNTEREXAMPLES. As in 3.2 Example 1 let $\mathcal{S}$ denote the class of all σ-compact spaces. In [3] Example 6.1 a separable space X has been presented with $\mathcal{S}\text{-Def}\,X = 1$ and $\mathcal{S}\text{-Ind}\,X = 0$. As will be shown in Section 11.2, $\mathcal{S}\text{-Ind}\,Z = \mathcal{S}\text{-Dim}\,Z$ for every separable space Z. So $\mathcal{S}\text{-Dim}\,X = 0$. Obviously $\mathcal{S}$ is not countably open multiplicative.

Another useful example is the following: Let $\mathcal{T} = \{\emptyset\}$. As is easily seen, we have

$$\mathcal{T}\text{-Def}\,\emptyset = -1\,, \text{ and } \mathcal{T}\text{-Def}\,X = \infty \text{ if and only if } X \neq \emptyset\,.$$

Obviously $\mathcal{T}$ is not universal.

7. Complementary dimension functions. In this section we shall show that the notions of kernels and hulls are complementary. We remind the reader of the basic assumptions of Section 2.1.

7.1. The relation between $\mathcal{T}$-Dim and $\mathcal{T}$-Sur has been discussed in Section 5, and the relation between $\mathcal{T}$-Dim and $\mathcal{T}$-Def is given in Section 6. Now we first discuss the relation between $\mathcal{T}$-Sur and $\mathcal{T}$-Def.

THEOREM. *Suppose $\mathcal{T}$ is F_σ monotone. Then* $\mathcal{T}\text{-Sur} \leqslant \mathcal{T}\text{-Def}$.

Proof. We may assume $\mathcal{T}\text{-Def}\,X < \infty$. Let Y be a $\mathcal{T}$-hull of X with $\operatorname{Ind} Y\setminus X = \mathcal{T}\text{-Def}\,X$. By virtue of [8] Theorem II. 9, there exists a G_δ subset G of Y with $G \supset Y\setminus X$ and $\operatorname{Ind} G = \operatorname{Ind} Y\setminus X$. Then $F = Y\setminus G$ is an F_σ subset of Y contained in X. Hence F is a $\mathcal{T}$-kernel of X with $\operatorname{Ind} X\setminus F \leqslant \mathcal{T}\text{-Def}\,X$. That is, $\mathcal{T}\text{-Sur}\,X \leqslant \mathcal{T}\text{-Def}\,X$.

7.2. For a further investigation of the relations between the large covering dimension modulo $\mathcal{T}$, the strong surplus and the strong deficiency we now discuss the interplay of such functions induced by two classes.

DEFINITIONS. We will say that two subsets X and Y of a space Z are *complementary in* Z if $Z = X \cup Y$ and $X \cap Y = \emptyset$. Let $\mathcal{T}$ and $\mathcal{Q}$ be classes of spaces. A space Z is called *ambiguous relative to* $\mathcal{T}$ *and* $\mathcal{Q}$ provided $X \in \mathcal{T}$ if and only if $Y \in \mathcal{Q}$ whenever X and Y are complementary in Z.

EXAMPLE. As before let $\mathcal{S}$ be the class of σ-compact spaces and $\mathcal{C}$ be the class of topologically complete spaces. Each compact space Z is ambiguous relative to $\mathcal{S}$ and $\mathcal{C}$.

The following theorem, which is almost evident, will be very important later on.

THEOREM. *Suppose Z is ambiguous relative to $\mathcal{T}$ and $\mathcal{Q}$. Then* $\mathcal{Q}\text{-Def}\,X \leqslant \mathcal{T}\text{-Sur}\,Y$ *whenever X and Y are complementary in Z.*

7.3. By combining Theorems 7.1 and 7.2 we get the following result.

THEOREM. *Suppose both $\mathcal{T}$ and $\mathcal{Q}$ are F_σ monotone. Suppose Z is ambiguous relative to $\mathcal{T}$ and $\mathcal{Q}$. Then* $\mathcal{T}\text{-Def}\,X = \mathcal{T}\text{-Sur}\,X = \mathcal{Q}\text{-Def}\,Y = \mathcal{Q}\text{-Sur}\,Y$, *whenever X and Y are complementary in Z.*

EXAMPLE. For every ordinal number α, $0 \leqslant \alpha < \Omega$, let $\mathcal{A}(\alpha)$ and $\mathcal{M}(\alpha)$ denote the families of *all absolute Borel sets of additive and multiplicative class α respectively.* It is a classical result that, for $\alpha \geqslant 2$, $X \in \mathcal{A}(\alpha)\big(\mathcal{M}(\alpha)\big)$ if and only if X is a Borel set of additive (multiplicative) class α in some complete space Y which contains X [6]. The same holds true for $\mathcal{M}(1)$. $\mathcal{M}(1)$ is the class of all topologically complete spaces (see also [6]). In [10] it has been proved that $\mathcal{A}(1)$ is the class of all σ-locally compact spaces. Evidently $\mathcal{M}(0)$ is the class of all compact spaces and $\mathcal{A}(0) = \{\emptyset\}$.

Using the characterization it is easily seen that every space $Z \in \mathcal{M}(\alpha) \cap \mathcal{A}(\alpha)$ is ambiguous with respect to $\mathcal{M}(\alpha)$ and $\mathcal{A}(\alpha)$, $2 \leqslant \alpha < \Omega$. Also, for $2 \leqslant \alpha < \Omega$, both $\mathcal{M}(\alpha)$ and $\mathcal{A}(\alpha)$ are G_δ monotone as well as F_σ monotone. Since, for $\alpha \geqslant 2$, $\mathcal{M}(1) \subset \mathcal{M}(\alpha) \cap \mathcal{A}(\alpha)$, the class $\mathcal{M}(\alpha) \cap \mathcal{A}(\alpha)$ is universal.

Hence we have the following proposition by the theorem above.

PROPOSITION. *For every ordinal α with $2 \leqslant \alpha < \Omega$ we have* $\mathcal{A}(\alpha)\text{-Sur} = \mathcal{A}(\alpha)\text{-Def}$ *and* $\mathcal{M}(\alpha)\text{-Sur} = \mathcal{M}(\alpha)\text{-Def}$.

7.4. LEMMA. *Suppose $\mathcal{T}$ and $\mathcal{Q}$ are closed monotone. Let Z be ambiguous relative to $\mathcal{T}$ and $\mathcal{Q}$ and let X and Y be complementary in Z. Let $\mathcal{U}$ be an open collection in Z. Then $\mathcal{U}|X$ is a $\mathcal{T}$-border cover if and only if $\mathcal{U}|Y$ is a $\mathcal{Q}$-border cover.*

Proof. Let $W = \bigcup \mathcal{U}$. If $X\setminus W \in \mathcal{T}$, then $Z\setminus(X\setminus W) \in \mathcal{Q}$. But $Z\setminus(X\setminus W) = Y \cup W$ and $Y\setminus W$ is a closed subset of $Y \cup W$. Hence $Y\setminus W \in \mathcal{Q}$, whenever $X\setminus W \in \mathcal{T}$.

THEOREM. *Suppose $\mathcal{T}$ and $\mathcal{Q}$ are closed monotone. Suppose Z is ambiguous relative to $\mathcal{T}$ and $\mathcal{Q}$. Then* $\mathcal{T}\text{-Dim}\,X = \mathcal{Q}\text{-Dim}\,Y$ *and* $\mathcal{T}\text{-dim}\,X = \mathcal{Q}\text{-dim}\,Y$ *whenever X and Y are complementary in Z.*

Proof. Since the proofs for the small and large covering dimensions are very similar, we only give one case. Due to symmetry we need only prove $\mathcal{T}\text{-Dim}\,X \leqslant \mathcal{Q}\text{-Dim}\,Y$ whenever X and Y are complementary in Z.

Suppose $\mathcal{Q}\text{-Dim}\,Y \leqslant n$. Let $\{U_\gamma|\ \gamma \in \Gamma\}$ be a $\mathcal{T}$-border cover of X. For each $\gamma \in \Gamma$, let U_γ^* be an open subset of Z with $U_\gamma^* \cap X = U_\gamma$. By virtue of the preceding lemma $\{U_\gamma^*|\ \gamma \in \Gamma\}|Y$ is a $\mathcal{Q}$-border cover of Y. Since $\mathcal{Q}\text{-Dim}\,Y \leqslant n$, this $\mathcal{Q}$-border cover has a $\mathcal{Q}$-border cover refinement $\{V_\delta|\ \delta \in \Delta\}$ of order $\leqslant n+1$. Using Lemma 5.1, we form an open collection

$\{V_\delta^*|\ \delta \in \Delta\}$ with $\{V_\delta^*|\ \delta \in \Delta\}|Y = \{V_\delta|\ \delta \in \Delta\}$, $\{V_\delta^*|\ \delta \in \Delta\} < \{U_\gamma^*|\ \gamma \in \Gamma\}$ and order $\{V_\delta^*|\ \delta \in \Delta\} \leqslant n+1$. From the lemma above it follows that $\{V_\delta^*|\ \delta \in \Delta\}|X$ is a $\mathcal{S}$-border cover of X. This border cover refines the given one and has order $\leqslant n+1$. So $\mathcal{S}\text{-Dim}X \leqslant n$.

EXAMPLE. From the preceding theorem, Example 1 in 3.2 and the example in 7.2 it follows that $\mathcal{C}\text{-dim}Q \times I^n = \mathcal{C}\text{-Dim}Q \times I^n = n$, where $\mathcal{C}$ is the class of topologically complete spaces, I the unit interval and Q the set of rational numbers in I.

7.5. THEOREM. *Suppose $\mathcal{S}$ and $\mathcal{Q}$ are F_σ monotone and weakly additive. Suppose Z is ambiguous relative to $\mathcal{S}$ and $\mathcal{Q}$. Then, if $\mathcal{S}$ is countably open multiplicative or if $\mathcal{Q}$ is countably closed additive,*

$$\mathcal{S}\text{-Def}X = \mathcal{S}\text{-Sur}X = \mathcal{S}\text{-Dim}X = \mathcal{Q}\text{-Dim}Y = \mathcal{Q}\text{-Sur}Y = \mathcal{Q}\text{-Def}Y,$$

whenever X and Y are complementary in Z.

Proof. $\mathcal{S}\text{-Def}X = \mathcal{S}\text{-Sur}X = \mathcal{Q}\text{-Def}Y = \mathcal{Q}\text{-Sur}Y$ by Theorem 7.3 and $\mathcal{S}\text{-Dim}X = \mathcal{Q}\text{-Dim}Y$ by Theorem 7.4.

If Z is ambiguous relative to $\mathcal{S}$ and $\mathcal{Q}$, then $Z \in \mathcal{S} \cap \mathcal{Q}$. Then, if $\mathcal{S}$ is countably open multiplicative, by Lemma 6.3 we have $\mathcal{S}\text{-Dim}X = \mathcal{S}\text{-Def}X$. If $\mathcal{Q}$ is countably closed additive, then $\mathcal{Q}\text{-Dim}Y = \mathcal{Q}\text{-Sur}Y$ by Theorem 5.4.

Remark. From the example in 7.3 and the proof above it follows that for each a with $2 \leqslant a < \Omega$, the class $\mathcal{A}(a) \cap \mathcal{M}(a)$ is precisely the class of all spaces which are ambiguous with respect to $\mathcal{A}(a)$ and $\mathcal{M}(a)$. Using the classical terminology [6], $\mathcal{A}(a) \cap \mathcal{M}(a)$ is the class of absolute ambiguous sets of class a (i.e., each member Z of $\mathcal{A}(a) \cap \mathcal{M}(a)$ is an ambiguous Borel set of class a in every Z containing space Y).

8. The finite sum theorem. In order for a finite sum theorem to hold one necessarily needs the class $\mathcal{S}$ to be finitely closed additive. The next theorem shows that the converse holds under closed monotonicity.

8.1. THEOREM. *Suppose $\mathcal{S}$ is closed monotone and finitely closed additive. Let $X = Y \cup Z$ with Y and Z closed in X. Then*

$$\mathcal{S}\text{-dim}X = \max\{\mathcal{S}\text{-dim}Y, \mathcal{S}\text{-dim}Z\}$$

and

$$\mathcal{S}\text{-Dim}X = \max\{\mathcal{S}\text{-Dim}Y, \mathcal{S}\text{-Dim}Z\}.$$

Proof. Let $n = \max\{\mathcal{S}\text{-Dim}Y, \mathcal{S}\text{-Dim}Z\}$. In view of Theorem 3.4 we have $\mathcal{S}\text{-Dim}X \geqslant n$. We shall show $\mathcal{S}\text{-Dim}X \leqslant n$. Let $\mathcal{U} = \{U_\gamma|\ \gamma \in \Gamma\}$ be a border cover of X with enclosure F. Then $\mathcal{U}|Z$ is a border cover of Z with enclosure $F \cap Z$. Let $\mathcal{V}$ be a border cover of Z with enclosure G such that $\mathcal{V} < \mathcal{U}|Z$ and order $\mathcal{V} \leqslant n+1$. We may assume that $\mathcal{V} = \{V_\gamma|\ \gamma \in \Gamma\}$ with $V_\gamma \subset U_\gamma$, that $V_\gamma = V_{\gamma'}$ whenever $U_\gamma = U_{\gamma'}$, and that V_γ and $V_{\gamma'}$ are distinct or both empty whenever $U_\gamma \neq U_{\gamma'}$. G is closed in X and $G \cup F$ is a closed $\mathcal{S}$-kernel of X. Let $W_\gamma = (U_\gamma \setminus Z) \cup V_\gamma$, $\gamma \in \Gamma$. It follows that $\mathcal{W} = \{W_\gamma|\ \gamma \in \Gamma\}$ is a border cover of X with enclosure $G \cup F$ such that order $\mathcal{W}|Z \leqslant n+1$ and $\mathcal{W} < \mathcal{U}$. Now we apply the same process to $\mathcal{W}|Y$. In this way we get a border cover of X which has order $\leqslant n+1$ at every point of Y and Z.

The proof for $\mathcal{S}$-dim is similar.

The proof given above is very suggestive for the proof of the locally finite sum theorem (Section 10).

8.2. PROBLEM. Is there a similar finite sum theorem for $\mathcal{S}$-Ind? This problem seems rather difficult even under the assumption that $\mathcal{S}$ is open monotone as well as closed monotone. But we do have a countable and a locally finite sum theorem for $\mathcal{S}$-Ind (see Sections 9 and 10).

9. The countable sum theorems. All classes are assumed to be *closed monotone* and *countably closed additive*. Open monotonicity and weak additivity are implied by our assumptions.

In this section we shall prove countable sum theorems for the dimension functions $\mathcal{S}$-Dim, $\mathcal{S}$-dim and $\mathcal{S}$-Ind. The interesting thing is that the proofs are totally different, thus illustrating various aspects of the dimension theory modulo a class $\mathcal{S}$.

9.1. THEOREM. *Let $\{F_i|\ i = 1, 2, \ldots\}$ be a countable closed covering of X such that $\mathcal{S}\text{-Dim}F_i \leqslant n$ for $i = 1, 2, \ldots$ Then $\mathcal{S}\text{-Dim}X \leqslant n$.*

Proof. The proof is by means of surplus techniques (cf. [3], Section 5). In view of Theorem 5.3 for each i there exists an F_σ $\mathcal{S}$-kernel G_i of F_i such that $\text{Ind}F_i \setminus G_i \leqslant \mathcal{S}\text{-Dim}F_i \leqslant n$.

$G = \bigcup\{G_i|\ i = 1, 2, \ldots\}$ is an F_σ $\mathcal{S}$-kernel of X and $\text{Ind}X \setminus G \leqslant \sup\{\text{Ind}F_i \setminus G|\ i = 1, 2, \ldots\}$ by the sum theorem of dimension [8], Theorem II.1. Since $G \supset G_i$ for each i, it follows that $\mathcal{S}\text{-Dim}X = \mathcal{S}\text{-Sur}X \leqslant n$ by Theorem 5.4.

9.2. THEOREM. *Let $\{F_i|\ i = 1, 2, \ldots\}$ be a countable closed covering of X such that $\mathcal{S}\text{-dim}F_i \leqslant n$ for $i = 1, 2, \ldots$ Then $\mathcal{S}\text{-dim}X \leqslant n$.*

Proof. In the proof the characterization of $\mathcal{S}$-dim in Section 4.4 is employed (cf. [7], Theorem 9.10). We shall show that S^n is an extensor of $X \bmod \mathcal{S}$. Let G_0 be a closed $\mathcal{S}$-kernel of X, C a closed subset of $X \setminus G_0$ and f a continuous mapping from C into S^n. Let $F_0 = \emptyset$.

We shall define an open collection $\{U_i|\ i = 0, 1, \ldots\}$ in X, a collection of closed $\mathcal{S}$-kernels $\{G_i|\ i = 0, 1, \ldots\}$ of X, and a collection of continuous mappings $\{g_i\colon \text{cl}_{X \setminus G_i}(U_i) \to S^n|\ i = 0, 1, \ldots\}$ such that, for each $i = 0, 1, \ldots$,

1) $C \subset U_k \subset U_i$, $k \leqslant i$;
2) $G_k \subset G_i$, $k \leqslant i$;

3) $U_i \cap G_i = \emptyset$, $U_i \cup G_i \supset F_0 \cup \ldots \cup F_i$;

4) $g_i | U_k = g_k$ for $k < i$.

Since S^n is an absolute neighborhood retract, there exists an open neighborhood U_0 of C in the space $X \setminus G_0$ and a continuous extension g_0: $\mathrm{cl}_{X\setminus G_0} U_0 \to S^n$ of f. Clearly 1) through 4) are satisfied when $i = 0$. Suppose U_i, G_i and g_i are defined for $i \leqslant j$. By virtue of Theorem 3.4 we have $\mathfrak{I}\text{-}\dim F_{j+1} \setminus U_j \leqslant n$. The mapping $g_j | B_{X\setminus G_j}(U_j) \cap F_{j+1}$ has a continuous extension h_{j+1} over $(F_{j+1} \setminus U_j) \setminus E_{j+1}$, where E_{j+1} is a closed $\mathfrak{I}$-kernel of $F_{j+1} \setminus U_j$ and $E_{j+1} \subset F_{j+1} \setminus \mathrm{cl}_{X\setminus G_j}(U_j)$. The combination of g_j and h_{j+1} yields a continuous map H_{j+1} of $\mathrm{cl}_{X\setminus G_j}(U_j) \cup (F_{j+1} \setminus E_{j+1})$ into S^n. Let $G_{j+1} = G_j \cup E_{j+1}$. Then $G_{j+1} \in \mathfrak{I}$ and is closed in X. Now as in the definition of U_0 and g_0, there exist an open neighborhood U_{j+1} of the domain of $H_{j+1} | (X \setminus G_{j+1})$ and a continuous extension g_{j+1}: $\mathrm{cl}_{X\setminus G_{j+1}}(U_{j+1}) \to S^n$ of $H_{j+1} | (X \setminus G_{j+1})$. It is easily seen that the conditions 1) through 4) are satisfied. Let $U = \bigcup \{U_i |\ i = 0, 1, \ldots\}$ and $G = \bigcup \{G_i |\ i = 0, 1, \ldots\}$. The maps g_i determine a unique continuous map g: $U \to S^n$ which is an extension of the map $f(1)$ and 4. In view of 1, 2 and 3 we have $U \cup G = X$ and $U \cap G = \emptyset$.

Since $\mathfrak{I}$ is countably closed additive $G \in \mathfrak{I}$ and consequently G is a closed $\mathfrak{I}$-kernel of X. $G \setminus G_0$ is a closed $\mathfrak{I}$-kernel of $X \setminus G_0$ which is disjoint from C. Hence $\mathfrak{I}\text{-}\dim X \leqslant n$.

9.3. THEOREM. *Let $\{F_i |\ i = 1, 2, \ldots\}$ be a countable closed covering of X such that $\mathfrak{I}\text{-}\mathrm{Ind} F_i \leqslant n$ for $i = 1, 2, \ldots$ Then $\mathfrak{I}\text{-}\mathrm{Ind} X \leqslant n$.*

Proof. The proof has the flavor of the theory of normal families of Hurewicz and Morita (see e.g. [8] Section II.8).

Let $\mathfrak{I}$ be a family of spaces. The family $\mathfrak{I}'$ is defined by

$$\mathfrak{I}' = \{X |\ \text{for any disjoint closed sets } F \text{ and } G \text{ there exists an open set } U \text{ of } X \text{ with } F \subset U \subset X \setminus G \text{ and } B_X(U) \in \mathfrak{I}\}.$$

In order to prove the theorem we need only show $\mathfrak{I}'$ is closed monotone and countably closed additive whenever $\mathfrak{I}$ is.

Suppose $\mathfrak{I}$ is closed monotone and countably closed additive. We shall show that $\mathfrak{I}'$ has the same properties in three steps.

LEMMA 1. *$\mathfrak{I}'$ is closed monotone.*

The proof of Lemma 1 is straightforward.

LEMMA 2. *$\mathfrak{I}'$ is open monotone.*

The proof of Lemma 2 is quite similar to the proof in [7], 11.3. Let $X \in \mathfrak{I}'$ and let V be an open subset of X. Let F and G be disjoint closed subsets of the subspace V. Write $V = \bigcup \{F_i |\ i = 1, 2, \ldots\}$ with F_i closed in X and $F_i \subset \mathrm{Int} F_{i+1}$ (Int denotes the interior in X). Let W_i be an open subset of X, $i = 1, 2, \ldots$, such that

$$F \subset \mathrm{cl}_V W_{i+1} \subset W_i \subset V \setminus G, \quad i = 1, 2, \ldots \quad \text{and} \quad \bigcap \{W_i |\ i = 1, 2, \ldots\} = F.$$

For each i

$$F \cap F_i \subset W_i \cap \mathrm{Int} F_{i+1} \subset F_{i+1}.$$

By virtue of Lemma 1 we have $F_{i+1} \in \mathfrak{I}'$. From the definition of $\mathfrak{I}'$ it follows that there exists an open set U_i with $F \cap F_i \subset U_i \subset \mathrm{cl}_V U_i \subset W_i \cap \mathrm{Int} F_{i+1}$ and $B_V(U_i) \in \mathfrak{I}$. Let $U = \bigcup \{U_i |\ i = 1, 2, \ldots\}$. Since the collection $\{U_i |\ i = 1, 2, \ldots\}$ is locally finite in $V \setminus F$, we have $B_V(U) \subset \bigcup \{B_V(U_i) |\ i = 1, 2, \ldots\}$. Since $\mathfrak{I}$ is closed monotone and countably closed additive, $B_V(U) \in \mathfrak{I}$. The inclusions $F \subset U \subset \mathrm{cl}_V U \subset V \setminus G$ are evident. Thus $V \in \mathfrak{I}'$.

LEMMA 3. *$\mathfrak{I}'$ is countably closed additive.*

Proof. Let $X = \bigcup \{X_i |\ i = 1, 2, \ldots\}$ where X_i is closed in X and $X_i \in \mathfrak{I}'$. Define $F_1 = X_1$ and $F_i = X_i \setminus \bigcup \{X_j |\ j = 1, \ldots, i-1\}$, $i \geqslant 2$. Then $X = \bigcup \{F_i |\ i = 1, 2, \ldots\}$ and the F_i are pairwise disjoint. Moreover

(i) $F_i \in \mathfrak{I}'$, $i = 1, 2, \ldots$ (Lemma 2),

(ij) $\bigcup \{F_j |\ j = 1, \ldots, i\}$ is closed in X for each i.

The proof of $X \in \mathfrak{I}'$ is now a slight modification of the proof of [7] Theorem 10.4 (Read "$\mathrm{Ind} X \leqslant n$" as $X \in \mathfrak{I}'$, "$\mathrm{Ind} X \leqslant n-1$" as $X \in \mathfrak{I}$. Observe that $\mathfrak{I}$ is weakly additive.)

10. Lining up the dimension functions.

All classes in this section are assumed to be *closed monotone*, *weakly additive* and *locally finitely closed additive*.

The main purpose of this section is to show

$$\mathfrak{I}\text{-}\mathrm{Ind} \geqslant \mathfrak{I}\text{-}\dim = \mathfrak{I}\text{-}\mathrm{Dim}.$$

As a byproduct we obtain the locally finite sum theorem for the three dimension functions. Observe that if $\mathfrak{I}$ is locally finitely closed additive, then $\mathfrak{I}$ is open monotone. This is clear since any open set V in X can be written as

$$V = \bigcup \{H_k |\ k = 0, 1, 2, \ldots\} \quad \text{where} \quad H_k = \left\{x \,\middle|\, \frac{1}{k+1} \leqslant \varrho(x, X \setminus V) \leqslant \frac{1}{k}\right\};$$

$\{H_k |\ k = 0, 1, \ldots\}$ is a locally finite closed cover of V.

10.1. We first prove the analogue of a theorem of Dowker. We shall generalize the proof of this theorem as given in [8], Theorem II.6. Compare also with Theorem 8.1.

THEOREM. *$\mathfrak{I}\text{-}\dim = \mathfrak{I}\text{-}\mathrm{Dim}$.*

Proof. We need only prove $\mathfrak{I}\text{-}\mathrm{Dim} \leqslant \mathfrak{I}\text{-}\dim$. Suppose $\mathfrak{I}\text{-}\dim X \leqslant n$. Observe that in view of Theorem 3.4 we have $\mathfrak{I}\text{-}\dim F \leqslant n$ for every closed subset F of X.

Let $\{U_\gamma|\ \gamma \in \Gamma\}$ be a $\mathcal{P}$-border cover of X with enclosure G_0. We may assume that $\{U_\gamma|\ \gamma \in \Gamma\}$ is a locally finite cover of $X \backslash G_0$. Let $\mathcal{F} = \{F_\nu|\ 1 \leqslant \nu < \tau\}$ be a locally finite closed cover of $X \backslash G_0$ (ν and τ are ordinal numbers) such that

(i) each F_ν meets at most finitely many elements of $\{U_\gamma|\ \gamma \in \Gamma\}$.

(ii) each F_ν is closed in X. Consequently $\mathcal{P}$-$\dim F_\nu \leqslant n$, for $1 \leqslant \nu < \tau$.

We assume that $F_\mu \neq F_\nu$, whenever $\mu \neq \nu$. In X we construct a transfinite sequence of border covers $\{U_{\nu,\gamma}|\ \gamma \in \Gamma\}$ with enclosures $\bigcup \{G_\mu|\ \mu \leqslant \nu\}$ such that

$$U_{0,\gamma} = U_\gamma \quad \text{and} \quad U_{\mu,\gamma} \supset U_{\nu,\gamma} \quad \text{for} \quad \mu < \nu;$$

$$G_\nu \subset F_\nu \quad \text{for} \quad 1 \leqslant \nu; \quad \text{and}$$

$$\{U_{\nu,\gamma}|\ \gamma \in \Gamma\}|F_\nu \text{ is a border cover of } F_\nu \text{ of order } \leqslant n+1.$$

Let $U_{\mu,\gamma}$ and G_μ be determined for $\mu < \nu$. Put $U^*_{\nu,\gamma} = \bigcap \{U_{\mu,\gamma}|\ \mu < \nu\}$. $\bigcup \{G_\mu|\ 1 \leqslant \mu < \nu\}$ is a closed $\mathcal{P}$-kernel of $X \backslash G_0$, since $\mathcal{P}$ is locally finitely closed additive. Since $\mathcal{P}$ is also weakly additive, $\bigcup \{G_\mu|\ 0 \leqslant \mu < \nu\}$ is a closed $\mathcal{P}$-kernel of X. Thus $\{U^*_{\nu,\gamma}|\ \gamma \in \Gamma\}|F_\nu$ is a $\mathcal{P}$-border cover of F_ν with enclosure $F_\nu \cap [\bigcup \{G_\mu|\ \mu < \nu\}]$.

Let $\mathcal{W}$ be a border cover of F_ν with enclosure G_ν such that $\mathcal{W} < \{U^*_{\nu,\gamma}|\ \gamma \in \Gamma\}|F_\nu$ and order $\mathcal{W} \leqslant n+1$. We may assume that $\mathcal{W} = \{W_\gamma|\ \gamma \in \Gamma\}$ with $W_\gamma \subset U^*_{\nu,\gamma}$, that $W_\gamma = W_{\gamma'}$ whenever $U^*_{\nu,\gamma} = U^*_{\nu,\gamma'}$, and that $W_\gamma \neq W_{\gamma'}$, or both empty, whenever $U^*_{\nu,\gamma} \neq U^*_{\nu,\gamma'}$. Let

$$U_{\nu,\gamma} = (U^*_{\nu,\gamma} \backslash F_\nu) \cup W_\gamma = U^*_{\nu,\gamma} \backslash (F_\nu \backslash W_\gamma).$$

Finally, let $V_\gamma = \bigcap \{U_{\nu,\gamma}|\ \nu < \tau\}$. Then, $\{V_\gamma|\ \gamma \in \Gamma\}$ is a $\mathcal{P}$-border cover of order $\leqslant n+1$ with enclosure $\bigcup \{G_\nu|\ \nu < \tau\}$, which refines $\{U_\gamma|\ \gamma \in \Gamma\}$. It follows that $\mathcal{P}$-$\operatorname{Dim} X \leqslant n$.

10.2. As a corollary to the preceding theorem we get the following.

Corollary (The locally finite sum theorem). *Let $\{F_\gamma|\ \gamma \in \Gamma\}$ be a locally finite closed covering of X such that $\mathcal{P}$-$\dim F_\gamma \leqslant n$ ($\mathcal{P}$-$\operatorname{Dim} F_\gamma \leqslant n$) for $\gamma \in \Gamma$. Then $\mathcal{P}$-$\dim X \leqslant n$ ($\mathcal{P}$-$\operatorname{Dim} X \leqslant n$).*

Proof. It is sufficient to prove the theorem for $\mathcal{P}$-dim. Let $\{F_\gamma|\ \gamma \in \Gamma\}$ be a locally finite closed cover of X with $\mathcal{P}$-$\dim F_\gamma \leqslant n$. Let $\{U_i|\ i = 1, \ldots, k\}$ be a border cover of X with enclosure G. Let $\{H_\nu|\ \nu < \tau\}$ be a locally finite closed covering of $X \backslash G$ such that

1) each H_ν meets at most finitely many elements of $\{F_\gamma|\ \gamma \in \Gamma\}$,

2) each H_ν is closed in X.

By virtue of Theorems 3.4 and 8.1 we have $\mathcal{P}$-$\dim H_\nu \leqslant n$. Now proceed as in the proof above.

10.3. In order to show $\mathcal{P}$-Ind $\geqslant \mathcal{P}$-dim we first prove the locally finite sum theorem for $\mathcal{P}$-Ind.

Theorem. *Let $\{F_\gamma|\ \gamma \in \Gamma\}$ be a locally finite closed covering of X such that $\mathcal{P}$-$\operatorname{Ind} F_\gamma \leqslant n$ for $\gamma \in \Gamma$. Then $\mathcal{P}$-$\operatorname{Ind} X \leqslant n$.*

Proof. The proof is by a normal family type argument as the proof of Theorem 9.3.

Suppose $\mathcal{P}$ is a closed monotone, weakly additive and locally finitely closed additive class. Let $\mathcal{P}'$ be defined as in the proof of Theorem 9.3. We need only show that $\mathcal{P}'$ is closed monotone, weakly additive and locally finitely closed additive.

Lemma 1. *$\mathcal{P}'$ is closed monotone.*

See 9.3 Lemma 1.

Lemma 2. *$\mathcal{P}'$ is open monotone.*

The proof of Lemma 2 is almost the same as the proof of Lemma 2 in 9.3. The collection $\{B_V(U_i)|\ i = 1, 2, \ldots\}$ constructed in that proof is locally finite in $V \backslash F$. So $B_V(U) \in \mathcal{P}$ since $\mathcal{P}$ is closed monotone, open monotone and locally finitely closed additive.

Lemma 3. *$\mathcal{P}'$ is weakly additive.*

The proof of Lemma 3 is almost identical to the proof of the first two steps of the inductive proof of Lemma 3 in 9.3. Observe that the weak additivity of $\mathcal{P}$ is essential!

Lemma 4. $\mathcal{P}'$ is locally finitely closed additive.

Proof. Let $X = \bigcup \{X_\gamma|\ \gamma \in \Gamma\}$, where X_γ is closed in X, $X_\gamma \in \mathcal{P}'$, and $\{X_\gamma|\ \gamma \in \Gamma\}$ locally finite. In a standard fashion we can find a locally finite open cover $\{U_\delta|\ \delta \in \Delta\}$ and a closed cover $\{F_\delta|\ \delta \in \Delta\}$ such that, for each δ, $F_\delta \subset U_\delta$ and $\operatorname{cl} U_\delta$ meets at most finitely many members of the collection $\{X_\gamma|\ \gamma \in \Gamma\}$. In view of Lemma 1 and Lemma 3 we have $\operatorname{cl} U_\delta \in \mathcal{P}'$ for each δ. Let A and B be two disjoint closed sets in X and $A_\delta = A \cap F_\delta$. Then A_δ and $[B(U_\delta) \cup B] \cap \operatorname{cl} U_\delta$ are disjoint closed sets of $\operatorname{cl} U_\delta$. Hence there exists an open set V_δ such that $A_\delta \subset V_\delta \subset \operatorname{cl} V_\delta \subset U_\delta \backslash B$ and $B(V_\delta) \in \mathcal{P}$.

Since $\{B(V_\delta)|\ \delta \in \Delta\}$ is a locally finite closed cover of $\bigcup \{B(V_\delta)|\ \delta \in \Delta\}$, we have $\bigcup \{B(V_\delta)|\ \delta \in \Delta\} \in \mathcal{P}$. Let $V = \bigcup \{V_\delta|\ \delta \in \Delta\}$. Then $A \subset V \subset X \backslash B$ and, by Lemma 1, $B(V) \in \mathcal{P}$. Hence $X \in \mathcal{P}'$.

Corollary. *If U is an open subset of X, then $\mathcal{P}$-$\operatorname{Ind} U \leqslant \mathcal{P}$-$\operatorname{Ind} X$.*

10.4. Theorem. $\mathcal{P}$-Ind $\geqslant \mathcal{P}$-dim.

Proof. The proof is similar to the usual proof of Ind $\geqslant$ dim (cf. [7] or [8]) and we will only indicate the crucial steps.

The proof is by induction on $\mathcal{P}$-Ind. Suppose that $\mathcal{P}$-$\operatorname{Ind} X \leqslant n$ ($n \geqslant 0$) and that the theorem holds for all spaces X with $\mathcal{P}$-$\operatorname{Ind} X \leqslant n-1$. Let

$\mathcal{G} = \{G_i|\ i = 1, \ldots, k\}$ be a finite border cover of X with enclosure F. Let $\{F_i|\ i = 1, \ldots, k\}$ be a closed covering of $X \setminus F$ such that $F_i \subset G_i$ for each i. In exactly the same way as in the proof of [7], Theorem 10.1 we can find an open border cover of $X \setminus F$ which refines $\mathcal{G}$ and has order not exceeding $n+1$ (using $\mathcal{P}$-dim and $\mathcal{P}$-Ind instead of dim and Ind respectively).

11. Summary. In this section the partial results of the preceding sections are put together. We remind the reader of the basic assumptions of Section 2.1.

11.1. THEOREM. *Suppose that a class $\mathcal{P}$ is*

1) *closed monotone, and*
2) *locally countably closed additive.*

Then $\mathcal{P}$-Dim $= \mathcal{P}$-dim $= \mathcal{P}$-Ind $= \mathcal{P}$-Sur.

Proof. First observe that if $\mathcal{P}$ is locally countably closed additive, then $\mathcal{P}$ is countably closed additive and locally finitely closed additive. The converse holds if $\mathcal{P}$ is closed monotone. The theorem now follows from the results in Sections 5 and 10. The family $\mathcal{A}(a)$ of absolute Borel sets of additive class a satisfies the conditions of the preceding theorem (see Example 7.3 and [4]).

The conditions of the theorem are also satisfied by the family $\mathcal{D}_k = \{X|\ \mathrm{Ind}\, X \leqslant k\}$. By a simple inductive proof it can be shown that $\mathcal{D}_k\text{-Ind}\, Y = \sup\{\mathrm{Ind}\, Y - (k+1), -1\}$ for every space Y. By the preceding theorem we have

$$\mathcal{D}_k\text{-Dim}\, Y = \mathcal{D}_k\text{-dim}\, Y = \sup\{\mathrm{Ind}\, Y - (k+1), -1\} \quad \text{for every space } Y.$$

11.2. COROLLARY. *Suppose that a class $\mathcal{P}$ is*

1) *closed monotone, and*
2) *countably closed additive.*

Then $\mathcal{P}$-Dim $= \mathcal{P}$-dim $= \mathcal{P}$-Ind $= \mathcal{P}$-Sur *on the class of separable metrizable spaces.*

11.3. THEOREM. *Suppose that a class $\mathcal{P}$ is*

1) *countably open multiplicative,*
2) *weakly additive, and*
3) *locally finitely closed additive.*

Then $\mathcal{P}$-Dim $= \mathcal{P}$-dim $= \mathcal{P}$-Ind $= \mathcal{P}$-Def *on the class of all spaces which have a $\mathcal{P}$-hull.*

The theorem follows from the results in Sections 6 and 10. The family $\mathcal{M}(a)$ of absolute Borel sets of multiplicative class $a \geqslant 1$ satisfies the conditions of the theorem (see Example 7.3 and [4]).

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DELFT INSTITUTE OF TECHNOLOGY, Delft, Netherlands
WAYNE STATE UNIVERSITY, Detroit, Michigan

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