Let $y_0 = \min\{y_1, y_2, \ldots, y_k\}$. Then $I = [c, d] \times [0, y_0] \in [I]$ and for $I(x_0)$ we have

$$|I(x_0) \cap G_\varepsilon|_2 = \sum_{k=1}^{\infty} |[a_{n_k}, b_{n_k}] \times [0, y_0] \cap G_\varepsilon|_2$$
$$> \sum_{k=1}^{K} |[a_{n_k}, b_{n_k}] \times [0, y_0] \cap G_\varepsilon|_2$$
$$> \tfrac{1}{2} y_0 \sum_{k=1}^{K} |(a_{n_k}, b_{n_k})|$$
$$> \tfrac{1}{4} y_0 r$$
$$= \tfrac{1}{4} |I(x_0)|_2 .$$

Hence it follows that $D_s^-(G_\varepsilon, x_0) \geqslant \frac{1}{4} > 0$.

EXAMPLE 2. In [2], pp. 299–300, there is an example of a continuous function g such that if θ is any fixed direction other than $\frac{1}{2}\pi$, and $c_\theta(f, x)$ is the directional essential cluster set in direction θ, then for every x, $c(f, x) \setminus c_\theta(f, x) \neq \emptyset$. By our theorem it follows that the strong essential cluster set can differ from the directional essential cluster set at every x for any direction other than $\frac{1}{2}\pi$.

EXAMPLE 3. Let $E \subset R^1$ be any set of measure zero and f the characteristic function of $E \times (0, \infty)$. $C_s(f, x) = \{0\}$ for every x, but $c(f, x) = \{1\}$ for every $x \in E$. So the exceptional set of the second part of the theorem can be any set of measure zero.

EXAMPLE 4. In [3], Sierpiński constructs an example of a non-measurable set S with the property that every line in the plane contains at most two points of S but for every measurable set E, $|S \cap E|_2 = |E|_2$. The characteristic function of S provides an example of a non-measurable function f for which $1 \in C(f, x)$ for every x, but $c(f, x) = \{0\}$ for every x.

The above examples indicate the possible differences between strong and linear metric density.

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On countably universal Boolean algebras and compact classes of models

by

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Abstract. In the first part of the paper we give a characterization of the set $\boldsymbol{T}$ of complete theories of Boolean algebras which has the following properties: 1° For every set I and every filter $\mathcal{F}$ of subsets of I the Boolean algebra $2^I/F$ is ω_1-universal provided its theory belongs to $\boldsymbol{T}$. 2° For every complete theory $T \notin \boldsymbol{T}$ there is a set I and a filter $\mathcal{F}$ such that $\mathrm{Th}(2^I/\mathcal{F}) = T$ and $2^I/\mathcal{F}$ is not ω_1-universal.

The second part of the paper contains a characterization of the class $\boldsymbol{C}_\varkappa$ of filters such that for every compact class $\boldsymbol{K}$ of similarity type of power $\leqslant \varkappa$ and for every $\mathcal{F} \in \boldsymbol{C}_\varkappa$ the class $\mathcal{F}(\boldsymbol{K})$ of all $\mathcal{F}$-reduced products of elements of $\boldsymbol{K}$ is compact. Let $\boldsymbol{F}$ be a class of filters and let $\boldsymbol{K}$ be a class of relational structures. By $\boldsymbol{F}(\boldsymbol{K})$ we denote the class $\bigcup \{\mathcal{F}(\boldsymbol{K})\colon \mathcal{F} \in \boldsymbol{F}\}$. As a corollary to the result of the second part we give a characterization of classes $\boldsymbol{F}$ of filters such that for every compact class $\boldsymbol{K}$ the class $\boldsymbol{F}(\boldsymbol{K})$ is compact.

The present paper is a continuation of [10]. In Section 1 we give the necessary background for Sections 2 and 3. In Section 2 we prove that for some class $\boldsymbol{T}$ of complete theories of Boolean algebras if $\mathrm{Th}(\mathbf{2}^I_{\mathcal{G}}) \in \boldsymbol{T}$ then $\mathbf{2}^I_{\mathcal{G}}$ is countably universal. Moreover, for every $T \notin \boldsymbol{T}$ we give an example of an ideal $\mathcal{G}$ such that $\mathrm{Th}(\mathbf{2}^I_{\mathcal{G}}) = T$ and $\mathbf{2}^I_{\mathcal{G}}$ is not countably universal. Section 3 contains a characterization of the class $\boldsymbol{C}_{\mathfrak{m}}$ of ideals such that, for every compact class $\boldsymbol{K}$ whose similarity type is of power $\leqslant \mathfrak{m}$ and every $\mathcal{G} \in \boldsymbol{C}$, the class $\mathcal{G}(\boldsymbol{K})$ of all $\mathcal{G}$-reduced products of elements of $\boldsymbol{K}$ is compact. If $\boldsymbol{C}$ is a class of ideals and $\boldsymbol{K}$ a class of relational structures, then $\boldsymbol{C}(\boldsymbol{K})$ is the class of relational structures such that $\mathfrak{A} \in \boldsymbol{C}(\boldsymbol{K})$ if and only if for some $\mathcal{G} \in \boldsymbol{C}$ and for some sequence $\langle \mathfrak{A}_i\colon i \in I\rangle$ of elements of $\boldsymbol{K}$, $\mathfrak{A} = \mathfrak{P}_{\mathcal{G}} \langle \mathfrak{A}_i\colon i \in I\rangle$ ($\mathfrak{P}_{\mathcal{G}}$ denotes the operation of a $\mathcal{G}$-reduced product). As a corollary to the results of Section 3, we give a characterization of classes $\boldsymbol{C}$ such that for every compact class $\boldsymbol{K}$ the class $\boldsymbol{C}(\boldsymbol{K})$ is compact. Finally, in Section 4, we give some results concerning separatistic Boolean algebras of the form $\mathbf{2}_{\mathcal{G}}$ (see Definition 3.1).

Investigations of operations which preserve the compactness of classes of relational structures were started by M. Makkai, who proved that the class of all direct products of a compact class is compact (see [6]). Further results were obtained by S. R. Kogalovskiĭ [5] and A. I. Omarov

[7], [8] and recently by L. Pacholski and J. Waszkiewicz. For more historical remarks see [10].

1. We use the standard terminology and notation, slightly simplified if there is no danger of misunderstanding. By $\mathfrak{A}$, sometimes with subscripts, we denote relational structures. If $\mathfrak{A}$ is a relational structure, then A is the universe of $\mathfrak{A}$. We use $\mathcal{F}$, $\mathcal{G}$ to denote ideals. If $\mathcal{F}$ is an ideal, then by F we denote the support of $\mathcal{F}$, i.e. the set X such that $\mathcal{F}$ is an ideal of subsets of X. If there is no danger of misunderstanding, we identify an ideal $\mathcal{F}$ with the pair $\langle F, \mathcal{F}\rangle$. By $\mathbf{2}$ we denote the two-element Boolean algebra. $\mathfrak{P}_{\mathcal{G}}$ is the operation of a $\mathcal{G}$-reduced product, i.e. if $\langle \mathfrak{A}_i\colon i \in G\rangle$ is a sequence of relational structures, then $\mathfrak{P}_{\mathcal{G}}\langle \mathfrak{A}_i\colon i \in G\rangle$ denotes the direct product of $\langle \mathfrak{A}_i\colon i \in G\rangle$ reduced by $\mathcal{G}$ ($\mathfrak{P}_{i\in I} \mathfrak{A}_i/\mathcal{G}$ in the standard notation). $\mathbf{2}^G_{\mathcal{G}}$ is the Boolean algebra of all subsets of G reduced by $\mathcal{G}$. We sometimes write $\mathbf{2}_{\mathcal{G}}$ instead of $\mathbf{2}^G_{\mathcal{G}}$. Bold-face letters denote classes, e.g. by $\boldsymbol{K}$ we denote classes of relational structures.

Now we shall recall some details from Ershov's paper [1] on the decidability of Boolean algebras. If $\mathfrak{B}$ is a Boolean algebra, then by $\mathfrak{J}(\mathfrak{B})$ we denote the ideal of all elements of $\mathfrak{B}$ which can be divided into atomic and atomless parts. We put $\mathfrak{B}_0 = \mathfrak{B}$ and $\mathfrak{B}_{n+1} = \mathfrak{B}_n/\mathfrak{J}(\mathfrak{B}_n)$. Let g_n be the natural homomorphism of $\mathfrak{B}_{n-1}$ onto $\mathfrak{B}_n$ and let $h_1 = g_0$ and $h_{n+1} = g_{n+2}\cdot h_n$. We put $\mathfrak{J}_n(\mathfrak{B}) = h_n^{-1}(0)$. Of course $\mathfrak{B}_n = \mathfrak{B}/\mathfrak{J}_{n-1}(\mathfrak{B})$. If $a \in \mathfrak{B}$, then by $(a)_n$ we denote the element $h_{n-1}(a)$ of $\mathfrak{B}_n$.

By $a_i(x)$ we denote the formula of the language of Boolean algebras which says $(x)_i$ *is an atomic element* and $\beta_{i,j}(x)$ denotes the formula which says $(x)_i$ *has at least* j *atoms* (see [9]). Let $L_{1,B}$ be the language of the elementary theory of Boolean algebras (i.e. the language $\{\cup, \cap, -, \mathbf{0}, \mathbf{1}\}$) extended by adding new relational symbols $\overline{a}_i$, $\overline{\beta}_{i,j}$. Let $T_{1,B}$ be the theory in $L_{1,B}$ obtained from the theory of Boolean algebras by adding the axioms: $\overline{a}_i \leftrightarrow a_i$ and $\overline{\beta}_{i,j} \leftrightarrow \beta_{i,j}$ for i, j. We shall use the following fact (see [9]).

THEOREM 1.0. *Every formula φ of $L_{1,B}$ is equivalent in $T_{1,B}$ to an open formula φ_0. Moreover, the procedure which gives the formula φ_0 is primitive recursive.*

Using Theorem 1.0 one can easily obtain a new proof of decidability of the elementary theory of Boolean algebras.

Following Tarski [11], with every Boolean algebra B we correlate a triple $\langle a, b, c\rangle$ where $a \leqslant \infty$, $b \leqslant \infty$, $c \leqslant 1$ in the following way:

1° if, for every natural number n, $\mathfrak{B}_n$ is non-trivial, then we put $a = \infty$, $b = 0$, $c = 0$,

2° otherwise $a = \max\{x\colon \mathfrak{B}_x$ is non-trivial$\}$, b is the number of atoms in $\mathfrak{B}_a$ and $c = 1$ if there is an atomless element in $\mathfrak{B}_a$; if $\mathfrak{B}_a$ is atomic, then $c = 0$.

Ershov ([1]) has proved that for every triple $\langle a, b, c\rangle$ such that $a \leqslant \infty$, $b \leqslant \infty$, $c \leqslant 1$ and $b = 0$, $c = 0$ if $a = \infty$ and $b+c > 0$ if $a < \infty$ there is a Boolean algebra $\mathfrak{B}$ such that $\langle a, b, c\rangle$ is correlated with $\mathfrak{B}$. Let $\langle a_0, b_0, c_0\rangle$ be the triple correlated with $\mathfrak{B}^0$ and let $\langle a_1, b_1, c_1\rangle$ be the triple correlated with $\mathfrak{B}^1$. One can easily check that if $\langle a_0, b_0, c_0\rangle \neq \langle a_1, b_1, c_1\rangle$ then $\mathrm{Th}(\mathfrak{B}^0) \neq \mathrm{Th}(\mathfrak{B}^1)$.

We say that a set Σ of formulas is *consistent with* T if for every finite $\Sigma_0 \subseteq \Sigma$ the sentence $\exists v_0 \ldots \exists v_n \bigwedge \Sigma_0$ is consistent with T ($v_0, \ldots, v_n$ are the free variables of Σ_0). Σ is complete over T if Σ is maximal consistent with T. Σ is finitely satisfiable in $\mathfrak{A}$ if for every finite $\Sigma_0 \subseteq \Sigma$ the existential closure of the conjunction of all elements of Σ_0 holds in $\mathfrak{A}$. Σ is complete over $\mathfrak{A}$ if Σ is maximal finitely satisfiable in $\mathfrak{A}$.

Now let us come back to the Tarski's classification of Boolean algebras. We shall write $\mathrm{tr}(\mathfrak{B}) = \langle a, b, c\rangle$ if $\langle a, b, c\rangle$ is the triple correlated with $\mathfrak{B}$; if b is an element of $\mathfrak{B}$, then $\mathrm{tr}(b) = \langle i, j, k\rangle$ where $\langle i, j, k\rangle = \mathrm{tr}(\mathfrak{B}^1)$, and $\mathfrak{B}^1$ denotes the algebra $\mathfrak{B}$ restricted to b (1). Let τ be a term in the language of Boolean algebras. We say that a set Σ *determines that* $\mathrm{tr}(\tau) = \langle i, j, k\rangle$ if, for every Boolean algebra $\mathfrak{B}$ and every set $\{x_0, x_1, \ldots, x_n\}$ of elements of $\mathfrak{B}$ such that $x_0, x_1, \ldots, x_n$ satisfy Σ in $\mathfrak{B}$, we have $\mathrm{tr}\big(\tau(x_0, \ldots, x_n)\big) = \langle i, j, k\rangle$. In this case we write $\mathrm{tr}(\Sigma{:}\tau) = \langle i, j, k\rangle$. If $\mathrm{tr}(\mathfrak{B}) = \langle i, j, k\rangle$, then $\mathrm{tr}_1(\mathfrak{B}) = i$, $\mathrm{tr}_2(\mathfrak{B}) = j$, $\mathrm{tr}_3(\mathfrak{B}) = k$. Similarly we introduce $\mathrm{tr}_1(x)$ and $\mathrm{tr}_1(\Sigma{:}\tau)$.

THEOREM 1.1. *If Σ is a set of formulas complete over T, then for every term τ of the language of Boolean algebras $T \cup \Sigma$ determines $\mathrm{tr}(\tau)$.*

Proof. By Theorem 1.0 every formula of Σ is equivalent to an open formula of $L_{1,B}$. Consequently, Σ is equivalent to a set of conjunctions of atomic formulas and negations of atomic formulas of $L_{1,B}$. Hence, since Σ is complete, it is equivalent to a complete set of atomic formulas and their negations. Let τ be a term. We put

$$i = \max\{t\colon \overline{\beta}_{t,u}(\tau) \in \Sigma \text{ for some } u > 0 \text{ or } \overline{a}_t \in \Sigma\},$$

$$j = \begin{cases} 0 & \text{if } i = \infty, \\ \max\{t\colon \overline{\beta}_{i,t}(\tau) \in \Sigma\} & \text{otherwise}, \end{cases}$$

$$k = \begin{cases} 0 & \text{if } \overline{a}_i(\tau) \in \Sigma, \\ 1 & \text{otherwise}. \end{cases}$$

It is a matter of simple computation to show that $\mathrm{tr}(\Sigma{:}\tau) = \langle i, j, k\rangle$, i.e., that for every Boolean algebra $\mathfrak{B}$ which is a model of T and every set $x_0, x_1 \ldots$ of elements of $\mathfrak{B}$ which satisfies Σ we have $\mathrm{tr}\big(\tau(x_0, \ldots, x_n)\big) = \langle i, j, k\rangle$.

(1) I.e. the algebra of all $x \leqslant b$.

As a corollary to Theorem 1.1 we obtain the following theorem of Tarski [11].

Corollary 1.1. *For every complete theory T of Boolean algebras there is a triple $\langle i, j, k\rangle$ such that if $\mathfrak{B}$ is a model of T, then* $\operatorname{tr}(\mathfrak{B}) = \langle i, j, k\rangle$.

We write $\operatorname{tr}(T) = \langle i, j, k\rangle$ if for every model $\mathfrak{B}$ of T we have $\operatorname{tr}(\mathfrak{B}) = \langle i, j, k\rangle$.

We say that a class $\boldsymbol{K}$ is *compact* if the compactness theorem holds in $\boldsymbol{K}$, i.e., if for every set Σ of sentences Σ has a model in $\boldsymbol{K}$ if and only if every finite subset of Σ has a model in $\boldsymbol{K}$. If $\boldsymbol{K}$ is a class of relational structures and $\mathcal{G}$ is an ideal, then by $\mathcal{G}(\boldsymbol{K})$ we denote the class $\{\mathfrak{P}_{\mathcal{G}}\langle\mathfrak{A}_i\colon i \in G\rangle\colon \mathfrak{A}_i \in \boldsymbol{K}\}$. If $\boldsymbol{C}$ is a family of ideals, then $\boldsymbol{C}(\boldsymbol{K}) = \bigcup\{\mathcal{G}(\boldsymbol{K})\colon \mathcal{G} \in \boldsymbol{C}\}$.

Now we shall recall some notions concerning ideals of sets. An ideal $\mathcal{G}$ of subsets of a set I is *countably incomplete* if I is the union of a countable subfamily of $\mathcal{G}$. $\mathcal{G}$ is $(\omega, \varkappa)$-*regular* if there is a subset X of $\mathcal{G}$ of power $\varkappa$ such that, for every infinite $Y \subseteq X$, $\bigcup Y = G$. Let $S_\omega(\varkappa)$ be the set of all finite subsets of $\varkappa$. $\mathcal{G}$ is $\varkappa^+$-*good* if for every monotonic function f on $S_\omega(\varkappa)$ into $\mathcal{G}$ (i.e. $x \subseteq y$ implies $f(x) \subseteq f(y)$) there is an additive function g (i.e. $g(x \cup y) = g(x) \cup g(y)$) such that $g(x) \supseteq f(y)$ for all $x \in S_\omega(\varkappa)$. For information on various types of ideals see [3] or [4].

If L is a first order language, then by $L(\varkappa)$ we denote the language L extended by adding new symbols of constants a_α for all $\alpha < \varkappa$. We say that $\mathfrak{A}$ is $\varkappa$-*universal* if, for every set Σ of sentences of $L(\varkappa)$ of power less than $\varkappa$, if $\operatorname{Th}(\mathfrak{A}) \cup \Sigma$ is consistent, then Σ has a model of the form $(\mathfrak{A}, \boldsymbol{a})$ where $\boldsymbol{a} \in A^\varkappa$. A is *countably universal* if it is ω_1-universal. One can easily show that $\mathfrak{A}$ is countably universal if and only if every set of formulas finitely satisfiable in $\mathfrak{A}$ is satisfiable in $\mathfrak{A}$.

2. The following theorem has been proved in [10].

Theorem 2.0. *If $\operatorname{tr}(\mathbf{2}_{\mathcal{G}}) = \langle 0, i, j\rangle$, then $\mathbf{2}_{\mathcal{G}}$ is countably universal.*

In this section we shall give an extension of Theorem 2.0.

Lemma 2.1. *Let $\operatorname{tr}(\mathbf{2}_{\mathcal{G}}) = \langle a, b, c\rangle$ and Σ be a set of formulas such that*

1. *Σ is finitely satisfiable in $\mathbf{2}_{\mathcal{G}}$,*
2. *$\operatorname{tr}(\Sigma \cup \operatorname{Th}(\mathbf{2}_{\mathcal{G}})\colon v_0) = \langle a_1, b_1, c_1\rangle$ and $\operatorname{tr}(\Sigma \cup \operatorname{Th}(\mathbf{2}_{\mathcal{G}})\colon -v_0) = \langle a_2, b_2, c_2\rangle$,*
3. *there is no element of $\mathbf{2}_{\mathcal{G}}$ which satisfies Σ.*

Then $b = 0$, $0 < a < \infty$ and we have $a_i = a-1$, $b_i = \infty$, $c_i = 0$ for $i = 1$ or $i = 2$.

Proof. Suppose that the conclusion of lemma is not true while 2.1.1 and 2.1.2 hold. We shall prove that there is an element of $\mathbf{2}_{\mathcal{G}}$ which satisfies Σ. We shall consider a few cases. One can easily check that if Σ is a set of formulas with one free variable v_0 then from $\operatorname{tr}(x) = \operatorname{tr}(T \cup \Sigma, v_0)$ and

$\operatorname{tr}(-x) = \operatorname{tr}(T \cup \Sigma, -v_0)$ it follows that x satisfies Σ. Hence it suffices to find an element x which satisfies the equations above.

Case 1. $a_1 = a_2$, $a \neq \infty$. Since Σ is finitely satisfiable in $\mathbf{2}_{\mathcal{G}}$ and, since the fact that $\operatorname{tr}_1(x) = i$ for $i < \infty$ can be described by a simple formula, we have $a_1 = a$. Of course $\operatorname{tr}((\mathbf{2}_{\mathcal{G}})_a) = \langle 0, b, c\rangle$ and, for some $\mathcal{F}$, $(\mathbf{2}_{\mathcal{G}})_a$ is of the form $\mathbf{2}_{\mathcal{F}}$. Hence by Theorem 2.0 there is an element x_1 of $\mathbf{2}_{\mathcal{F}}$ such that $\operatorname{tr}(x_1) = \langle 0, b_1, c_1\rangle$ and $\operatorname{tr}(-x_1) = \langle 0, b_2, c_2\rangle$. Let X be a subset of G $(G = F)$ such that $x_1 = X/\mathcal{F}$. Then $\operatorname{tr}((X/\mathcal{G})_a) = \langle 0, b_1, c_1\rangle$ and $\operatorname{tr}((-X/\mathcal{G})_a) = \langle 0, b_2, c_2\rangle$; hence $x = X/\mathcal{G}$ satisfies Σ in $\mathbf{2}_{\mathcal{G}}$.

Case 2. $a_1 = a_2$, $a = \infty$. We shall define two sequences $\langle X_n\colon n < \omega\rangle$ and $\langle Y_n\colon n < \omega\rangle$ of subsets of G such that $X_n \subseteq X_{n+1}$, $Y_n \subseteq Y_{n+1}$, $X_n \cap Y_n = 0$ and $\beta_{n,1}(X_n/\mathcal{G}) \wedge \neg\beta_{n,2}(X_n/G) \wedge \beta_{n,1}(Y_n/\mathcal{G}) \wedge \neg\beta_{n,2}(Y_n/\mathcal{G})$ holds in $\mathbf{2}_{\mathcal{G}}$. If X_n and Y_n are defined, then $\operatorname{tr}(-((X_n \cup Y_n)/\mathcal{G})) = \langle \infty, 0, 0,\rangle$; hence there are sets X'_n and Y'_n such that $X'_n \cap Y'_n = 0$, $(X'_n \cup Y'_n) \cap (X_n \cup Y_n) = 0$ and $(\beta_{n,1} \wedge \neg\beta_{n,2})(X'_n/\mathcal{G})$ and $\beta_{n,1} \wedge \neg\beta_{n,2}(Y'_n/\mathcal{G})$ hold in $\mathbf{2}_{\mathcal{G}}$. We put $X_{n+1} = X_n \cup X'_n$ and $Y_{x+1} = Y_n \cup Y'_n$. It is easy to show that if $x = (\bigcup X_n)/G$, then $\operatorname{tr}(x) = \langle \infty, 0, 0\rangle$ and $\operatorname{tr}(-x) = \langle \infty, 0, 0\rangle$.

Case 3. $a_1 \neq a_2$, $0 < a$. Without loss of generality we assume that $a_1 < a_2$. In this case it suffices to find an element x such that $\operatorname{tr}(x) = \langle a_1, b_1, c_1\rangle$ because $\langle a_2, b_2, c_2\rangle = \langle a, b, c\rangle$ and, moreover, $\operatorname{tr}_1(y) = a_1$ implies $\operatorname{tr}(-y) = \langle a, b, c\rangle$. If b_1 is finite, then Σ is equivalent in $\mathbf{2}_{\mathcal{G}}$ to the simple formula $(\beta_{a_1,b_1} \wedge \neg\beta_{a_1,b_1+1})(v_0)$; hence we assume that $b_1 = \infty$.

Moreover, we can assume that $c_1 = 0$. In fact, since $a_1 < a$, there is an element y of $\mathbf{2}_{\mathcal{G}}$ such that $(y)_{a_1}$ is atomless. Hence, if for some $x \in \mathbf{2}_{\mathcal{G}}$ $\operatorname{tr}(x) = \langle a_1, b_1, 0\rangle$, then $\operatorname{tr}(x \cup y) = \langle a_1, b_1, 1\rangle$.

Now we shall prove that there is an atom in $(\mathbf{2}_{\mathcal{G}})_{a_1+1}$. We consider two subcases.

Subcase 3.1. $b \neq 0$ or $a = \infty$. Then for every $n < a$ there is an atom in $(\mathbf{2}_{\mathcal{G}})_n$.

Subcase 3.2. $b = 0$ and $a < \infty$. Since 2.1.3 does not hold and $b = 0$, $0 < a < \infty$, we have $a_1 \neq a-1$ or $b_1 \neq \infty$ or $c_1 = 1$. But we have assumed that $b = \infty$ and $c_1 \neq 1$; hence $a_1 \neq a-1$. Moreover, $a_1 < a$, hence $a_1 + 2 \leqslant a$. Consequently the Boolean algebra $(\mathbf{2}_{\mathcal{G}})_{a_1+2}$ is non-trivial and hence there is an atom in $(\mathbf{2}_{\mathcal{G}})_{a_1+1}$.

Now let X be a subset of G such that $(X/\mathcal{G})_{a_1+1}$ is an atom of $(\mathbf{2}_{\mathcal{G}})_{a_1+1}$. Since $X/\mathcal{G} \notin \mathfrak{J}_{a_1+1}(\mathbf{2}_{\mathcal{G}})$, there is an infinite set $\{X_i\colon i < \omega\}$ of subsets of X such that for $i < \omega$ $(X_i/\mathcal{G})_{a_1}$ is an atom of $(\mathbf{2}_{\mathcal{G}})_{a_1}$ and $(X_i/\mathcal{G})_{a_1} \cap (X_j/\mathcal{G})_{a_1} = \mathbf{0}$ for $i \neq j$. By easy induction one can prove that we can assume that $X_i \cap X_j = 0$. Let $X' = \bigcup\{X_i\colon i < \omega\}$. Then $(X'/\mathcal{G})_{a_1}$ has infinitely many atoms. We shall find a set $Y \subseteq X'$ such that $\operatorname{tr}(Y/\mathcal{G}) = \langle a_1, \infty, 0\rangle$, i.e., $(X/\mathcal{G})_{a_1}$ is atomic and has infinitely many atoms. Let $Y_1 = \bigcup\{X_{2i}\colon i < \omega\}$

and $Y_2 = \bigcup\{X_{2i+1}\colon\ i<\omega\}$. We claim that $Y_1/\mathcal{G} \in \mathfrak{J}_{a_1+1}(\mathbf{2}_\mathcal{G})$ or $Y_2/\mathcal{G} \in \mathfrak{J}_{a_1+1}(\mathbf{2}_\mathcal{G})$. In fact, if $Y_1/\mathcal{G} \notin \mathfrak{J}_{a_1+1}(\mathbf{2}_\mathcal{G})$, then

(1) $$(Y_2/\mathcal{G})_{a_1+1} < (Y_2/\mathcal{G})_{a_1+1} \cup (Y_1/\mathcal{G})_{a_1+1}\,.$$

Since $Y_1 \cup Y_2 \subseteq X$ by (1), we obtain $(Y_2/\mathcal{G})_{a_1+1} < (X/\mathcal{G})_{a_1+1}$, but $(X/\mathcal{G})_{a_1+1}$ is an atom of $(\mathbf{2}_\mathcal{G})_{a_1+1}$, whence $(Y_2/\mathcal{G})_{a_1+1}$ is the zero element of $(\mathbf{2}_\mathcal{G})_{a_1+1}$, which means that $(Y_2/\mathcal{G}) \in \mathfrak{J}_{a_1+1}(\mathbf{2}_\mathcal{G})$. It follows immediately from the definitions of Y_i that for $i = 1, 2$ $(Y_i/\mathcal{G})_{a_1}$ has infinitely many atoms, which completes the proof that there is a set Y' such that $\mathrm{tr}(Y'/\mathcal{G}) = \langle a_1, \infty, l\rangle$. If $l = 0$ we put $Y = Y'$. Otherwise Y' can be divided into two sets Y and Z such that $(Y/\mathcal{G})_{a_1}$ is atomic and $(Z/\mathcal{G})_{a_1}$ is atomless, whence $\mathrm{tr}(Y/\mathcal{G}) = \langle a_1, \infty, 0\rangle$.

THEOREM 2.2. *If* $\mathrm{tr}(\mathbf{2}_\mathcal{G}) = \langle a, b, c\rangle$ *and* $b \neq 0$ *or* $a = 0$ *or* $a = \infty$, *then* $\mathbf{2}_\mathcal{G}$ *is countably universal.*

Proof. Let Σ be a set of formulas of the language of Boolean algebras which is finitely satisfiable in $\mathbf{2}_\mathcal{G}$. We can assume that Σ is complete and finitely satisfiable in $\mathbf{2}_\mathcal{G}$, i.e., for every formula σ, if $\{\sigma\} \cup \Sigma$ is finitely satisfiable in $\mathbf{2}_\mathcal{G}$, then $\sigma \in \Sigma$. Let $\varepsilon\colon n+1 \to \mathbf{2}$ (²). If φ is a formula of the language of Boolean algebras, then $\varphi(\varepsilon)$ denotes the formula φ in which every free occurrence of v_i is replaced by $\varepsilon(i)$. Let $\Sigma_\varepsilon = \{\varphi(\varepsilon)\colon\ \varphi \in \Sigma\}$. We put $X^0 = -X$, $X^1 = X$, $2^\infty = \{\varepsilon\colon\ \varepsilon \in \mathbf{2}^n,\ n<\omega\}$. If $\mathfrak{X} = \langle X_i\colon\ i<\omega\rangle$ then $\varepsilon\mathfrak{X} = \bigcap\{X_i^{\varepsilon(i)}\colon\ i \in \mathrm{dom}(\varepsilon)\}$.

Now we shall define by induction a sequence $\mathfrak{X} = \langle X_i\colon\ i<\omega\rangle$ of subsets of G such that:

(2) *the sequence* $\langle X_i/\mathcal{G}\colon\ i<\omega\rangle$ *satisfies* Σ *in* $\mathbf{2}_\mathcal{G}$, *and*

(3) *if, for some* $\varepsilon \in \mathbf{2}^\infty$, $\mathrm{tr}\big((\varepsilon\mathfrak{X})/\mathcal{G}\big) = \langle n, 0, 1\rangle$, *then is a subset* Y_ε *of* $\varepsilon\mathfrak{X}$ *such that* $\mathrm{tr}(Y_\varepsilon/\mathcal{G}) = \langle n-1, \infty, 0\rangle$.

To simplify the notation, let us introduce the following symbol:

$$\langle k_1, l_1, m_1\rangle + \langle k_2, l_2, m_2\rangle = \begin{cases} \langle k_2, l_2, m_2\rangle & \text{if } k_1 < k_2\,,\\ \langle k_1, l_1+l_2, \max\{m_1, m_2\}\rangle & \text{if } k_1 = k_2\,,\\ \langle k_1, l_1, m_1\rangle & \text{if } k_2 < k_1\,.\end{cases}$$

If $\mathfrak{A}$ and $\mathfrak{B}$ are Boolean algebras, then $\mathrm{tr}(\mathfrak{A}) + \mathrm{tr}(\mathfrak{B}) = \mathrm{tr}(\mathfrak{A}\times\mathfrak{B})$.

From Lemma 2.1 and (3) we obtain the following stronger version of (3).

(4) *If* $\langle k_1, l_1, m_1\rangle + \langle k_2, l_2, m_2\rangle = \mathrm{tr}(\varepsilon X/\mathcal{G})$, *then there is a subset* Z *of* $\varepsilon\mathfrak{X}$ *such that* $\mathrm{tr}(Z/\mathcal{G}) = \langle k_1, l_1, m_1\rangle$ *and* $\mathrm{tr}(\varepsilon\mathfrak{X} - Z/\mathcal{G}) = \langle k_2, l_2, m_2\rangle$.

Since we have assumed that Σ is complete, we have

$$\mathrm{tr}(\mathrm{Th}(\mathbf{2}_\mathcal{G}) \cup \Sigma\colon\ v_0) = \langle a_1, b_1, c_1\rangle \text{ and } \mathrm{tr}(\mathrm{Th}(\mathbf{2}_\mathcal{G}) \cup \Sigma\colon\ -v_0) = \langle a_2, b_2, c_2\rangle\,.$$

(²) $n = \{i\colon\ i<n\}$.

Hence, since $b \neq 0$ or $a = 0$ or $a = \infty$, by Lemma 2.1 there is a set X_0' such that $\mathrm{tr}(X_0'/\mathcal{G}) = \langle a_1, b_1, c_1\rangle$ and $\mathrm{tr}(-X_0'/\mathcal{G}) = \langle a_2, b_2, c_2\rangle$. To define X_0 we shall consider three cases: $a_1 = a_2$, $a_1 < a_2$ and $a_1 > a_2$. If $a_1 = a_2$ then $b_1 + b_2 > 0$. By Lemma 2.1 there is an element Y_0' such that $\mathrm{tr}(Y_0'/\mathcal{G}) = \langle a-1, \infty, 0\rangle$. We put $X_0 = X_0' \cup Y_0'$ if $b_1 = 0$ and $X_0 = X_0' - Y_0'$ if $b_2 = 0$. If $a_1 < a_2$ then we put $X_0 = X_0' \cup Y_0$, where Y_0 is a set such that $\mathrm{tr}(Y_0/\mathcal{G}) = \langle a_1 - 1, \infty, 0\rangle$; if $a_1 > a_2$ then $X_0 = X_0' - Y_0$, where $\mathrm{tr}(Y_0/\mathcal{G}) = \langle a_2 - 1, \infty, 0\rangle$. It is a matter of simple computation to check that X_0 has the desired properties.

Now let us assume that a sequence $\mathfrak{X}_n = \langle X_0, \ldots, X_n\rangle$ is defined. For $\varepsilon \in \mathbf{2}^n$ let $X_\varepsilon = \varepsilon\mathfrak{X}_n$ and let $\mathcal{G}(\varepsilon)$ denote the ideal $\mathcal{G}$ restricted to X_ε. Since Σ is complete,

$$\mathrm{tr}\big(\mathrm{Th}(\mathbf{2}_{\mathcal{G}(\varepsilon)}) \cup \Sigma\colon\ v_{n+1}\big) = \langle a_1, b_1, c_1\rangle \quad \textit{and}$$
$$\mathrm{tr}\big(\mathrm{Th}(\mathbf{2}_{\mathcal{G}(\varepsilon)}) \cup \Sigma_\varepsilon\colon -v_{n+1}\big) = \langle a_2, b_2, c_2\rangle\,.$$

If

(5) $$\mathrm{tr}(\mathbf{2}_{\mathcal{G}(\varepsilon)}) = \langle a', b', c'\rangle \quad \textit{and} \quad \langle a_i, b_i, c_i\rangle = \langle a'-1, \infty, c'\rangle \quad \textit{for } i = 1 \textit{ or } i = 2\,,$$

then by the inductive hypothesis there is a set $Z_\varepsilon' \subseteq X_\varepsilon$ such that

$$\mathrm{tr}\big(Z_\varepsilon'/\mathcal{G}(\varepsilon)\big) = \langle a_1, b_1, c_1\rangle \quad \textit{and} \quad \mathrm{tr}\big((X_\varepsilon - Z_\varepsilon')/\mathcal{G}(\varepsilon)\big) = \langle a_2, b_2, c_2\rangle\,.$$

If (5) does not hold, then the existence of such a set is a consequence of Lemma 2.1 applied to $\mathbf{2}_{\mathcal{G}(\varepsilon)}$. In the same way as X_0 was obtained from X_0', we obtain from Z_ε' a set Z_ε such that $Z_\varepsilon' \Delta Z_\varepsilon \in \mathfrak{J}_{\mathrm{tr}_1(Z_\varepsilon'/\mathcal{G})}(\mathbf{2}_{\mathcal{G}(\varepsilon)})$ and Z_ε contains a set Y_ε' such that $\mathrm{tr}\big(Y_\varepsilon'/\mathcal{G}(\varepsilon)\big) = \langle \mathrm{tr}(Z_\varepsilon/\mathcal{G}(\varepsilon)) - 1, \infty, 0\rangle$ and $X_\varepsilon - Z_\varepsilon$ contains a set Y_ε'' such that $\mathrm{tr}\big(Y_\varepsilon''/\mathcal{G}(\varepsilon)\big) = \langle \mathrm{tr}_1\big((X_\varepsilon - Z_\varepsilon)/\mathcal{G}(\varepsilon)\big) - 1, \infty, 0\rangle$.

We put $X_{n+1} = \bigcup\{Z_\varepsilon\colon\ \varepsilon \in \mathbf{2}^n\}$. Obviously, the sequence $\langle X_i\colon\ i \leqslant n+1\rangle$ satisfies (3) and the sequence $\langle X_1/\mathcal{G}\colon\ i \leqslant n+1\rangle$ satisfies all the formulas of Σ whose free variables are among $v_0, \ldots, v_{n+1}$.

Slightly modifying the proof of Theorem 2.2, we can obtain the following fact, which will be used later.

THEOREM 2.3. *If* $\mathrm{tr}(\mathbf{2}_\mathcal{G}) = \langle m, 0, 1\rangle$ *and there is an element* x *of* $\mathbf{2}_\mathcal{G}$ *such that* $\mathrm{tr}(x) = \langle m-1, \infty, 0\rangle$, *then* $\mathbf{2}_\mathcal{G}$ *is countably universal.*

From Theorem 2.2 we obtain the following corollaries.

COROLLARY 2.4. *If, for all* $m > 0$, $\mathrm{tr}(\mathbf{2}_\mathcal{G}) \neq \langle m, 0, 1\rangle$, *then for every finite relational structure* $\mathfrak{A}$, *the reduced power* $\mathfrak{A}_\mathcal{G}$ *is countably universal. If, moreover,* $\mathcal{G}$ *is countably incomplete, then for every relational structure* $\mathfrak{A}$ *(not necessarily finite) the reduced power* $\mathfrak{A}_\mathcal{G}$ *is countably universal.*

COROLLARY 2.5. *If, for all $m>0$, $\mathrm{tr}(\mathbf{2}_{\mathcal{G}}) \neq \langle m, 0, 1\rangle$, then for every compact class $\boldsymbol{K}$ of countable similarity type, the class $\mathcal{G}(\boldsymbol{K})$ is compact. If, moreover, $\mathcal{G}$ is countably incomplete, then we can omit the assumption of compactness of the class $\boldsymbol{K}$.*

Corollary 2.4 follows from Theorems 1 and 2 of [10], Corollary 2.5 is a consequence of the theorems of Section 4 of [9].

Now we shall give another consequence of Theorem 2.2, namely we shall prove that in Theorems 1 and 2 of [10] the assumption that $\mathbf{2}_{\mathcal{G}}$ is countably universal is necessary.

THEOREM 2.6. *If, for every compact class $\boldsymbol{K}$ of countable similarity type, the class $\mathcal{G}(\boldsymbol{K})$ is compact, then $\mathbf{2}_{\mathcal{G}}$ is countably universal.*

Proof. Let $\mathcal{G}$ be an ideal such that $\mathbf{2}_{\mathcal{G}}$ is not countably universal. By Theorem 2.2 there is a natural number m such that $\mathrm{tr}(\mathbf{2}_{\mathcal{G}}) = \langle m, 0, 1\rangle$. Also, by Theorem 2.3 there is no element x in $\mathbf{2}_{\mathcal{G}}$ such that $\mathrm{tr}(x) = \langle m-1, \infty, 0\rangle$. Let L be a language with one binary relational symbol $\leqslant$ and let $\mathfrak{A}_1 = \langle\{0\}, \leqslant\rangle$ and $\mathfrak{A}_2 = \langle\{0, 1\}, \leqslant\rangle$. We put $\boldsymbol{K}_1 = \{\mathfrak{A}_1, \mathfrak{A}_2\}$.

Let

$$\Pi_m = \{\alpha_{m-1}(\mathbf{1})\} \cup \{\beta_{m-1,i}(\mathbf{1}) \colon i<\omega\} .$$

We claim that Π_m is finitely satisfiable in $\mathcal{G}(\boldsymbol{K}_1)$. In fact, let Π' be a finite subset of Π_m. We can assume that $\Pi' = \{\alpha_{m-1}(\mathbf{1})\} \cup \{\beta_{m-1,i}(\mathbf{1}) \colon i<n\}$. Since $\mathrm{tr}(\mathbf{2}_{\mathcal{G}}) = \langle m, 0, 1\rangle$, there is an element y of $\mathbf{2}_{\mathcal{G}}$ such that $\mathrm{tr}(y) = \langle m-1, n, 0\rangle$. Let Y be a subset of G such that $Y/\mathcal{G} = y$. Let $\mathcal{A} = \langle \mathfrak{A}_j \colon j \in G\rangle$ be the sequence of elements of $\boldsymbol{K}_1$ such that $\mathfrak{A}_j = \mathfrak{A}_2$ if and only if $j \in Y$. Of course, $\mathfrak{P}_{\mathcal{G}}\mathcal{A}$ is an element of $\mathcal{G}(\boldsymbol{K}_1)$ and Π' holds in $\mathfrak{P}_{\mathcal{G}}\mathcal{A}$. On the other hand, $\mathcal{G}(\boldsymbol{K}_1)$ is not compact. In fact, if $\mathcal{C}$ is a sequence of elements of $\boldsymbol{K}_1$ such that $\mathfrak{P}_{\mathcal{G}}\mathcal{C}$ is a model of Π_m, then we put $X = \{j \in G \colon \mathfrak{C}_j = \mathfrak{A}_2\}$. It is easy to check that if $x = X/\mathcal{G}$, then x is an element of $\mathbf{2}_{\mathcal{G}}$ such that $\mathrm{tr}(x) = \langle m-1, \infty, 0\rangle$.

From Theorem 2.6 and Theorem 1 of [10] we obtain the following result.

COROLLARY 2.7. *$\mathbf{2}_{\mathcal{G}}$ is countably universal if and only if, for every compact class $\boldsymbol{K}$ of countable similarity type, the class $\mathcal{G}(\boldsymbol{K})$ is compact.*

Now we shall prove that Theorem 2.2 cannot be extended to every Boolean algebra of the form $\mathbf{2}_{\mathcal{G}}$.

THEOREM 2.8. *For each positive integer m there is an ideal $\mathcal{F}_m$ of subsets of ω such that $\mathrm{tr}(\mathbf{2}_{\mathcal{F}_m}) = \langle m, 0, 1\rangle$ and $\mathbf{2}_{\mathcal{F}_m}$ is not countably universal.*

Proof. Let m be a positive integer. Let $\{E_i \colon i<\omega\}$ be a partition of ω into infinite sets and, for each $i<\omega$, let $\mathcal{G}_i$ denote an ideal of subsets of E_i such that

(6) $$\mathrm{tr}(\mathbf{2}_{\mathcal{G}_i}) = \langle m-1, 1, 0\rangle$$

and

(7) $$\mathcal{G}_i \text{ is non-principal}.$$

The existence of such an ideal is a consequence of the theorem of Ershov [1] mentioned in Section 1.

We put

$$\mathcal{F}_m = \{X \subseteq \omega \colon E_i \cap X \in \mathcal{G}_i \text{ for } i<\omega \text{ and } \{i \colon E_i \cap X \neq 0\} \text{ is finite}\} .$$

Let

$$\Sigma_m = \{\alpha_{m-1}(v_0)\} \cup \{\beta_{m-1,i}(v_0) \colon i<\omega\} .$$

We shall prove that, for every $m<\omega$,

(8) $$\Sigma_m \text{ is finitely satisfiable in } \mathbf{2}_{\mathcal{F}_m},$$

and

(9) $$\Sigma_m \text{ is not satisfiable in } \mathbf{2}_{\mathcal{F}_m}.$$

It is easy to remark that for every $i<\omega$ the equivalence class of E_i with respect to $\mathcal{F}_m$ forms an atom of $(\mathbf{2}_{\mathcal{F}_m})_{m-1}$. Consequently $\bigcup\{E_i \colon i<n\}/F_m$ has n atoms and is atomic in $(\mathbf{2}_{\mathcal{F}_m})_{m-1}$; hence (8) holds.

Moreover, by easy induction one can prove that

(10) $$\text{if } Y \subseteq E_i, \text{ then } \mathrm{tr}(Y/\mathcal{G}_i) = \mathrm{tr}(Y/\mathcal{F}_m) .$$

Now we shall prove an auxiliary fact.

2.9. *For every $n \leqslant m$ and every set $X \subseteq \omega$, we have* (i) *if and only if* (ii) *and* (iii), *where*

(i) *$X/\mathcal{F}_m \in \mathfrak{J}_n(\mathbf{2}_{\mathcal{F}_m})$,*

(ii) *$(X \cap E_i)/\mathcal{G}_i \in \mathfrak{J}_n(\mathbf{2}_{\mathcal{G}_i})$, for $i<\omega$,*

(iii) *$\{i \colon (X \cap E_i)/\mathcal{G}_i$ contains an atom of $(\mathbf{2}_{\mathcal{G}_i})_{n-1}\}$, is finite.*

Proof. We will proceed by induction.

a. (ii)$\wedge$(iii)$\Rightarrow$(i). Since (ii) holds, for each $i<\omega$ there are sets X_i and Y_i such that $X \cap E_i = X_i \cup Y_i$ and $(X_i/\mathcal{G}_i)_n$ is an atomic and $(Y_i/\mathcal{G}_i)_n$ is an atomless element of $(\mathbf{2}_{\mathcal{G}_i})_n$. On the other hand, by (iii) we can assume that for $i>i_0$ $X_i = 0$. By (10), for $i<\omega$, $(Y_i/F_m)_n$ is an atomless element of $(\mathbf{2}_{\mathcal{F}_m})_n$ and, for $i \leqslant i_0$, $(X_i/\mathcal{F}_m)_n$ is an atomic element of $(\mathbf{2}_{\mathcal{F}_m})_n$. Let $X' = \bigcup\{X_i \colon i \leqslant i_0\}$ and $Y = \bigcup\{Y_i \colon i<\omega\}$. Of course $(X'/\mathcal{F}_m)_n$ is a union of a finite number of atomic elements of $(\mathbf{2}_{\mathcal{F}_m})_n$; hence it is an atomic element of $(\mathbf{2}_{\mathcal{F}_m})_n$. Also $(Y/\mathcal{F}_m)_n$ is an atomless element of $(\mathbf{2}_{\mathcal{F}_m})_n$. In fact, if $Y' \subseteq Y$ and $Y'/\mathcal{F}_m \notin \mathfrak{J}_n(\mathbf{2}_{\mathcal{F}_m})$, then by the inductive hypothesis

(11) $$(Y' \cap E_i)/\mathcal{G}_i \notin \mathfrak{J}_n(\mathbf{2}_{\mathcal{G}_i}) \quad \text{for some } i<\omega ,$$

or

(12) $$I = \{i \colon \big((Y' \cap E_i)/\mathcal{G}_i\big)_{n-1} \text{ has an atom of } (\mathbf{2}_{\mathcal{G}_i})_{n-1}\} \text{ is infinite.}$$

If (11) holds, then by (10) $((Y'\cap E_i)/\mathcal{F}_m)_n$ is atomless; hence $(Y'/\mathcal{F}_m)_n$ contains an atomless element. Let us assume that (11) does not hold and (12) holds. Let $Y_Z=\bigcup\{Y'\cap E_i\colon i\in Z\}$ for $Z\subseteq I$. Then if Z is infinite, then by the definition of I and by the inductive hypothesis $(Y_Z/\mathcal{F}_m)\notin\mathfrak{J}_n(\mathbf{2}_{\mathcal{F}_m})$ and since (11) does not hold, for every finite Z, $(Y_Z/\mathcal{F}_m)\in\mathfrak{J}_n(\mathbf{2}_{\mathcal{F}_m})$. Consequently $(Y_Z/\mathcal{F}_m)\notin\mathfrak{J}_n(\mathbf{2}_{\mathcal{F}_m})$ if and only if Z is infinite and hence Y' is atomless.

b. (i) $\Rightarrow$ (ii). It is an immediate consequence of (10).

c. (i) $\Rightarrow$ (iii). It suffices to prove that

(13) *for every set Z such that the set $D_n(Z)=\{i\colon (E_i\cap Z)/\mathcal{G}_i$ belongs to an equivalence class of an atom of $(\mathbf{2}_{\mathcal{G}_i})_n\}$ is infinite, there is a set $Y\subseteq Z$ such that $(Y/\mathcal{F}_m)_n$ is an atomless element of $(\mathbf{2}_{\mathcal{F}_m})_n$* $(\mathfrak{J}_0=0)$.

In fact, let us suppose that (13) holds. Then if $X/\mathcal{F}_m\in\mathfrak{J}_{n+1}(\mathbf{2}_{\mathcal{F}_m})$ then there are sets X_1 and X_2 such that $X=X_1\cup X_2$, $(X_1/\mathcal{F}_m)_n$ is an atomic element of $(\mathbf{2}_{\mathcal{F}_m})_n$ and $(X_2/\mathcal{F}_m)_n$ is an atomless element of $(\mathbf{2}_{\mathcal{F}_m})_n$. Hence, for each $i<\omega$, $((X_2\cap E_i)/\mathcal{G}_i)_n$ is atomless. Since, for every $i<\omega$ $((X_2\cap E_i)/\mathcal{G}_i)_n$ is atomless, it remains to prove that $D_n(X_1)$ is finite. In fact, if we assume that $D_n(X_1)$ is infinite, then by (13) there is a subset Y of X_1 such that $(Y/\mathcal{F}_m)_n$ is atomless, which contradicts the fact that $(X_1/\mathcal{F}_m)_n$ is atomic.

To finish the proof of 2.9 it remains to prove (13). We consider two cases: $n=0$ and $n>0$.

$n=0$. Let Z be a set such that $D_0(X)$ is infinite. Let Y be a subset of X such that $Y\cap E_i$ has exactly one element if $i\in D_0(X)$ and $Y\cap E_i$ is empty otherwise. We have $\{i\colon E_i\cap Y\neq 0\}=D_0(X)$. Hence the set $\{i\colon E_i\cap Y\neq 0\}$ is infinite. By the definition of $\mathcal{F}_m$, for every infinite $Y_0\subseteq Y$, $Y_0/F_m\neq 0$. Moreover, since for each i, $\mathcal{G}_i$ is non-principal, for every finite $Y_1\subseteq Y$, $Y_1/\mathcal{F}_m=0$; hence $Y/\mathcal{F}_m$ is an atomless element of $\mathbf{2}_{\mathcal{F}_m}$.

$n>0$. Now let X be a set such that $D_n(X)$ is infinite. Then there is a set Y such that $Y\subseteq X$, $Y\cap E_i$ is empty if $i\notin D_n(X)$, and if $i\in D_n(X)$, then $((Y\cap E_i)/\mathcal{G}_i)_{n-1}$ forms an atom of $(\mathbf{2}_{\mathcal{G}_i})_{n-1}$. Let $X_Z=\bigcup\{E_i\cap Y\colon i\in Z\}$ for $Z\subseteq D_n(X)$. By the inductive hypothesis one can show that $X_Z/F_m\in\mathfrak{J}_n(\mathbf{2}_{\mathcal{F}_m})$ if and only if Z is finite; hence $(Y/\mathcal{F}_m)_n$ is atomless.

Now we are ready to prove that Σ_m is not satisfiable in $\mathbf{2}_{\mathcal{F}_m}$. Let us suppose that X is a set such that $(X/\mathcal{F}_m)_{m-1}$ is atomic. Hence $X/\mathcal{F}_m\in\mathfrak{J}_m(\mathbf{2}_{\mathcal{F}_m})$. By 2.9 we have

(14) $$D_{m-1}(X)\text{ is finite}.$$

Let $Y=\bigcup\{X\cap E_i\colon i\notin D_{m-1}(X)\}$. We claim that $Y/\mathcal{F}_m\in\mathfrak{J}_{m-1}(\mathbf{2}_{\mathcal{F}_m})$. In fact, for every $i<\omega$ $((E_i\cap Y)/\mathcal{G}_i)_{m-1}$ is atomless, and hence by (6) we have $(Y\cap E_i)/\mathcal{G}_i\in\mathfrak{J}_{m-1}(\mathbf{2}_{\mathcal{G}_i})$. Moreover, if we suppose that the set $H=\{i\colon (Y\cap E_i/\mathcal{G}_i)_{m-2}$ has an atom$\}$ is infinite, then one can prove that $(Y/\mathcal{F}_m)_{m-1}$ contains an atomless element. But $Y\subseteq X$ and $(X/\mathcal{F}_m)_{m-1}$ is atomic; hence H is finite and consequently, by 2.9, $Y/\mathcal{F}_m\in\mathfrak{J}_{m-1}(\mathbf{2}_{\mathcal{F}_m})$.

Now, by (6), $(E_i/\mathcal{G}_i)_{m-1}$ has exactly one atom; hence, by (14), $(X/\mathcal{F}_m)_{m-1}$ has a finite number of atoms, which contradicts the fact that $X/\mathcal{F}_m$ satisfies Σ_m.

Now it remains to prove that $\operatorname{tr}(\mathbf{2}_{\mathcal{F}_m})=\langle m,0,1\rangle$. Since Σ_m is finitely satisfiable and is not satisfiable in $\mathbf{2}_{\mathcal{F}_m}$, by Theorem 2.2, for some natural number m_1, we have $\operatorname{tr}(\mathbf{2}_{\mathcal{F}_m})=\langle m_1,0,1\rangle$. Also by Theorem 2.3 we have $m_1=m$.

3. In this section we give a characterization of ideals $\mathcal{G}$ such that for every compact class $\boldsymbol{K}$ the class $\mathcal{G}(\boldsymbol{K})$ is compact, and also some related results.

DEFINITION 3.1. An ideal $\mathcal{G}$ is *$\varkappa$-separatistic* if, for every set Σ of sentences of $L_b(\varkappa)$ (L_b denotes the language of Boolean algebras) which is satisfiable in $\mathbf{2}_{\mathcal{G}}$, there is a sequence $(X_i\colon i<\varkappa)$ of subsets of $\mathcal{G}$ such that the sequence $(X_i/\mathcal{G}\colon i<\varkappa)$ satisfies Σ and for every $s,t\in S_\omega(\varkappa)$ we have

(15) $$\bigcap\{X_i\colon i\in s\}\cap\bigcap\{G-X_i\colon i\in t\}=0$$
$$\text{if}\quad \bigcap\{X_i/\mathcal{G}\colon i\in s\}\cap\bigcap\{-(X_i/\mathcal{G})\colon i\in t\}=\mathbf{0}.$$

LEMMA 3.2. *If $\mathcal{G}$ is $\varkappa$-separatistic and $(\omega,\varkappa)$-regular, then, for every set Σ of sentences of $L_b(\varkappa)$ which is satisfiable in $\mathbf{2}_{\mathcal{G}}$, there are sequences $\langle X_i^+\colon i<\varkappa\rangle$ and $\langle X_i^-\colon i<\varkappa\rangle$ such that*

.1. the sequence $\langle X_i^+/\mathcal{G}\colon i<\varkappa\rangle$ satisfies Σ in $\mathbf{2}_{\mathcal{G}}$,

.2. $G-(X_i^+\cup X_i^-)\in\mathcal{G}$ for $i<\varkappa$, $X_i^-\subseteq G-X_i^+$,

.3. if $I\subseteq\varkappa$ is infinite, then $\bigcap\{X_i^+\colon i\in I\}=\bigcap\{X_i^-\colon i\in I\}=0$,

.4. if $\bigcap\{X_i^+/\mathcal{G}\colon i\in s\}\cap\bigcap\{-(X_i^+/\mathcal{G})\colon i\in t\}=\mathbf{0}$, then $\bigcap\{X_i^+\colon i\in s\}\cap\bigcap\{X_i^-\colon i\in t\}=0$ for $s,t\in S_\omega(\varkappa)$.

Proof. Since $\mathcal{G}$ is $\varkappa$-separatistic, there is a sequence $\langle X_i\colon i<\varkappa\rangle$ such that the sequence $\langle X_i/\mathcal{G}\colon i<\varkappa\rangle$ satisfies Σ and (15) holds. Since $\mathcal{G}$ is $(\omega,\varkappa)$-regular, there is a subfamily $(E_i\colon i<\varkappa)$ of $\mathcal{G}$ such that, for every infinite set $I\subseteq\varkappa$, $\bigcup\{E_i\colon i\in I\}=G$. We put $X_i^+=X_i-E_i$ and $X_i^-=(G-X_i)-E_i$.

Let L be a fixed language of power $\varkappa$. We assume that all sentences of L are enumerated by elements of $\varkappa$. If $i<\varkappa$ then by a_i we denote the ith sentence of L. We say that the sequence $\langle\sigma,a_{i_0},\ldots,a_{i_n}\rangle$ is *acceptable* if σ is a formula of the language of Boolean algebras (L_b) and a_{i_j} is a sentence of L ($j\leqslant n$). If π is a formula of L, then we say that the acceptable sequence $\langle\sigma,a_{i_0},\ldots,a_{i_n}\rangle$ is *connected with* π if $\langle\sigma,a_{i_0},\ldots,a_{i_n}\rangle$

is the sequence defined for π in the proof of the theorem of Feferman and Vaught (Th. 3.1. of [2]). Let us recall that if $\langle \sigma, a_{i_0}, \ldots, a_{i_n}\rangle$ is the sequence connected with π, then for every ideal $\mathfrak{G}$ and every sequence $\langle \mathfrak{A}_i\colon i \in G\rangle$ of relational structures of similarity type L we have

(16) $\mathfrak{P}_{\mathfrak{G}}\langle \mathfrak{A}_j\colon j \in G\rangle \models \pi$ *if and only if* $\mathbf{2}_{\mathfrak{G}} \models \sigma[[a_{i_0}], \ldots, [a_{i_n}]]$,

where $[a_{i_k}] = \{j \in G\colon \mathfrak{A}_j \models a_{i_k}\}/\mathfrak{G}$.

LEMMA 3.3. *Let Σ be a set of formulas of L and $\boldsymbol{K}$ a class of structures of L. With L and $\boldsymbol{K}$ we connect a set $\Pi = \Pi_0 \cup \Pi_1$ of formulas of $L_b(\varkappa)$, where*

$$\Pi_0 = \{\sigma(c_{i_0}, \ldots, c_{i_n})\colon \langle \sigma, a_{i_0}, \ldots, a_{i_n}\rangle \text{ is an acceptable sequence connected with } \pi,\ \pi \in \Sigma\},$$

and

$$\Pi_1 = \{\textstyle\bigcap\{h(i)c_i\colon i \in \mathrm{dom}(h)\} = \mathbf{0}\colon s \in S_\omega(\varkappa),\ h\colon s \to 2,\ \boldsymbol{K} \models \neg\bigwedge\{h(i)a_i\colon i \in \mathrm{dom}(h)\}\}\ (^3).$$

Then, if Σ is finitely satisfiable in $\mathfrak{G}(\boldsymbol{K})$, then Π is finitely satisfiable in $\mathbf{2}_{\mathfrak{G}}$.

Proof. Let Π' be a finite subset of Π and let $\Sigma_0 \subseteq \Sigma$ be a finite set such that if $\sigma(c_{i_0}, \ldots, c_{i_n}) \in \Pi'$, then there is a sentence $\pi \in \Sigma_0$ such that $\langle \sigma, a_{i_0}, \ldots, a_{i_n}\rangle$ is connected with π. Since Σ is finitely satisfiable in $\mathfrak{G}(\boldsymbol{K})$, there is a sequence $\mathcal{A} = \langle \mathfrak{A}_i\colon i \in G\rangle$ such that $\mathfrak{P}_{\mathfrak{G}}(\mathcal{A}) \models \bigwedge \Sigma_0$. If c_i appears in any formula of Π', then we put $C_i = \{k \in G\colon \mathfrak{A}_k \models a_i\}$. Let $\sigma(c_{i_0}, \ldots, c_{i_n}) \in \Pi'$. Then, by the definition of Σ_0, for some $\pi \in \Sigma_0$ $\langle \sigma, a_{i_0}, \ldots, a_{i_n}\rangle$ is connected with π. Hence, since $\mathfrak{P}_{\mathfrak{G}}(\mathcal{A}) \models \pi$, by (16) we obtain

$$\mathbf{2}_{\mathfrak{G}} \models \sigma[C_{i_0}/\mathfrak{G}, \ldots, C_{i_n}/\mathfrak{G}].$$

Moreover, if $\boldsymbol{K} \models \neg\bigwedge\{h(i)a_i\colon i \in \mathrm{dom}(h)\}$, then by the definition of the sequence $\langle C_i\colon i < \varkappa\rangle$ we have $\bigcap\{h(i)C_i\colon i \in \mathrm{dom}(h)\} = 0$; hence the sequence $\langle C_i\colon i < \varkappa\rangle$ satisfies Π'.

THEOREM 3.4. *If $\mathfrak{G}$ is $\varkappa$-separatistic and $\mathbf{2}_{\mathfrak{G}}$ is $\varkappa^+$-universal, then for every compact class $\boldsymbol{K}$ the class $\mathfrak{G}(\boldsymbol{K})$ is compact provided the similarity type of $\boldsymbol{K}$ has power $< \varkappa$.*

Proof. Let Σ be a set of sentences of L which is finitely satisfiable in $\mathfrak{G}(\boldsymbol{K})$. Hence by Lemma 3.3 the set Π defined in Lemma 3.3 is finitely satisfiable in $\mathbf{2}_{\mathfrak{G}}$. Since $\mathbf{2}_{\mathfrak{G}}$ is $\varkappa^+$-universal, Π is satisfiable in $\mathbf{2}_{\mathfrak{G}}$. Since $\mathfrak{G}$ is $\varkappa$-separatistic, there exists a sequence $\langle C_i\colon i < \varkappa\rangle$ of elements subsets of G such that if $c_i = C_i/\mathfrak{G}$, and $\boldsymbol{c} = \langle c_i\colon i < \varkappa\rangle$, then $(\mathbf{2}_{\mathfrak{G}}, \boldsymbol{c})$ is a model

(^{3}) $1c \doteq c$, $0c = -c$, $1a = a$, $0a = \neg a$.

of Π and if $s, t \in S_\omega(\varkappa)$ and $\bigcap\{c_i\colon i \in s\} \cap \bigcap\{-c_i\colon i \in t\} = \mathbf{0}$, then $\bigcap\{C_i\colon i \in s\} \cap \bigcap\{G - C_i\colon i \in t\} = 0$.

Now we are going to define a sequence $\langle \mathfrak{A}_i\colon i \in G\rangle$ such that $\mathfrak{A}_i \in \boldsymbol{K}$ and $\mathfrak{P}_{\mathfrak{G}}(\mathfrak{A}_i\colon i \in G) \models \Sigma$. Let i_0 be a fixed element of G; we shall define $\mathfrak{A}_{i_0}$. Let $\Sigma_{i_0} = \{a_k\colon i_0 \in C_k\}$ and $\Sigma'_{i_0} = \{\neg a_k\colon i_0 \notin C_k\}$. We claim that $\Sigma_{i_0} \cup \Sigma'_{i_0}$ is finitely satisfiable in $\boldsymbol{K}$. In fact, if $s, t \in S_\omega(\varkappa)$ and $\{a_j\colon j \in s\} \subseteq \Sigma_{i_0}$ and $\{\neg a_j\colon j \in t\} \subseteq \Sigma'_{i_0}$, then $i_0 \in \bigcap\{C_k\colon k \in s\} \cap \bigcap\{G - C_k\colon k \in t\}$; hence $\bigwedge\{a_k\colon k \in s\} \wedge \bigwedge\{\neg a_k\colon k \in t\}$ is satisfiable in $\boldsymbol{K}$. Hence, since $\boldsymbol{K}$ is a compact class $\Sigma_{i_0} \cup \Sigma'_{i_0}$ has a model in $\boldsymbol{K}$. We select $\mathfrak{A}_{i_0}$ such that $\mathfrak{A}_{i_0}$ is a model of $\Sigma_{i_0} \cup \Sigma'_{i_0}$.

Of course $\mathfrak{P}_{\mathfrak{G}}\langle \mathfrak{A}_i\colon i \in G\rangle \in \mathfrak{G}(\boldsymbol{K})$. Let $\sigma \in \Sigma$ and let $\langle \pi, a_{i_0}, \ldots, a_{i_n}\rangle$ be an acceptable sequence correlated with σ. By the definition of Π_0 in Lemma 3.3, $\pi(c_{i_0}, \ldots, c_{i_n})$ belongs to Π_0; hence by the definition of the sequence $\langle C_i\colon i < \varkappa\rangle$ we have

(17) $$\mathbf{2}_{\mathfrak{G}} \models \pi[C_{i_0}/\mathfrak{G}, \ldots, C_{i_n}/\mathfrak{G}].$$

On the other hand, if $k \in C_i$, then a_i holds in $\mathfrak{A}_k$ and if $k \notin C_i$, then $\mathfrak{A}_k \models \neg a_i$; hence $C_i/\mathfrak{G} = [a_i]$. This, by (16) and (17), completes the proof that $\mathfrak{P}_{\mathfrak{G}}\langle \mathfrak{A}_i\colon i \in G\rangle$ is a model of Σ.

THEOREM 3.5. *If $\mathbf{2}_{\mathfrak{G}}$ is $\varkappa^+$-universal, $\varkappa$-separatistic and $(\omega, \varkappa)$-regular, then for every class $\boldsymbol{K}$ of relational structures of similarity type of power $\leqslant \varkappa$ the class $\mathfrak{G}(\boldsymbol{K})$ is compact.*

Proof. Let Σ be a set of sentences of L which is finitely satisfiable in $\mathfrak{G}(\boldsymbol{K})$. Hence the set Π defined in Lemma 3.3 is finitely satisfiable. Since $\mathbf{2}_{\mathfrak{G}}$ is $\varkappa^+$-universal, $\varkappa$-separatistic and $(\omega, \varkappa)$-regular by Lemma 3.2, there are sequences $\langle X_i^+\colon i < \varkappa\rangle$, $\langle X_i^-\colon < \varkappa\rangle$ which satisfy the conditions 3.2.1–3.2.4. Let $Y_{i_0} = \{k < \varkappa\colon i_0 \in X_k^+\}$ and $Z_{i_0} = \{k < \varkappa\colon i_0 \in X_k^-\}$. We claim that Y_{i_0} and Z_{i_0} are finite. Of course $i_0 \in \bigcap\{X_k^+\colon k \in Y_{i_0}\}$; hence $\bigcap\{X_k^+\colon k \in Y_{i_0}\} \neq 0$ and consequently by 3.2.3 Y_{i_0} is finite. Moreover, $i_0 \in \bigcap\{X_k^+\colon k \in Y_{i_0}\} \cap \bigcap\{X_k^-\colon k \in Z_{i_0}\}$; hence by 3.2.1, 3.2.4 and the definition of the set Π_1, there is a relational structure $\mathfrak{A}_{i_0} \in \boldsymbol{K}$ such that $\mathfrak{A}_{i_0}$ is a model of $\bigwedge\{a_k\colon k \in Y_{i_0}\} \wedge \bigwedge\{\neg a_k\colon k \in Z_{i_0}\}$. The proof that the reduced product $\mathfrak{P}_{\mathfrak{G}}\langle \mathfrak{A}_i\colon i \in G\rangle$ is a model of Σ is similar to that of the proof of Theorem 3.4. The only difference is in the proof of the fact that $X_i^+/\mathfrak{G} = [a_i]$. We have

(18) $$X_i^+ \subseteq \{k\colon \mathfrak{A}_k \models a_i\} \subseteq -X_i^-.$$

On the other hand, $-(X_i^+ \cup X_i^-) \in \mathfrak{G}$; hence by a simple computation from (18) we obtain $\{k\colon \mathfrak{A}_k \models a_i\} \Delta X_i^+ \in \mathfrak{G}$ (Δ is the symmetric difference symbol).

Now we shall prove that the assumptions of Theorems 3.4 and 3.5 are necessary.

THEOREM 3.6. *If the class $\mathcal{G}(\boldsymbol{K})$ is compact for every compact class whose similarity type is of power $\leqslant \varkappa$, then the Boolean algebra $\mathbf{2}_{\mathcal{G}}$ is $\varkappa^+$-universal and $\varkappa$-separatistic.*

Proof. Let Σ be a complete set of sentences of $L_b(\varkappa)$ such that $\mathrm{Th}(\mathbf{2}_{\mathcal{G}}) \cup \Sigma$ is consistent. To prove that Σ has a model of the form $(\mathbf{2}_{\mathcal{G}}, \boldsymbol{c})$ where $\boldsymbol{c} = \langle C_i/\mathcal{G}\colon i < \varkappa\rangle$ and the sequence $\langle C_i\colon i < \varkappa\rangle$ satisfies (15), we consider the class:

$$\boldsymbol{K}_1 = \{\mathfrak{A}\colon \mathfrak{A} = (\mathbf{2}, \boldsymbol{d}),\ \boldsymbol{d} \in \mathbf{2}^{\varkappa},\ \textit{and if}\ (\bigcap \{h(i)c_i\colon i \in \mathrm{dom}(h)\} = 0) \in \Sigma,\ \textit{then} \bigcap \{h(i)d_i\colon i \in \mathrm{dom}(h)\} = \mathbf{0}\}.$$

The compactness of class $\boldsymbol{K}_1$ is an immediate consequence of the compactness theorem for the propositional calculus. Since $\boldsymbol{K}_1$ is compact, the class $\mathcal{G}(\boldsymbol{K}_1)$ is also compact. Moreover, we can identify $L_b(\varkappa)$ with the language of $\boldsymbol{K}_1$. Hence we can consider Σ as a set of sentences of the language of $\boldsymbol{K}_1$. Since $\mathrm{Th}(\mathbf{2}_{\mathcal{G}}) \cup \Sigma$ is consistent, Σ is finitely satisfiable in $\mathcal{G}(\boldsymbol{K}_1)$. Let $\mathfrak{A}$ be an element of $\mathcal{G}(\boldsymbol{K}_1)$ which is a model of Σ. Then $\mathfrak{A} = \mathfrak{P}_{\mathcal{G}}\langle(\mathbf{2}, \boldsymbol{c}^i)\colon i \in G\rangle$, where, for $i \in G$, $\boldsymbol{c}^i \in \mathbf{2}^{\varkappa}$. We put $C_k = \{i \in G\colon c_k^i = \mathbf{1}\}$ and $\boldsymbol{c} = \langle C_k/\mathcal{G}\colon k < \varkappa\rangle$. Of course $(\mathbf{2}_{\mathcal{G}}, \boldsymbol{c})$ is a model of Σ (now considered as a set of sentences of $L_b(\varkappa)$). Moreover, if $\bigcap \{h(k)(C_k/\mathcal{G})\colon k \in \mathrm{dom}(h)\} = \mathbf{0}$, then by the completeness of Σ the sentence $\bigcap \{h(k)c_k\colon k \in \mathrm{dom}(h)\} = \mathbf{0}$ is an element of Σ, and hence, by the definition of $\boldsymbol{K}_1$, we have

$$\mathfrak{B} \models \bigcap \{h(k)d_k\colon k \in \mathrm{dom}(h)\} = \mathbf{0} \quad \text{for every} \quad \mathfrak{B} \in \boldsymbol{K}_1.$$

Consequently $\bigcap \{h(k)C_k\colon k \in \mathrm{dom}(h)\} = 0$, which completes the proo that the sequence $\langle C_k\colon k < \varkappa\rangle$ has the desired properties.

THEOREM 3.7. *If the class $\mathcal{G}(\boldsymbol{K})$ is compact for every class with similarity type of power $\leqslant \varkappa$, then $\mathcal{G}$ is $(\omega, \varkappa)$-regular.*

Proof. We consider the class

$$\boldsymbol{K}_2 = \{\mathfrak{A}\colon \mathfrak{A} = (\mathbf{2}, c),\ \boldsymbol{c} \in \mathbf{2}^{\varkappa}\ \textit{and}\ \{i \in \varkappa\colon c_i = 0\}\ \textit{is finite}\}.$$

By the assumptions of the theorem the class $\mathcal{G}(\boldsymbol{K}_2)$ is compact. Let $\Sigma = \{c_k = 0\colon k < \varkappa\}$ be the set of sentences of $L_b(\varkappa)$ (which is also the language of $\boldsymbol{K}_2$). Of course every finite subset of Σ has a model in $\mathcal{G}(\boldsymbol{K}_2)$. Hence, by the compactness of $\mathcal{G}(\boldsymbol{K}_2)$, there is in $\mathcal{G}(\boldsymbol{K}_2)$ a model of Σ. Let $\mathfrak{A} = \mathfrak{P}_{\mathcal{G}}\langle(\mathbf{2}, \boldsymbol{c}^i)\colon i \in G\rangle$ be a model of Σ which belongs to $\mathcal{G}(\boldsymbol{K}_2)$. For $i < \varkappa$ we put $C_k = \{i \in G\colon c^i = 1\}$. Since $\mathfrak{A}$ is a model of Σ, C_k is an element of $\mathcal{G}$. We claim that for every infinite $I \subseteq \varkappa$ $\bigcup\{C_k\colon k \in I\} = G$. In fact, let $i \in G$. Then the set $\{k\colon c_k^i = \mathbf{0}\}$ if finite by the definition of $\boldsymbol{K}_2$; hence, for some $k_0 \in I$, $c_{k_0}^i = \mathbf{1}$ and consequently $i \in C_{k_0}$.

COROLLARY 3.8. *Let $\mathcal{G}$ be an ideal. $\mathcal{G}$ is $\varkappa$-separatistic and $\mathbf{2}_{\mathcal{G}}$ is $\varkappa^+$-universal if and only if, for every compact class $\boldsymbol{K}$ of similarity type of power $\leqslant \varkappa$, the class $\mathcal{G}(\boldsymbol{K})$ is compact.*

COROLLARY 3.9. *$\mathcal{G}$ is $\varkappa$-separatistic and $(\omega, \varkappa)$-regular and $\mathbf{2}_{\mathcal{G}}$ is $\varkappa^+$-universal if and only if, for every class $\boldsymbol{K}$ of similarity type of power $\leqslant \varkappa$, the class $\mathcal{G}(\boldsymbol{K})$ is compact.*

Corollary 3.8 is a consequence of Theorem 3.4 and Theorem 3.6. Corollary 3.9 is a consequence of Theorems 3.5, 3.6 and 3.7.

Sometimes it seems to be more natural to consider classes of operations on classes of relational structures instead of a simple operation (see e.g. [6]). The method developed above enables us to give the following characterization of classes of operations which preserve the compactness of classes relational structures.

DEFINITION 3.10. Let $\boldsymbol{C}$ be a family of ideals. We say that $\boldsymbol{C}$ is *$\varkappa$-strongly compact* if the class $B(\boldsymbol{C}) = \{\mathbf{2}_{\mathcal{G}}\colon \mathcal{G} \in \boldsymbol{C}\}$ is compact and for every Boolean algebra $\mathfrak{A}$ in $B(\boldsymbol{C})$ there is a Boolean algebra $\mathfrak{B}$ in $B(\boldsymbol{C})$ such that $\mathfrak{A} \equiv \mathfrak{B}$ and $\mathfrak{B}$ is $\varkappa^+$-universal and $\varkappa$-separatistic.

DEFINITION 3.11. A class $\boldsymbol{C}$ of ideals is *$\varkappa$-powerfull* if, for every element $\mathfrak{A}$ of $B(\boldsymbol{C})$, there is an element $\mathcal{G}$ of $\boldsymbol{C}$ such that $\mathbf{2}_{\mathcal{G}}$ is $\varkappa^+$-universal, $\varkappa$-separatistic and $\mathfrak{A} \equiv \mathbf{2}_{\mathcal{G}}$ and, moreover, $\mathcal{G}$ is $(\omega, \varkappa)$-regular.

THEOREM 3.12. *$\boldsymbol{C}$ is $\varkappa$-strongly compact if and only if, for every compact class $\boldsymbol{K}$ of relational structures of similarity type of power $\leqslant \varkappa$, the class $\boldsymbol{C}(\boldsymbol{K})$ is compact.*

THEOREM 3.13. *$\boldsymbol{C}$ is $\varkappa$-powerfull if and only if, for every class $\boldsymbol{K}$ of relational structures which similarity type is of power $\leqslant \varkappa$, the class $\boldsymbol{C}(\boldsymbol{K})$ is compact.*

Proofs of Theorems 3.13 and 3.12 can be obtained by a modification of the proofs of Corollaries 3.8 and 3.9.

Finally let us give the following corollary, which by a theorem of Keisler [3] implies the main result of paper [7] of Omarov.

COROLLARY 3.14. *$\boldsymbol{C}$ is $\varkappa$-strongly compact for every $\varkappa$ if and only if, for every compact class $\boldsymbol{K}$, the class $\boldsymbol{C}(\boldsymbol{K})$ is compact.*

DEFINITION 3.15. A class $\boldsymbol{C}$ of ideals is *elementary* if the class $B(\boldsymbol{C}) = \{\mathbf{2}_{\mathcal{G}}\colon \mathcal{G} \in \boldsymbol{C}\}$ is an elementary subclass of the class of all Boolean algebras of the form $\mathbf{2}_{\mathcal{G}}$.

THEOREM 3.16. (Omarov [7]). *If $\boldsymbol{C}$ is an elementary class of ideals, then, for every compact class $\boldsymbol{K}$, the class $\boldsymbol{C}(\boldsymbol{K})$ is compact.*

Let us conclude this section by a remark that all the theorems of this section can be extended to limit reduced powers and some of the results can be extended to the case of strongly compact classes or relational structures (in the sense of [7]).

4. In this section we try to find properties of an ideal $\mathcal{G}$ which imply that the Boolean algebra $\mathbf{2}_{\mathcal{G}}$ is $\varkappa$-separatistic.

THEOREM 4.1. *If $\mathcal{G}$ is $\varkappa^+$-good, then, for every sequence $\langle b_i: i<\varkappa\rangle$ of elements of $\mathbf{2}_{\mathcal{G}}$, there is a sequence $\langle B_i: i<\varkappa\rangle$ of subsets of G such that:*

1. *$b_i = B_i/\mathcal{G}$ for $i<\varkappa$ and*
2. *for every $s \in S_\omega(\varkappa)$ and every h: $s \to 2$ we have*

$$\bigcap\{h(i)b_i:\ i\in s\} = \mathbf{0}\ \textit{implies}\ \bigcap\{h(i)B_i:\ i\in s\} = 0\,.$$

Proof. Let $\langle B_i': i>\varkappa\rangle$ be a sequence of subsets of G such that for $i<\varkappa$ $B_i'/\mathcal{G} = b_i$. Let for $s \in S_\omega(\varkappa)$

$$f(s) = \bigcup\{\bigcap\{h(i)B_j':\ i\in t\}:\ h\in 2^t,\ t\subseteq s,\ \bigcap\{h(i)b_i:\ i\in t\} = \mathbf{0}\}\,.$$

Clearly f is a monotonic function and, for each $s\in S_\omega(\varkappa)$, $f(s)\in\mathcal{G}$. By the $\varkappa$-goodness of $\mathcal{G}$ there is a function g: $S_\omega(\varkappa)\to\mathcal{G}$ such that, for every $s, t\in S_\omega(\varkappa)$, $g(s)\supseteq f(s)$ and $g(s\cup t) = g(s)\cup g(t)$. We shall define a sequence B_i such that

$$B_i \Delta B_i' \subseteq g(\{i\}) \tag{19}$$

and 4.1.2 holds. (19) and 4.1.2 are equivalent to conditions (20) and (21) below:

$$\textit{if } m\notin g\{i\} \textit{ then } m\notin B_i\Delta B_i' \textit{ for } m\in G\,, \tag{20}$$

$$\textit{if } m\in\bigcap\{h(i)B_i:\ i\in\mathrm{dom}(h)\},\ \textit{then}\ \bigcap\{h(i)b_i:\ i\in\mathrm{dom}(h)\}\neq\mathbf{0}\ \textit{for}\ m\in G\,. \tag{21}$$

It suffices to show that for a given particular m_0 we can satisfy conditions (20) and (21).

Let $X = 2^\varkappa$ be the topological space with the product topology. With every sequence $\langle E_i: i<\varkappa\rangle$ of subsets of G we connect an element x of X such that $x(i) = 1$ if and only if $m_0\in E_i$. Conversely, every element of X can be treated as a code of a set of statements $\{m_0\in E_i: i<\varkappa\}$. Let y be the element of X which is connected with the sequence $\langle B_i': i<\varkappa\rangle$. To satisfy (20) and (21) it suffices to find an element x of X which belongs to the intersection

$$\bigcap\{Y_j:\ j<\varkappa\}\cap\bigcap\{Y_{s,t}:\ s,t\in S_\omega(\varkappa)\} \tag{22}$$

where

$$Y_j = \{x:\ m\notin g(\{j\})\to y(j) = x(j)\}$$

and

$$Y_{s,t} = \{x:\ \Pi\{x_i:\ i\in s\}\cdot\prod\{(1-x(i)):\ i\in t\} = 1 \to\bigcap\{b_i:\ i\in s\}\cap\bigcap\{-b_i:\ i\in t\}\neq\mathbf{0}\}\,.$$

For every $j\in\varkappa$ and every $s, t\in S_\omega(\varkappa)$, the sets Y_j and $Y_{s,t}$ are closed subsets of X; hence, since X is a compact space, it suffices to check that the intersection

$$\bigcap\{Y_j:\ i\in I_0\}\cap\bigcap\{Y_{s,t}:\ \langle s,t\rangle\in I_1\subseteq(S_\omega(\varkappa))^2\} \tag{23}$$

is non-empty if I_0 and I_1 are finite.

Let

$$u = I_0\cup\{i\in\varkappa:\ i\in s\ \textit{or}\ i\in t\ \textit{and}\ \langle s,t\rangle\in I_1\}\,,$$

$$s_0 = \{k\in u:\ k\in I_0,\ y(k) = 1\ \textit{and}\ m\notin g(\{k\})\}$$

and let

$$t_0 = \{k\in u:\ k\in I_0,\ y(k) = 0\ \textit{and}\ m\notin g(\{k\})\}\,.$$

Since g is an additive function, we have

$$m_0\notin g(s_0\cup t_0)\,. \tag{24}$$

Moreover, for $k\in s_0$, $y(k) = 1$ and for $k\in t_0$ $y(k) = 0$; hence

$$\prod\{y(k):\ k\in s_0\}\cdot\prod\{(1-y(k)):\ k\in t_0\} = 1,$$

and consequently

$$m_0\in\bigcap\{B_k':\ k\in s_0\}\cap\bigcap\{-B_k':\ k\in t_0\}\,. \tag{25}$$

Since $g\supseteq f$, by (24), we obtain $m_0\notin f(s_0\cup t_0)$; hence, by the definition of f and (25), we have

$$\bigcap\{b_k:\ k\in s_0\}\cap\bigcap\{-b_k:\ k\in t_0\}\neq\mathbf{0}\,. \tag{26}$$

Since $s_0\cup t_0\subseteq u$ and $s_0\cap t_0 = 0$, by (26) there are sets $s_1, t_1\in S_\omega(\varkappa)$ such that $s_0\subseteq s_1$, $t_0\subseteq t_1$, $t_1\cup s_1 = u$ and

$$\bigcap\{b_k:\ k\in s_1\}\cap\bigcap\{-b_k:\ k\in t_1\}\neq\mathbf{0}\,. \tag{27}$$

For $k\in u$ we put

$$x_u(k) = \begin{cases}1 & \textit{if}\quad k\in s_1\,,\\ 0 & \textit{if}\quad k\in t_1\end{cases}$$

and if $k\notin u$, then $x_u(k)$ is arbitrary.

We claim that x_u is an element of the intersection (23). In fact, for every $k\in I_0$ we have $x_u\in Y_k$ because $s_0\subseteq s_1$ and $t_0\subseteq t_1$. Moreover, if for $\langle s,t\rangle\in I$ we have

$$\prod\{x_n(i):\ i\in s\}\cdot\prod\{(1-x_n(i)):\ i\in t\} = 1\,,$$

then $s\subseteq s_1$ and $t\subseteq t_1$ and consequently by (27)

$$\bigcap\{b_k:\ k\in s\}\cap\bigcap\{-b_k:\ k\in t\}\neq\mathbf{0}\,.$$

COROLLARY 4.2. *If $\mathcal{G}$ is $\varkappa^+$-good and $\mathbf{2}_{\mathcal{G}}$ is $\varkappa^+$-universal, then, for every compact class $\boldsymbol{K}$ whose similarity type is of power $\leqslant\varkappa$, the class $\mathcal{G}(\boldsymbol{K})$ is compact.*

COROLLARY 4.3. *For every complete theory of Boolean algebras and every cardinal number $\varkappa$, there is an ideal $\mathcal{G}$ such that, for every compact class $\boldsymbol{K}$ of similarity type of power $\leqslant\varkappa$, the class $\mathcal{G}(\boldsymbol{K})$ is compact.*

Corollary 4.3 is a consequence of the existence of good ideals (see eg. [4]).

The following fact was stated in [4].

LEMMA 4.4. *If $\mathcal{G}$ is $\varkappa^+$-good, then it is ω-incomplete if and only if it is $(\omega, \varkappa)$-regular.*

COROLLARY 4.5. *If $\mathcal{G}$ is $\varkappa^+$-good and ω-incomplete and $\mathbf{2}_{\mathcal{G}}$ is $\varkappa^+$-universal, then, for every class $\boldsymbol{K}$ whose similarity type is of power $\leqslant \varkappa$, the class $\mathcal{G}(\boldsymbol{K})$ is compact.*

Finally, let us remark that the assumption of $\varkappa^+$-goodness in Theorem 4.1 is not necessary. The proof of it is easy. Also there is a $\varkappa$-separatistic ideal $\mathcal{G}$ such that the Boolean algebra is $\varkappa^+$-universal and $\mathcal{G}$ is not $(\omega, \varkappa)$-regular.

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Concerning closed quasi-orders on hereditarily unicoherent continua

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Abstract. The purpose of this paper is to define and study a class of hereditarily unicoherent Hausdorff continua, called *nearly smooth*, which admit a closed quasi-order that is closely related to the weak cutpoint order. The major result is that for each point p of a nearly smooth continuum M there exists a decomposition $\mathfrak{D}$ of M such that (i) $\mathfrak{D}$ is upper semicontinuous, (ii) the elements of $\mathfrak{D}$ are continua, (iii) the decomposition space of $\mathfrak{D}$ is a generalized tree which is smooth at the element of $\mathfrak{D}$ containing p, and (iv) $\mathfrak{D}$ is the finest decomposition satisfying (i), (ii), and (iii). In addition, characterizations of nearly smooth continua, smooth continua, and generalized trees are obtained in terms of closed quasi-orders and the set-valued function T. A preliminary result of independent interest is that every semi-aposyndetic, hereditarily unicoherent continuum is a dendroid.

The notion of weak cutpoint order has been useful in studying the structure of arcwise connected, hereditarily unicoherent continua. For example, Koch and Krule [10] have shown that a hereditarily unicoherent continuum is a generalized tree [12] if and only if there exists a point p such that the weak cutpoint order with respect to p is a closed partial order. Charatonik and Eberhart [3] have applied the notion of weak cutpoint order to obtain characterizations of smooth dendroids and to study their mapping properties.

It is the purpose of this paper to study hereditarily unicoherent continua admitting a closed quasi-order which is closely related to the weak cutpoint order. One should observe that for non-arcwise connected, hereditarily unicoherent continua the weak cutpoint order is a quasi-order and not a partial order.

It is shown that a hereditarily unicoherent continuum is smooth at a point p [6] if and only if the weak cutpoint order with respect to p is closed. This result motivates the definition of a *nearly smooth* continuum as a hereditarily unicoherent continuum admitting a closed quasi-order which "approximates" the weak cutpoint order. Characterizations of nearly smooth continua, smooth continua, and generalized trees are obtained in terms of closed quasi-orders and the set-valued function T [4].