Strong essential cluster sets

by

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Abstract. Let H be the upper half-space of R^2, $l^-(x, A)$ the upper orthogonal linear metric density of A at point $x \in \partial H$. Let $D^-(x, A)$ be the upper strong metric density of A at point $x \in \partial H$. For a measurable function $f\colon H \to R$ the linear essential cluster set of f at $x \in \partial H$ is the set of λ such that for every $\varepsilon > 0$, $l^-(x, [y\colon \lambda - \varepsilon < f(y) < \lambda + \varepsilon]) > 0$. The strong essential cluster set of f at $x \in \partial H$ is defined similarly. From two lemmas about the structure of measurable sets, the following theorem can be established. For a continuous function $f\colon H \to R$, the strong essential cluster set of f contains the linear essential cluster set of f at every point $x \in \partial H$. Equality actually holds between these sets except at a set of $x \in \partial H$ of the first category. For a measurable function, equality holds except for a set of $x \in \partial H$ of measure zero. Examples are given to show that this is the best possible result.

In obtaining an analogue for essential cluster sets of a theorem of Collingwood [1] on cluster sets, Goffman and Sledd [2] show that if f is a continuous function in the upper half plane, H, the ordinary essential cluster set of f is contained in the vertical essential cluster set of f, at each point in ∂H except at a set which is both of the first category and measure zero. If f is measurable the exceptional set is of measure zero, but not necessarily of the first category. In the same paper it is shown that the "contains in" relation cannot be strengthened to equality.

Our purpose is to show that this containment is replaced by equality if strong essential cluster sets are considered instead of ordinary essential cluster sets.

Our result follows immediately from two lemmas about the metric structure of measurable sets. One of these lemmas establishes a relation between strong and linear metric density. This relation while, perhaps, not unexpected, appears not to have been discussed.

Let $|\cdot|_i$ denote i-dimensional Lebesgue measure, $i = 1, 2$ and let $[I]$ be the collection of closed rectangles of the form $[a, b] \times [0, k]$, $a < 0 < b$, a, b and k rational. For $I \in [I]$ let $I(x_0)$ denote the closed rectangle obtained

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by mapping (x, y) into $(x+x_0, y)$. We define the strong upper density of a measurable set $E \subset H$ at $x \in \partial H$ as

$$D_s^-(E, x) = \lim_{n\to\infty}\left[\sup_{\delta(I)<1/n}\left\{\frac{|I(x)\cap E|_2}{|I(x)|_2} : I \in [I]\right\}\right],$$

where $\delta(I)$ is the diameter of I; and the upper metric density of E at $x \in \partial H$ in the vertical direction as

$$d^-(E, x) = \lim_{n\to\infty}\left[\sup_{h<1/n}\left\{\frac{|l(x, h)\cap E|_1}{h} : h \text{ rational}\right\}\right]$$

where $l(x, h)$ denotes the vertical open line segment in H of length h, and end point x. $D_s^-(E, x)$ and $d^-(E, x)$ are measurable functions of x. We need one more definition.

For f measurable, a real number y is in the strong essential cluster set, $C_s(f, x)$, of f at x if, for each $\varepsilon > 0$, the set $f^{-1}\{(y-\varepsilon, y+\varepsilon)\}$ has positive strong upper density at x. The essential vertical cluster set, $c(f, x)$, is defined similarly.

LEMMA 1. *If $E \subset H$ is open, then $D_s^-(E, x) \geqslant d^-(E, x)$ for every x and $A = \{x: D_s^-(E, x) > d^-(E, x)\}$ is of the first category.*

Proof. Let x be fixed, $d^-(E, x) = a$. Then there is a sequence of rational numbers h_n decreasing to 0 such that

$$\frac{|l(x, h_n)\cap E|_1}{h_n} > a - \frac{1}{n}.$$

Let n be fixed and let $h_n = h$. Then, since E is open, $l(x, h)\cap E$, considered as a subset of R^1, is open. So

$$l(x, h)\cap E = \bigcup_{m=1}^{\infty}(a_m, b_m)$$

where the (a_m, b_m) are the components of the open set, and

$$\sum_{m=1}^{\infty}(b_m - a_m) > (a-1/n)h.$$

We pick a finite collection of closed intervals $[c_m, d_m]$, $m = 1, \ldots, M$, such that $[c_m, d_m] \subset (a_m, b_m)$, $m = 1, \ldots, M$ and

$$\sum_{m=1}^{M}(d_m - c_m) > (a-1/n)h.$$

Since $[c_m, d_m] \subset l(x, h)\cap E$, there are closed rectangles $J_m \subset E$ such that

$$J_m = [x-\varepsilon_m, x+\varepsilon_m]\times[c_m, d_m]$$

with $\varepsilon_m > 0$, rational. Let $\varepsilon = \min[\varepsilon_m, m = 1, \ldots, M]$. Then for $I = [-\varepsilon, \varepsilon]\times[0, h]$ we have $I \in [I]$, and

$$|I(x)\cap E|_2 \geqslant \sum_{m=1}^{M} 2\varepsilon(d_m - c_m) \geqslant 2\varepsilon(a-1/n)h = (a-1/n)|I|_2.$$

It follows that $D_s^-(E, x) \geqslant a$, and we have $D_s^-(E, x) \geqslant d^-(E, x)$, for every x. The proof that A is of the first category follows the same outline as [2], p. 297 and is included here for completeness. Let:

$$A_{\alpha\beta n} = \{x: D_s^-(E, x) \geqslant \beta > \alpha \geqslant \frac{|l(x, h)\cap E|_1}{h} \text{ for all } h < 1/n,\ h \text{ rational}\},$$

for each natural number n and pair of rational numbers α, β, $\alpha < \beta$. Suppose $A_{\alpha\beta n}$ is dense in an open interval L. Then, since E is open, for every $x \in L$ and $h < 1/n$,

$$|l(x, h)\cap E| \leqslant \alpha h.$$

Let $x_0 \in A_{\alpha\beta n}\cap L$. Let $I \in [I]$ be such that $I(x_0) = J\times[0, h]$ where $h < 1/n$ and $J \subset L$. Then

$$|I(x_0)\cap E|_2 = \int_J |l(x, h)\cap E|_1 dx \leqslant \alpha h\cdot|J|_1 = \alpha\cdot|I|_2.$$

Therefore $D_s^-(E, x_0) \leqslant \alpha$. This contradiction shows that $A_{\alpha\beta n}$ is nowhere dense, so that A is of first category.

LEMMA 2. *If $E \subset H$ is measurable, then $D_s^-(E, x) = d^-(E, x)$ for a.e. x.*

Proof. Let A and $A_{\alpha\beta n}$ be as in Lemma 1; these sets are measurable. Suppose some $A_{\alpha\beta n}$ has positive measure. Then we select any point x_0 of $A_{\alpha\beta n}$ such that the metric density of $A_{\alpha\beta n}$ at x_0 is 1. Let $\varepsilon = \frac{1}{3}(\beta-\alpha)$. There is an $\eta > 0$ for which

$$|[x_0+c, x_0+d]\cap A_{\alpha\beta n}|_1 > (1-\varepsilon)|[x_0+c, x_0+d]|_1$$

for any interval, $[x_0+c, x_0+d]$, with $d > 0 > c$ and $d-c < \eta$. This implies that for any $I \in [I]$ satisfying $I = [c, d]\times[0, h]$ with $d-c < \eta$ and $\delta(I) < 1/n$ we have, letting $J = [x_0+c, x_0+d]$,

$$\begin{aligned}|I(x_0)\cap E|_2 &= \int_J |l(x, h)\cap E|_1 dx\\ &= \int_{J\cap A_{\alpha\beta n}} |l(x, h)\cap E|_1 dx + \int_{J-A_{\alpha\beta n}} |l(x, h)\cap E|_1 dx\\ &\leqslant \alpha h|J\cap A_{\alpha\beta n}|_1 + h\varepsilon|J|_1\\ &\leqslant (\alpha+\varepsilon)|I(x_0)|_2.\end{aligned}$$

From the definition of $D_s^-(E, x_0)$ we have

$$D_s^-(E, x_0) \leqslant \alpha + \frac{\beta - \alpha}{3} < \beta .$$

This contradiction shows that A has measure zero.

We also let

$$B = \{x\colon d^-(E, x) > D_s^-(E, x)\}$$

and

$$B_{\alpha\beta n} = \left\{x\colon \frac{|I(x) \cap E|_2}{|I(x)|_2} \leqslant \alpha < \beta \leqslant d^-(E, x) \text{ for all } I \subset [I],\ \delta(I) < \frac{1}{n}\right\}.$$

B and $B_{\alpha\beta n}$ are measurable. To complete the proof of Lemma 2 we need only show that $B_{\alpha\beta n}$ has measure zero for each α, β, n. To accomplish this, we introduce the auxiliary sets W_h for h rational and positive.

$$W_h = \left\{x\colon \lim_{\substack{r\to x+\\ s\to x-}} \frac{1}{(r-s)} \int_s^r \frac{|l(t, h) \cap E|_1}{h}\, dt = \frac{|l(x, h) \cap E|_1}{h}\right\}.$$

That is, W_h is the set of points where the indefinite integral of $\dfrac{|l(x, h) \cap E|_1}{h}$ is differentiable. For h fixed, W_h is measurable. Since $\dfrac{|l(x, h) \cap E|_1}{h}$ is a bounded measurable function, the measure of the complement of W_h is zero. Setting $W = \bigcap_{\substack{h \text{ rational}\\ h>0}} W_h$ we have

$$|W \cap B_{\alpha\beta n}|_1 = |B_{\alpha\beta n}|_1, \quad \text{for all } \alpha,\ \beta,\ n.$$

We show that $W \cap B_{\alpha\beta n} = \emptyset$. If $x \in B_{\alpha\beta n}$, then

$$\sup_{\substack{I\in[I]\\ \delta(I)<1/n}} \left[\frac{|I(x) \cap E|_2}{|I(x)|_2}\right] \leqslant \alpha .$$

Then, for $0 < h < 1/n$, h rational,

$$\overline{\lim_{\substack{r\to x+\\ s\to x-}}} \frac{1}{(r-s)} \int_s^r \frac{|l(t, h) \cap E|_2}{h}\, dt \leqslant \alpha .$$

Accordingly, if x belongs to W, then

$$\frac{|l(x, h) \cap E|_1}{h} \leqslant \alpha ,$$

for every $h < 1/n$, h rational. Hence $d^-(E, x) \leqslant \alpha$ for $x \in B_{\alpha\beta n} \cap W$. This contradiction shows that $B_{\alpha\beta n} \cap W = \emptyset$, and so $|B_{\alpha\beta n}|_1 = 0$.

THEOREM. *If $f\colon H \to R$ is continuous, then $C_s(f, x) \supset c(f, x)$, for every x, and $C_s(f, x) = c(f, x)$ except for a set of first category and measure zero. If f is measurable, then $C_s(f, x) = c(f, x)$ except for a set of measure zero.*

The proof follows immediately by applying the definitions of $C_s(f, x)$ and $c(f, x)$ and the lemmas to the inverse images of the rational open intervals. Further we show that this is the best possible result.

EXAMPLE 1. Let P be the Cantor ternary set and let U be the complement of P relative to $[0, 1]$. We denote U as the countable union of disjoint open intervals, (a_n, b_n), $n = 1, 2, \ldots$ For $0 < \alpha \leqslant 1$ we let

$$E_{\alpha n} = \{(x, y)\colon a_n < x < b_n,\ y = \alpha(x - a_n)(b_n - x)\}$$

and

$$E_\alpha = \bigcup_{n=1}^{\infty} E_{\alpha n} .$$

We define a continuous function f as follows:

$$f(x, y) = \begin{cases} \alpha & \text{for } (x, y) \in E_\alpha , \\ 1 & \text{elsewhere} . \end{cases}$$

For $x \in P$ we have that $c(f, x) = \{1\}$, but we show that $0 \in C_s(f, x)$ for $x \in P$. This will establish that the exceptional set of the first part of the theorem can be uncountable. Let $0 < \varepsilon < 1$. Let $x_0 \in P$. We need only show that if $G_\varepsilon = \{(x, y)\colon f(x, y) < \varepsilon\}$, then $D_s^-(G_\varepsilon, x_0) > 0$. If $x_0 = a_n$ or b_n for some n, then $D_s^-(G_\varepsilon, x_0) = 1$ clearly. We therefore assume $x_0 \neq a_n$ or b_n for any n. For any $\delta > 0$ we can find an interval $J = (x_0 + c, x_0 + d)$ such that $c < 0$, $d > 0$, $d - c = r < \delta$, d, c rational, and $J \cap U = \bigcup_{k=1}^{\infty} (a_{n_k}, b_{n_k})$. Since $|P|_1 = 0$ we have

$$\sum_{k=1}^{\infty} |(a_{n_k}, b_{n_k})| = r .$$

We pick $K > 0$ so that

$$\sum_{k=1}^{K} |(a_{n_k}, b_{n_k})| > \tfrac{1}{2} r .$$

Consider $1 \leqslant k \leqslant K$ fixed. For $a_{n_k} < x < b_{n_k}$,

$$G_\varepsilon = \{(x, y)\colon y < \varepsilon(x - a_{n_k})(b_{n_k} - x)\} ,$$

and there is a $y_k > 0$ such that y_k rational and for $0 < y < y_k$ we have

$$|[a_{n_k}, b_{n_k}] \times [0, y] \cap G_\varepsilon|_2 > \tfrac{1}{2} y (b_{n_k} - a_{n_k}) .$$

Let $y_0 = \min\{y_1, y_2, \ldots, y_k\}$. Then $I = [c, d] \times [0, y_0] \in [I]$ and for $I(x_0)$ we have

$$|I(x_0) \cap G_\varepsilon|_2 = \sum_{k=1}^{\infty} |[a_{n_k}, b_{n_k}] \times [0, y_0] \cap G_\varepsilon|_2$$
$$> \sum_{k=1}^{K} |[a_{n_k}, b_{n_k}] \times [0, y_0] \cap G_\varepsilon|_2$$
$$> \tfrac{1}{2} y_0 \sum_{k=1}^{K} |(a_{n_k}, b_{n_k})|$$
$$> \tfrac{1}{4} y_0 r$$
$$= \tfrac{1}{4} |I(x_0)|_2 .$$

Hence it follows that $D_s^-(G_\varepsilon, x_0) \geqslant \frac{1}{4} > 0$.

Example 2. In [2], pp. 299–300, there is an example of a continuous function g such that if θ is any fixed direction other than $\frac{1}{2}\pi$, and $c_\theta(f, x)$ is the directional essential cluster set in direction θ, then for every x, $c(f, x) \setminus c_\theta(f, x) \neq \emptyset$. By our theorem it follows that the strong essential cluster set can differ from the directional essential cluster set at every x for any direction other than $\frac{1}{2}\pi$.

Example 3. Let $E \subset R^1$ be any set of measure zero and f the characteristic function of $E \times (0, \infty)$. $C_s(f, x) = \{0\}$ for every x, but $c(f, x) = \{1\}$ for every $x \in E$. So the exceptional set of the second part of the theorem can be any set of measure zero.

Example 4. In [3], Sierpiński constructs an example of a non-measurable set S with the property that every line in the plane contains at most two points of S but for every measurable set E, $|S \cap E|_2 = |E|_2$. The characteristic function of S provides an example of a non-measurable function f for which $1 \in C(f, x)$ for every x, but $c(f, x) = \{0\}$ for every x.

The above examples indicate the possible differences between strong and linear metric density.

References

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On countably universal Boolean algebras and compact classes of models

by

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Abstract. In the first part of the paper we give a characterization of the set $\boldsymbol{T}$ of complete theories of Boolean algebras which has the following properties: 1° For every set I and every filter $\mathcal{F}$ of subsets of I the Boolean algebra $2^I/F$ is ω_1-universal provided its theory belongs to $\boldsymbol{T}$. 2° For every complete theory $T \notin \boldsymbol{T}$ there is a set I and a filter $\mathcal{F}$ such that $\mathrm{Th}(2^I/\mathcal{F}) = T$ and $2^I/\mathcal{F}$ is not ω_1-universal.

The second part of the paper contains a characterization of the class $\boldsymbol{C}_\varkappa$ of filters such that for every compact class $\boldsymbol{K}$ of similarity type of power $\leqslant \varkappa$ and for every $\mathcal{F} \in \boldsymbol{C}_\varkappa$ the class $\mathcal{F}(\boldsymbol{K})$ of all $\mathcal{F}$-reduced products of elements of $\boldsymbol{K}$ is compact. Let $\boldsymbol{F}$ be a class of filters and let $\boldsymbol{K}$ be a class of relational structures. By $\boldsymbol{F}(\boldsymbol{K})$ we denote the class $\bigcup \{\mathcal{F}(\boldsymbol{K}) \colon \mathcal{F} \in \boldsymbol{F}\}$. As a corollary to the result of the second part we give a characterization of classes $\boldsymbol{F}$ of filters such that for every compact class $\boldsymbol{K}$ the class $\boldsymbol{F}(\boldsymbol{K})$ is compact.

The present paper is a continuation of [10]. In Section 1 we give the necessary background for Sections 2 and 3. In Section 2 we prove that for some class $\boldsymbol{T}$ of complete theories of Boolean algebras if $\mathrm{Th}(\boldsymbol{2}^I_{\mathcal{G}}) \in \boldsymbol{T}$ then $\boldsymbol{2}^I_{\mathcal{G}}$ is countably universal. Moreover, for every $T \notin \boldsymbol{T}$ we give an example of an ideal $\mathcal{G}$ such that $\mathrm{Th}(\boldsymbol{2}^I_{\mathcal{G}}) = T$ and $\boldsymbol{2}^I_{\mathcal{G}}$ is not countably universal. Section 3 contains a characterization of the class $\boldsymbol{C}_\mathfrak{m}$ of ideals such that, for every compact class $\boldsymbol{K}$ whose similarity type is of power $\leqslant \mathfrak{m}$ and every $\mathcal{G} \in \boldsymbol{C}$, the class $\mathcal{G}(\boldsymbol{K})$ of all $\mathcal{G}$-reduced products of elements of $\boldsymbol{K}$ is compact. If $\boldsymbol{C}$ is a class of ideals and $\boldsymbol{K}$ a class of relational structures, then $\boldsymbol{C}(\boldsymbol{K})$ is the class of relational structures such that $\mathfrak{A} \in \boldsymbol{C}(\boldsymbol{K})$ if and only if for some $\mathcal{G} \in \boldsymbol{C}$ and for some sequence $\langle \mathfrak{A}_i \colon i \in I \rangle$ of elements of $\boldsymbol{K}$, $\mathfrak{A} = \mathfrak{P}_{\mathcal{G}} \langle \mathfrak{A}_i \colon i \in I \rangle$ ($\mathfrak{P}_{\mathcal{G}}$ denotes the operation of a $\mathcal{G}$-reduced product). As a corollary to the results of Section 3, we give a characterization of classes $\boldsymbol{C}$ such that for every compact class $\boldsymbol{K}$ the class $\boldsymbol{C}(\boldsymbol{K})$ is compact. Finally, in Section 4, we give some results concerning separatistic Boolean algebras of the form $\boldsymbol{2}_{\mathcal{G}}$ (see Definition 3.1).

Investigations of operations which preserve the compactness of classes of relational structures were started by M. Makkai, who proved that the class of all direct products of a compact class is compact (see [6]). Further results were obtained by S. R. Kogalovskiĭ [5] and A. I. Omarov