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Some mapping characterizations of unicoherence

by

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Abstract. In this paper we characterize unicoherence in terms of certain real-valued mappings. The following theorems are typical of the results obtained: (1) Let X be a separable, locally connected, connected, perfectly normal space. Then X is unicoherent if and only if for every pair of disjoint non-empty closed sets A and B of X there exists a mapping f of X onto $I=[0,1]$ such that $0\in f(A)$, $1\in f(B)$ and $I\setminus f(A\cup B)$ contains a dense subset D of I such that for every $d\in D$, $f^{-1}(d)$ is connected. (2) Let X be a separable, locally connected, connected, compact normal space. Then X is unicoherent if and only if for every pair of disjoint non-empty continua A and B of X there exists a monotone mapping f of X onto I such that $f(A)=0$ and $f(B)=1$.

The concept of non-alternating mappings was introduced by G. T. Whyburn in [8] and in [9] he showed that if M is a locally connected, compact connected metric space and J is any arc in M, there exists a non-alternating retraction $r\colon M\to J$ which, when M was unicoherent, was monotone. His proofs depended heavily upon cyclic element theory for compact locally connected continua. In [1], K. Borsuk characterized unicoherence for compact, locally connected metric continua in terms of mappings into the circle. More recently, K. Kuratowski proved that when X is a compact and locally connected space and Y is a metric space, $\mathcal{N}$, the set of all non-alternating mappings of X onto Y, is a G_δ-set in the space of all continuous maps of X into Y.

In this paper we characterize unicoherence for separable, perfectly normal, locally connected, connected spaces in terms of non-alternating mappings onto $[0,1]$.

Notation and terminology. Throughout this paper let X denote a connected, locally connected normal space. By a continuum we mean a closed and connected set and a region is an open connected set. By a mapping we will always mean a continuous function. We will use I to denote $[0,1]$ and a surjection f of X onto a space Y will be denoted by $f\colon X\Rightarrow Y$. A perfectly normal space is a normal space in which every closed subset is a G_δ-set.

Definitions. We say that X is *unicoherent* provided whenever $X=H\cup K$, where H and K are continua, $H\cap K$ is a continuum.

We say that a mapping $f: X \Rightarrow Y$ is *non-alternating* provided that whenever $y \in Y$ and $X \setminus f^{-1}(y) = A_1 \cup A_2$ is a separation, $f(A_1) \cap f(A_2) = \emptyset$. We use $\mathcal{C}(f)$ to denote $\{y \in Y: f^{-1}(y) \text{ is connected}\}$ and say that $f: X \Rightarrow Y$ is *monotone* (respectively, *d-monotone*) provided that $\mathcal{C}(f) = Y$ (respectively, $\overline{\mathcal{C}(f)} = Y$).

A mapping $f: X \to Y$ is said to be interior at $y \in Y$ provided that whenever U is an open subset of X that meets $f^{-1}(y)$, y is interior to $f(U)$. For a mapping $f: X \Rightarrow I$ we use $\mathcal{J}(f)$ to denote $\{y \in (0,1): f \text{ is interior at } y\}$.

Lemma 1. *A mapping $f: X \Rightarrow I$ is non-alternating if and only if for every $y \in (0,1)$, $X \setminus f^{-1}(y)$ has exactly two components.*

Proof of the necessity. Suppose that P, Q and R are components of $X \setminus f^{-1}(y)$. Then for some pair, say P and Q, $f(P) \cap f(Q) \neq \emptyset$. But then $X \setminus f^{-1}(y) = P \cup H$, where $H = X \setminus (f^{-1}(y) \cup P)$, is a separation with $f(P) \cap f(H) \neq \emptyset$. Of course this implies that $X \setminus f^{-1}(y)$ has exactly two components.

The sufficiency. For any point $y \in (0,1)$, let A_y (respectively, B_y) denote the component of $X \setminus f^{-1}(y)$ that maps onto $[0, y)$ (respectively, $(y, 1]$). Then $X \setminus f^{-1}(1) = \bigcup A_y$, $y \in (0,1)$ and $X \setminus f^{-1}(0) = \bigcup B_y$, $y \in (0,1)$ are connected. Now suppose that $X \setminus f^{-1}(x) = H \cup K$ is a separation. By the above $x \in (0,1)$ and by our hypothesis H and K are connected and so $f(H) \cap f(K)$ must be empty.

Lemma 2. *If $f: X \Rightarrow I$ is d-monotone, it is also non-alternating.*

Proof. Let $y \in (0,1)$ and suppose that P_1, P_2 and P_3 are components of $X \setminus f^{-1}(y)$. Now each of the sets P_i, $i = 1, 2, 3$, has limit points in $f^{-1}(y)$ and so for some pair, say P_1 and P_2, we have that $\mathrm{int}\big(f(P_1) \cap f(P_2)\big) \neq \emptyset$. Let $c \in \mathcal{C}(f) \cap f(P_1) \cap f(P_2)$. Now $f^{-1}(c)$ is connected and so it must lie entirely in any component of $X \setminus f^{-1}(y)$ that it intersects. But $f^{-1}(c) \cap P_1 \neq \emptyset \neq P_2 \cap f^{-1}(c)$ and this contradiction implies that $X \setminus f^{-1}(y)$ has exactly two components. Then by Lemma 1, f is non-alternating.

Lemma 3. *Let A and B be disjoint closed sets in X, let C be a component of A and let D be a component of B. There exist disjoint regions H and K of X such that $C \subset H$, $D \subset K$, $\mathrm{Fr}\,H = \mathrm{Fr}\,K$ and $\mathrm{Fr}\,H$ misses $A \cup B$.*

Proof. Let W be any open set containing A such that $\overline{W} \cap B = \emptyset$ and let P be the component of $X \setminus \mathrm{Fr}\,W$ that contains C. Let K be the component of $X \setminus \overline{P}$ that contains D and let H be the component of $X \setminus \overline{K}$ that contains C. Then since X is locally connected, $\mathrm{Fr}\,H = \mathrm{Fr}\,K$ and since $\mathrm{Fr}\,H \subset \mathrm{Fr}\,W$, $\mathrm{Fr}\,H$ misses $A \cup B$.

Lemma 4. *Suppose that X is perfectly normal and let A and B be non-empty disjoint closed subsets of X. Then there exists a non-alternating mapping $f: X \Rightarrow I$ such that $0 \in f(A)$, $1 \in f(B)$ and f is not interior at any point of $f(A \cup B)$.*

Proof. We will use induction to construct a sequence $\{f_n\}_{n=1}^{\infty}$ of mappings which converge uniformly to the desired map. To this end let U and V be open sets containing A and B respectively such that $\overline{U} \cap \overline{V} = \emptyset$. Let A_0 be any component of $\overline{U}$ that meets A and let B_0 be any component of $\overline{V}$ that meets B. Finally let $T = \overline{U} \cup \overline{V}$. By Lemma 3 there exist disjoint regions $H_{1,1}$ and $K_{1,1}$ containing A_0 and B_0 respectively such that $F_{1,1} = \mathrm{Fr}\,H_{1,1} = \mathrm{Fr}\,K_{1,1}$ and $F_{1,1} \cap T = \emptyset$. Let $f_1: X \Rightarrow I$ be any mapping such:

(1) $f_1(A_0) = 0$ and $f_1(B_0) = 1$,

(2) $f_1^{-1}[0, 1/2) = H_{1,1}$ and $f_1^{-1}(1/2, 1] = K_{1,1}$,

(3) $X \setminus (H_{1,1} \cup K_{1,1}) = f_1^{-1}(1/2)$.

Now suppose that for every positive integer $n \leqslant k$ we have chosen a mapping $f_n: X \Rightarrow I$ such that:

(a) $f_n(A_0) = 0$ and $f_n(B_0) = 1$.

(b) For every integer m, $1 \leqslant m \leqslant 2^{n-1}$, there exist disjoint regions $H_{m,n}$ and $K_{m,n}$ such that $A_0 \subset H_{m,n}$ and $B_0 \subset K_{m,n}$, $F_{m,n} = \mathrm{Fr}\,H_{m,n} = \mathrm{Fr}\,K_{m,n}$ misses T, $f_n^{-1}[0, m/2^n) = H_{m,n}$ and $f_n^{-1}(m/2^n] = K_{m,n}$, and $X \setminus (H_{m,n} \cup K_{m,n}) = f_n^{-1}(m/2^n)$.

(c) $f_n | f_{n-1}^{-1}(S_{n-1}) = f_{n-1}$ where $S_{n-1} = \{1/2^{n-1}, 2/2^{n-1}, \ldots, (2^{n-1}-1)/2^{n-1}\}$.

(d) For every integer m, $1 < m \leqslant 2^{n-1}$, $f_n^{-1} f_n\big[f_{n-1}^{-1}\big((m-1)/2^{n-1}, m/2^{n-1}\big)\big] = f_{n-1}^{-1}\big((m-1)/2^{n-1}, m/2^{n-1}\big)$.

We define f_{k+1} as follows: Let $A_{0,k} = A_0$ and let $B_{k,k} = B_0$. Now for each m, $1 \leqslant m < 2^k$, $H_{m,k}$ is connected and since every component of $f_k^{-1}(m/2^k)$ meets $H_{m,k}$, $A_{m,k} = H_{m,k} \cup f_k^{-1}(m/2^k)$ is a continuum. Likewise for each m, $1 \leqslant m < 2^k$, $B_{m,k} = K_{m,k} \cup f_k^{-1}(m/2^k)$ is a continuum. Furthermore for each m, $1 \leqslant m < 2^k$, $T \cap H_{m,k} = T \cap \overline{H}_{m,k}$ and $B_{m,k}$ are disjoint closed sets, $T \subset (T \cap H_{m,k}) \cup B_{m,k}$, and $A_{m-1,k}$ is a component of $T \cap H_{m,k}$. Similarly for $m = 2^k$, $B_{k,k}$ is a component of $T \cap K_{m-1,k}$, $T \subset (T \cap K_{m-1,k}) \cup A_{m-1,k}$ and $T \cap K_{m-1,k}$ and $A_{m-1,k}$ are disjoint closed sets. So that finally, by Lemma 3, for each m, $1 \leqslant m \leqslant 2^k$ there exist disjoint regions $H_{2m-1,k+1}$ and $K_{2m-1,k+1}$ containing $A_{m-1,k}$ and $B_{m,k}$ respectively such that $\mathrm{Fr}\,H_{2m-1,k+1} = \mathrm{Fr}\,K_{2m-1,k+1}$ and $\mathrm{Fr}\,H_{2m-1,k+1}$ misses T. Choose $f_{k+1}: X \Rightarrow I$ to be any mapping such that for each m, $1 < m \leqslant 2^k$,

(i) $f_{k+1} f_k^{-1}[(m-1)/2^k, m/2^k] = [(m-1)/2^k, m/2^k]$,

(ii) $f_{k+1}^{-1}\big((m-1)/2^k\big) = f_k^{-1}\big((m-1)/2^k\big)$ and

(iii) $f_{k+1}^{-1}\big((2m-1)/2^{k+1}\big) = X \setminus (H_{2m-1,k+1} \cup K_{2m-1,k+1})$.

Note such a selection is possible since X is perfectly normal. Clearly f_{k+1} so chosen satisfies (a)-(d).

Now there exists a mapping $f: X \Rightarrow I$ such that $\{f_n\}_{n=1}^{\infty}$ converges uniformly to f since for all integers p, n, m, where $p \geqslant n > 0$ and $1 \leqslant m$

$< 2^n$, $f_p f_n^{-1}[(m-1)/2^n, m/2^n] = [(m-1)/2^n, m/2^n]$. In order to see that f is non-alternating let $y \in (0, 1)$ and let $\{p_i\}_{i=1}^{\infty}$ and $\{q_i\}_{i=1}^{\infty}$ be sequences of positive integers such that for each i, $1 \leqslant p_i \leqslant 2^{q_i}-1$, $q_{i+1} > q_i$, $p_i/2^{q_i} < y$ and $\{p_i/2^{q_i}\}_{i=1}^{\infty}$ is an increasing sequence which converges to y. Then

$$f^{-1}[0, y) = \bigcup_{i=1}^{\infty} f^{-1}[0, p_i/2^{q_i}) = \bigcup_{i=1}^{\infty} f_{q_i}^{-1}[0, p_i/2^{q_i})$$

since

$$f_{q_i}^{-1}[0, p_i/2^{q_i}) \subset f^{-1}[0, p_{i+1}/2^{q_{i+1}}) \subset f_{q_{i+1}}^{-1}[0, p_{i+1}/2^{q_{i+1}})$$

and since each of the sets $f_{q_i}^{-1}[0, p_i/2^{q_i}) = H_{p_i, q_i}$ is connected, $f^{-1}[0, y)$ is connected. Similarly $f^{-1}(y, 1]$ is connected and by Lemma 1, f is non-alternating.

Let $x \in (A \cup B)$ and let Q be the component of T containing x. Let $\{p_i\}_{i=1}^{\infty}$ and $\{q_i\}_{i=1}^{\infty}$ be sequence of positive integers such that $1 \leqslant p_i \leqslant 2^{q_i}-1$ and $q_{i+1} > q_i$ and such that

$$f(x) = \bigcap_{i=1}^{\infty} \left(p_i/2^{q_i}, (p_i+1)/2^{q_i}\right).$$

Recall that if any component of T met a set of the form $f_n^{-1}(m/2^n)$ it lie entirely within the set, so that for each $i \geqslant 1$,

$$Q \subset f_{q_i}^{-1}(p_i/2^{q_i}, (p_i+1)/2^{q_i}) \subset f^{-1}[p_i/2^{q_i}, (p_i+1)/2^{q_i}].$$

Hence $f(Q) = f(x)$ and since $\operatorname{int} Q \neq \emptyset$, f is not interior at $f(x)$.

Remark 5. When A and B are continua we need only assume that X is normal and modify our construction slightly to obtain a non-alternating map onto I separating A and B.

THEOREM 1. *Let X be a separable and perfectly normal space. Then X is unicoherent if and only if for every non-alternating map $f\colon X \Rightarrow I$, $\mathfrak{I}(f) \subset \mathfrak{C}(f)$.*

Proof of the necessity. Let $x \in \mathfrak{I}(f)$ and let $H = f^{-1}[0, x)$ and $K = f^{-1}(x, 1]$. We assert that $f^{-1}(x) = \bar{H} \cap \bar{K}$. In order to see this suppose that $z \in (f^{-1}(x) \setminus \bar{H})$. Then there exists a region U containing z such that $U \cap H = \emptyset$. But then $f(U) \cap [0, x) = \emptyset$ and this is a contradiction since f is interior at x. Hence $f^{-1}(x) = \bar{H} \cap \bar{K}$ and since X is unicoherent and $X = \bar{H} \cup \bar{K}$, $x \in \mathfrak{C}(f)$.

The sufficiency. Suppose $X = H \cup K$ where H and K are continua and $H \cap K = A \cup B$ is a separation. Let $f\colon X \Rightarrow I$ be a non-alternating map such that $0 \in f(A)$, $1 \in f(B)$ and $\mathfrak{I}(f) \cap f(A \cup B) = \emptyset$. By the main result of [10], there exists $x \in \mathfrak{I}(f)$ and by our hypothesis, $f^{-1}(x) = C$ is connected. Then C separates any $a \in f^{-1}(0) \cap A$ from any point $b \in f^{-1}(1) \cap B$. But since $x \notin f(A \cup B)$, C is a subset of $H \setminus H \cap K$ or $K \setminus H \cap K$ and as such, C cannot separate a and b in X. This contradiction implies that X is unicoherent.

EXAMPLES (1). Let $X = \{(x, y) \in E^2\colon (x-1/2)^2+y^2 = 1/4\}$ and let $f(x, y) = x$. Then $f\colon X \Rightarrow I$ is non-alternating, but $\mathfrak{C}(f) = \{0, 1\}$. However X is not unicoherent.

(2). Let $Y = \{(x, y) \in E^2\colon 0 \leqslant x \leqslant 1$ and $0 \leqslant y \leqslant 1\}$ and let $X = Y \setminus \{(0, y)\colon 1/4 \leqslant y \leqslant 3/4\}$. Define $f\colon X \Rightarrow I$ by $f(x, y) = x$. Then X is unicoherent and f is interior at 0, but $0 \notin \mathfrak{C}(f)$. Thus we can not enlarge $\mathfrak{I}(f)$ to include 0 or 1 in Theorem 1.

(3). Let Y be as in (2), let $Z = \{(x, y)\colon 1/2 < x \leqslant 1$ and $1/2 < y \leqslant 1\}$ and let $X = Y \setminus (Z \cup \{(1/2, y)\colon 1/2 \leqslant y \leqslant 3/4\}$ and again let $f(x, y) = x$. Then X is unicoherent and $1/2 \notin \mathfrak{C}(f)$ so that our construction in Lemma 4 does not in general allow us to choose f to be monotone. However we can prove the following:

COROLLARY (1.1). *Let X be a separable and perfectly normal space. Then X is unicoherent if and only if for every pair of disjoint non-empty closed sets A and B of X, there exists a d-monotone map such that $0 \in f(A)$, $1 \in f(B)$ and $\mathfrak{C}(f) \cap (I \setminus f(A \cup B))$ is dense in I.*

Proof. The proof of the sufficiency is similar to that for Theorem 1.

The necessity. By Lemma 4, there exists a non-alternating map $f\colon X \Rightarrow I$ such that $\mathfrak{I}(f) \cap f(A \cup B) = \emptyset$ and by Whyburn's result in [10], $\mathfrak{I}(f)$ is dense in I. It then follows from Theorem 1, that $\mathfrak{C}(f) \cap f(A \cup B)$ is dense in I.

COROLLARY (1.2). *If X is separable and unicoherent, then a mapping $f\colon X \Rightarrow I$ is non-alternating if and only if it is d-monotone.*

This is a consequence of the fact that $\mathfrak{I}(f)$ is dense in I.

DEFINITION. We say that a subset A of X is *C-separated* provided that there exists disjoint continua L and M of X such that $A \subset (L \cup M)$ and $A \cap L \neq \emptyset \neq A \cap M$. We say that X has *Property C* provided that every separated closed subset of X is C-separated.

LEMMA 6. *If X is normal, then X has Property C if and only if for every pair of non-empty disjoint closed sets A and B of X there exists a non-alternating map $f\colon X \Rightarrow I$ such that $f(A \cup B) = \{0, 1\}$.*

Proof. Suppose that X has Property C and let A and B be disjoint closed subsets of X. Let L and M be disjoint continua such that $(A \cup B) \subset (L \cup M)$ and $(A \cup B) \cap L \neq \emptyset \neq (A \cup B) \cap M$. By Remark 5 there exists a non-alternating map $f\colon X \Rightarrow I$ such that $f(L) = 0$ and $f(M) = 1$. Then $f(A \cup B) = \{0, 1\}$ as required.

The sufficiency. Let A be a separated closed subset of X, say $A = H \cup K$ where H and K are disjoint closed sets. By our hypothesis there exists a non-alternating map $f\colon X \Rightarrow I$ such that $f(H \cup K) = \{0, 1\}$. By Lemma 1, each of the sets $f^{-1}[0, 1/4)$ and $f^{-1}(3/4, 1]$ are connected and so $L = \overline{f^{-1}[0, 1/4)}$ and $M = \overline{f^{-1}(3/4, 1]}$ are the required continua.

THEOREM 2. *If X is separable, normal and has Property C, then X is unicoherent if and only if every pair of disjoint non-empty continua A and B of X there exists a d-monotone map $f\colon X\Rightarrow I$ such that $f(A)=0$ and $f(B)=1$.*

Proof of the sufficiency. Suppose that $X=H\cup K$ where H and K are continua and $H\cup K=A\cup B$ is a separation. Since X has Property C, there exist disjoint continua L and M of X such that $(A\cup B)\subset(L\cup M)$ and $(A\cup B)\cap L\neq\emptyset\neq(A\cup B)\cap M$. By our hypothesis there exists a d-monotone map $f\colon X\Rightarrow I$ such that $f(L)=0$ and $f(M)=1$. Let $x\in(0,1)\cap\mathcal{C}(f)$. Then $f^{-1}(x)$ must separate L and M. But it must lie entirely within $H\backslash(L\cup M)$ or $K\backslash(L\cup M)$ and cannot separate L and M in X. Hence X is unicoherent.

Proof of the necessity. Let A and B be disjoint subcontinua of X. By Remark 5 there exists a non-alternating map $f\colon X\Rightarrow I$ such that $f(A)=0$ and $f(B)=1$ and it follows from Corollary (1.1), that f is d-monotone.

COROLLARY (2.1). *If X is separable and normal, then X is unicoherent and has Property C if and only if for every pair of non-empty disjoint closed sets A and B of X, there exists a d-monotone map $f\colon X\Rightarrow I$ such that $f(A\cup B)=\{0,1\}$.*

Proof of the sufficiency. By Lemmas 2 and 7, X has Property C and by Theorem 2, X is unicoherent.

The necessity follows from Corollary (1.1) and Lemma 6.

LEMMA 7. *If X is normal and has Property C, then X is unicoherent if and only if every pair of disjoint continua can be separated by a continuum.*

Proof. The necessity follows immediately from Theorem (4.7) of [11].

The sufficiency. Suppose that $X=H\cup K$ where H and K are continua and $H\cap K=A\cup B$ is a separation. Since X has Property C, there exists disjoint continua L and M such that $(A\cup B)\subset(L\cup M)$ and $(A\cup B)\cap L\neq\emptyset\neq(A\cup B)\cap M$. By our hypothesis there exists a continuum T such that T separates L and M in X. But then T would be a subset of $H\backslash(A\cup B)$ or $K\backslash(A\cup B)$ and thus could not separate L and M. This contradiction implies that X is unicoherent.

DEFINITION. Let A and B non-empty subsets of X. We say that a finite collection of subsets of X, $\{S_1, S_2, \ldots, S_n\}$, is a *simple chain from A to B provided*, $A\cap S_1\neq\emptyset\neq B\cap S_n$ and $S_i\cap S_j\neq\emptyset$ if and only if $|i-j|\leqslant 1$.

LEMMA 8. *If X is compact, X has Property C.*

Proof. Let A be a separated closed set in X, say $A=H\cup K$, where H and K are non-empty and closed. Let P and Q be disjoint open sets containing H and K respectively such that $\overline{P}\cap Q=\emptyset$ and every component of $P\cup Q$ meets A. Then since A is compact, $P\cup Q$ has only finitely many components, say $C_1, C_2, \ldots, C_n$. With loss of generality, we may and do assume that $\overline{C}_i\cap\overline{C}_j=\emptyset$ if $i\neq j$. Let $\{W_\alpha\}$, $\alpha\in\Gamma$, be an open covering of $X\backslash A$ by connected sets such that no $\overline{W}_\alpha$ intersects more than one $\overline{C}_i$. Let $\mathcal{K}=\{C_i\}_{i=1}^{n}\cup\{W_\alpha\colon \alpha\in\Gamma\}$. Since X is connected, $\mathcal{K}$ contains a simple chain $\mathcal{S}$ from C_1 to C_2. Let $\mathcal{S}_1$ be the subchain of $\mathcal{S}$ such that $\mathcal{S}_1$ contains exactly one C_i different than C_1, say C_s. Now if C denotes the closure of the union of the elements of $\mathcal{S}_1$, then $\bigcup\{\overline{C}_i\colon i\neq 1$ and $i\neq s\}\cup C$ is a closed set with $\leqslant(n-1)$ components. Clearly the result now follows by induction.

LEMMA 9. *If $f\colon X\Rightarrow I$ is closed and d-monotone, f is monotone.*

Proof. Let $y\in I$ and suppose that $f^{-1}(y)=H\cup K$ is a separation. Let U and V be disjoint open sets containing H and K respectively and let $F=\mathrm{Fr}(U\cup V)$. Now $R=I\backslash f(F)$ is an open set containing y. Let $U_0=f^{-1}(R)\cap U$ and $V_0=f^{-1}(R)\cap V$ and let Q_1 and Q_2 be components of U_0 that meet H. Suppose that $f(Q_1)\cap[0,y)\neq\emptyset$ and $f(Q_2)\cap(y,1]\neq\emptyset$. Let P be any component of V_0 that meets K. Then $f(P)\cap[0,y)\neq\emptyset$ or $f(P)\cap(y,1]\neq\emptyset$. Suppose $f(P)\cap(y,1]\neq\emptyset$. Then $f(Q_2)\cap f(P)\neq\emptyset$ and in particular $f(Q_2)\cap f(P)$ contains a point $c\in\mathcal{C}(f)$. But then $f^{-1}(c)\subset(X\backslash F)$ and $f^{-1}(c)\cap U\neq\emptyset\neq f^{-1}(c)\cap V$. This contradicts the connectedness of $f^{-1}(c)$. Thus we must have that every component of $U_0\backslash H$ maps into $[0,y)$ or every component of $U_0\backslash H$ maps into $(y,1]$. Suppose the latter case holds. Then by an argument similar to that above, we must have that every component of $V_0\backslash K$ maps into $[0,y)$. But then $X=[f^{-1}[0,y)\cup V_0]\cup[f^{-1}(y,1]\cup U_0]$ is a separation of X and of course this contradicts the connectedness of X. Therefore $\mathcal{C}(f)=X$ and f is monotone.

COROLLARY (2.2). *Suppose that X is compact, separable and normal. Then the following are equivalent:*

(a) *X is unicoherent.*

(b) *Every pair of disjoint continua can be separated by a continuum.*

(c) *For every pair disjoint non-empty continua A and B of X, there exists a monotone map $f\colon X\Rightarrow I$ such that $f(A)=0$ and $f(B)=1$.*

(d) *For every pair of disjoint non-empty closed sets A and B of X there exists a monotone map $f\colon X\Rightarrow I$ such that $f(A\cup B)=\{0,1\}$.*

Remarks. We do not know whether every locally connected, connected normal space X has Property C, however we have shown that every Lindelöf, locally compact, locally connected, connected Hausdorff space has Property C [4].

A. H. Stone has made the following conjecture: (a) Let X be a locally connected, connected normal space and let $n>2$ be an integer. If X is not unicoherent, there exist continua $A_1, A_2, \ldots, A_n$ in X such that

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$X = \bigcup_{i=1}^{n} A_i$, $A_i \cap A_j \neq \emptyset$ if and only if $|i(\bmod n) - j(\bmod n)| \leqslant 1$ and no three of the A_i's have a point in common. We have been able to show that (a) (for $n = 4$) is equivalent to (b): Let X be a locally connected, connected normal space. Then X is unicoherent if and only if for every pair of disjoint subcontinua A and B of X there exists a sub-continuum C of X such that C separates A and B in X [4].

A. H. Stone has proved (a) for $n = 3$ [6] and (b) is true whenever X has Property C. The equivalence of (a) for $n = 3$ and unicoherence for compact metric continua is due to A. D. Wallace [7].

G. T. Whyburn proved that if X is a compact locally connected metric continuum and J is an arc in X from a to b, then there exists a non-alternating retraction $f\colon X \Rightarrow J$ of X onto J which was also monotone when X was unicoherent [9]. His proofs leaned heavily on cyclic element theory for compact metric continua.

Related results. In [3] we defined a set X to be *weakly-unicoherent* provided that whenever $X = H \cup K$ where H and K are continua and K is compact $H \cap K$ is a continuum. A set $A \subset X$ is *γ-closed* provided it is closed and $\operatorname{Fr} A$ is compact. In [2], we defined a set X to be *γ-unicoherent* provided that whenever $X = H \cup K$ where H and K are γ-closed and connected sets, $H \cap K$ is connected and in [2] we showed that a locally compact Hausdorff space is γ-unicoherent if and only if it is weakly-unicoherent, but in general the two concepts do not coincide. Finally we say that a space X has the *Complementation Property* provided that for every compact set K in X, $X \setminus K$ has exactly one component with a non-compact closure. In [3] we related the Complementation Property to weak-unicoherence and characterized those spaces which enjoyed both of these properties.

The following propositions can be proved by the techniques of this paper:

THEOREM 3. *Let X be a separable and normal space. Then X is γ-unicoherent if and only if for every pair of disjoint, non-empty γ-closed subsets A and B of X there exists a d-monotone map $f\colon X \Rightarrow I$ such that $f(A \cup B) = \{0, 1\}$.*

THEOREM 4. *Let X be a locally compact, normal, separable Hausdorff space. Then*

(a) *X has the Complementation Property if and only if for all compact sets K in X there exists a non-alternating map $f\colon X \Rightarrow I$ such that $f(K) = 0$ and for all $c \in [0, 1)$, $f^{-1}[0, c]$ is compact.*

(b) *X is weakly-unicoherent and has the Complementation Property if and only if for every pair of disjoint compact non-empty continua A and B of X there exists a monotone map $f\colon X \Rightarrow [0, 1]$ such that $f(A) = 0$ and $f(B) = 1$ and for all $c \in [0, 1)$, $f^{-1}[0, c]$ is compact.*

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