Locally compact groups and β-varieties of topological groups

by

Sidney A. Morris (Sydney)

Abstract. It has previously been shown by the author that any abelian variety containing a connected compact group is a β-variety. As a consequence, it is seen that the varieties $V(R)$ and $V(T)$, respectively generated by the additive group of reals and the circle group, are β-varieties. It is proved here that any variety generated by arcwise connected Hausdorff groups is a β-variety. From this it is deduced that connected locally compact groups generate β-varieties. It is also noted that any variety containing a non-abelian connected compact group is a β-variety.

§ 1. Preliminaries. A non-empty class V of topological groups (not necessarily Hausdorff) is said to be a *variety* if it is closed under the operations of taking subgroups, quotients, arbitrary cartesian products and isomorphic images. (See [8]–[11].)

We note that a variety (of topological groups) determines a variety of groups [15]; the latter is simply the class of groups which with some topology appear in the former.

The smallest variety containing a class Ω of topological groups is said to be the *variety generated by* Ω and is denoted by $V(\Omega)$. (See [2], [3] and [12]–[14].)

A *full variety* is a variety V which contains every topological group algebraically isomorphic to a member of V.

If V is a variety, X is a topological space and F is a member of V, then F is said to be a *free topological group of* V *on* X, denoted by $F(X, V)$, if it has the properties:

(a) X is a subspace of F.

(b) X generates F algebraically.

(c) For any continuous mapping γ of X into any group H in V, there exists a continuous homomorphism Γ of F into H such that $\Gamma | X = \gamma$.

The following results on free topological groups are proved in [8]: (a) $F(X, V)$ is unique (up to isomorphism) if it exists (b) $F(X, V)$ exists if and only if there is a member of V which has X as a subspace, (c) $F(X, V)$ is the free group of the underlying variety of groups on the set X.

If V is a variety such that $F(X, V)$ exists and is Hausdorff for each Tychonoff space X, then V is called a *β-variety*.

Using Świerczkowski [16] it is shown in [9] that every full variety is a β-variety. Finally we note that in [10] it is proved that β-varieties exist in profusion.

§ 2. The theorems.

THEOREM 1. *If Ω is any class of arcwise connected Hausdorff groups and Ω contains a non-trivial group then $V(\Omega)$ is a β-variety.*

Proof. If X is any Tychonoff space, then Lemma 5, p. 116 of [6] implies that X can be embedded in a product of copies of any member of Ω. Therefore $F(X, V(\Omega))$ exists.

To show that $F(X, V(\Omega))$ is Hausdorff it is sufficient to find for each element $a \neq e_1$, a member H of Ω and a continuous homomorphism Γ of $F(X, V(\Omega))$ into H such that $\Gamma(a) \neq e$, where e and e_1 denote the identity elements of H and $F(X, V(\Omega))$ respectively.

Suppose $a = x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ where $x_i \in X$ and ε_i is an integer for $i = 1, \dots, n$. Since $a \neq e_1$, $x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}$ is not a law [15] of the underlying variety $\overline{V(\Omega)}$ of groups. Noting also that $F(X, V(\Omega))$ is the free group on the set X of $\overline{V(\Omega)}$, we see that there is a member H of Ω and a (not neccessarily continuous) homomorphism φ of $F(X, V(\Omega))$ into H such that $\varphi(x_1^{\varepsilon_1} \dots x_n^{\varepsilon_n}) \neq e$.

Since X is a Tychonoff space and H is arcwise connected Hausdorff, Theorem 3.6 of [5] implies that there exists a continuous mapping γ of X into H such that $\gamma(x_i) = \varphi(x_i)$ for $i = 1, \dots, n$. This implies that there exists a continuous homomorphism Γ of $F(X, V(\Omega))$ into H such that $\Gamma | X = \gamma$. Clearly $\Gamma(a) = \varphi(a) \neq e$ and the proof is complete.

If the "arcwise connected Hausdorff" condition is weakened to "connected Hausdorff" we expect, but have failed to prove, that the resulting proposition is false.

It would be interesting to know the answer to the following:

OPEN QUESTION (1). Can every β-variety be generated by its arcwise connected Hausdorff groups?

Two remarks are relevant. Firstly, we note that every β-variety contains non-trivial arcwise connected Hausdorff groups. This is easily seen by letting X be any arcwise connected Tychonoff space and observing that the subgroup of $F(X, V)$ algebraically generated by $x^{-1}X$, for any $x \in X$, is an arcwise connected Hausdorff group. Secondly, we point out that it is clear from [4] that every full variety is generated by its arcwise connected Hausdorff groups.

(1) This question is answered in the negative in [2].

THEOREM 2. *If Ω is any class of connected locally compact groups then $V(\Omega)$ is a β-variety.*

Proof. It is clear from § 4.6 of [7] that any connected locally compact group H is a subgroup of a product of Lie groups, each of these Lie groups being a quotient of H. This implies that $V(\Omega) = V(\Gamma)$, where Γ is a class of connected Lie groups. Noting that any connected Lie group is arcwise connected, the result immediately follows from Theorem 1.

We note that the "connected" condition in the above theorem cannot be removed. For example, it is shown in [10] that a variety generated by discrete groups is not a β-variety.

It is (essentially) shown in [10] that any abelian variety containing a connected compact group is a β-variety. Example 3.2 of [10] shows that the "abelian" condition cannot be omitted. However our final theorem, which is a simple consequence of Theorem 3 of [1] and Lemma 4.8 of [10], is a companion to this.

THEOREM 3. *If the variety V contains a locally compact connected non-solvable group (in particular, a compact connected non-abelian group) then V is a β-variety.*

References

[1] S. Balcerzyk and Jan Mycielski, *On the existence of free subgroups in topological groups*, Fund. Math. 44 (1957), pp. 303–308.
[2] M. S. Brooks, Sidney A. Morris and Stephen A. Saxon, *Generating varieties of topological groups*, Proc. Edinburgh Math. Soc. (to appear).
[3] Su-shing Chen and Sidney A. Morris, *Varieties of topological groups generated by Lie groups*, Proc. Edin. Math. Soc. (to appear).
[4] S. Hartman and Jan Mycielski, *On the imbedding of topological groups into connected topological groups*, Colloq. Math. 5 (1958), pp. 167–169.
[5] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis* I, New York 1963.
[6] J. L. Kelly, *General Topology*, New York 1955.
[7] D. Montgomery and L. Zippin, *Topological Transformation Groups*, New York 1955.
[8] Sidney A. Morris, *Varieties of topological groups*, Bull. Austral. Math. Soc. 1 (1969), pp. 145–160.
[9] — *Varieties of topological groups* II, Bull. Austral. Math. Soc. 2 (1970), pp. 1–13.
[10] — *Varieties of topological groups* III, Bull. Austral. Math. Soc. 2 (1970), pp. 165–178.
[11] — *Varieties of topological groups and left adjoint functors*, J. Austral. Math. Soc. (1973).
[12] — *On varieties of topological groups generated by solvable groups*, Colloq. Math. (1972).
[13] — *Varieties of topological groups generated by solvable and nilpotent groups*, Colloq. Math. (to appear).
[14] — *Locally compact abelian groups and the variety of topological groups generated by the reals*, Proc. Amer. Math. Soc. 34 (1972), pp. 290–292.

[15] Hanna Neumann, *Varieties of groups*, Berlin 1967.
[16] S. Świerczkowski, *Topologies in free algebras*, Proc. London Math. Soc. (3) 14 (1964), pp. 566–576.

THE UNIVERSITY OF NEW SOUTH WALES
Kensington, Australia

Reçu par la Rédaction le 25. 11. 1971

Some mapping characterizations of unicoherence

by

R. F. Dickman, Jr. (Blacksburg, Va.)

Abstract. In this paper we characterize unicoherence in terms of certain real-valued mappings. The following theorems are typical of the results obtained: (1) Let X be a separable, locally connected, connected, perfectly normal space. Then X is unicoherent if and only if for every pair of disjoint non-empty closed sets A and B of X there exists a mapping f of X onto $I = [0, 1]$ such that $0 \in f(A)$, $1 \in f(B)$ and $I \setminus f(A \cup B)$ contains a dense subset D of I such that for every $d \in D$, $f^{-1}(d)$ is connected. (2) Let X be a separable, locally connected, connected, compact normal space. Then X is unicoherent if and only if for every pair of disjoint non-empty continua A and B of X there exists a monotone mapping f of X onto I such that $f(A) = 0$ and $f(B) = 1$.

The concept of non-alternating mappings was introduced by G. T. Whyburn in [8] and in [9] he showed that if M is a locally connected, compact connected metric space and J is any arc in M, there exists a non-alternating retraction $r\colon M \to J$ which, when M was unicoherent, was monotone. His proofs depended heavily upon cyclic element theory for compact locally connected continua. In [1], K. Borsuk characterized unicoherence for compact, locally connected metric continua in terms of mappings into the circle. More recently, K. Kuratowski proved that when X is a compact and locally connected space and Y is a metric space, $\mathcal{N}$, the set of all non-alternating mappings of X onto Y, is a G_δ-set in the space of all continuous maps of X into Y.

In this paper we characterize unicoherence for separable, perfectly normal, locally connected, connected spaces in terms of non-alternating mappings onto $[0, 1]$.

Notation and terminology. Throughout this paper let X denote a connected, locally connected normal space. By a continuum we mean a closed and connected set and a region is an open connected set. By a mapping we will always mean a continuous function. We will use I to denote $[0, 1]$ and a surjection f of X onto a space Y will be denoted by $f\colon X \Rightarrow Y$. A perfectly normal space is a normal space in which every closed subset is a G_δ-set.

DEFINITIONS. We say that X is *unicoherent* provided whenever $X = H \cup K$, where H and K are continua, $H \cap K$ is a continuum.