Cardinalities of metric completions

by

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Abstract. $|A|$ denotes the cardinality of A. Suppose X is a metrizable space of infinite weight k and that $\widetilde{X}$ is a completion of X in some compatible metric. $|\widetilde{X}|$ can only be k or $k^{\aleph_0}$. For $k^{\aleph_0} > k$, we discuss the possible values of $|\widetilde{X}|$ for different compatible metrics on X. Since $|\widetilde{X}|$ must be $k^{\aleph_0}$ if $|X| > k$, we also assume that $|X| = k$.

Theorem. *X has a completion of cardinality $k^{\aleph_0}$ iff X contains a closed discrete subset of cardinality p, where p is the least infinite cardinal such that $p^{\aleph_0} = k^{\aleph_0}$.*

Theorem. *X has a completion of cardinality k iff every nonempty set in X contains a nonempty (relatively) open set of cardinality $< p$, where p is the least infinite cardinal such that $p^{\aleph_0} = k^{\aleph_0}$.*

These theorems also obviously give characterizations of those spaces X which have only completions of cardinality k or only completions of cardinality $k^{\aleph_0}$. In the latter case, the theorem can be reformulated roughly as follows: X has every completion of cardinality $k^{\aleph_0}$ iff $K_p(X) \neq \emptyset$, where $K_p(X)$ is a certain transfinitely defined "kernel" of X, reducing, in the case $k = \aleph_0$, to the ordinary perfect kernel of X.

1. Introduction and notation. All spaces considered here are metrizable. We denote the weight of X (the minimal cardinality of a dense subset of X) by $w(X)$; $|X|$ denotes the cardinality of X. A "completion of X", denoted $\widetilde{X}$, means the metric completion $(\widetilde{X}, \widetilde{d})$ of (X, d), where d is some compatible metric on X.

If $w(X) = k$, then for any completion we clearly have $k \leqslant |\widetilde{X}| \leqslant k^{\aleph_0}$. Our main purpose is to characterize the cardinalities of completions of X in terms of topological properties of X. Since the case of k finite or $k^{\aleph_0} = k$ is trivial, k will hereafter denote, without further mention except for special emphasis, an infinite cardinal satisfying $k^{\aleph_0} > k$. (There are arbitrarily large cardinals with this property — for example, all sequential cardinals. On the generalized continuum hypothesis, if $k^{\aleph_0} > k$ and k is infinite, then k is sequential.) Further, given k, p will always denote the least infinite cardinal satisfying $p^{\aleph_0} = k^{\aleph_0}$.

It turns out, from known results, that if $w(X) = k$, then every completion of X has cardinality either k or $k^{\aleph_0}$, never an intermediate value. In section 2 we show that if $w(X) = k$, then X has a completion of cardinality $k^{\aleph_0}$ if and only if X has a closed discrete subset of cardinal p. This of course also provides a characterization of those X for which every completion is "small". In section 3, we prove further that every completion of X has cardinality $k^{\aleph_0}$ if and only if a certain transfinitely defined "kernel" in X is non-empty. The kernel construction there is an instance of the general kernel constructions discussed in [6].

For any infinite cardinal m, $B(m)$ denotes the product of countably many discrete spaces of cardinality m.

2. The existence of "large" completions. If $w(X) = k$, we first note that for any completion, $|\widetilde{X}|$ is either k or $k^{\aleph_0}$. This is immediate from the following theorem of Stone [5], applied to the absolute G_δ set $\widetilde{X}$. m denotes an infinite cardinal.

THEOREM 2.1. *If Y is an absolute Borel set of weight $\leqslant m$ and $|Y| > m$, then $|Y| = m^{\aleph_0}$ and Y contains a closed subspace C, $|C| = m^{\aleph_0}$, where C is homeomorphic either to the Cantor set or some $B(q)$.*

In view of this theorem, it is also clear that if $w(X) = k$ and $|X| > k$, then $|\widetilde{X}| = k^{\aleph_0}$ for every completion, so that our attention should focus on spaces of both weight and cardinality k.

There are spaces of weight k for which every completion has cardinality k — for example, the compact countable spaces. On the other hand, every completion of Q, the rational numbers, has cardinality c. Indeed, if some $\widetilde{Q}$ were countable, then Q would be a G_δ in $\widetilde{Q}$ and hence be completely metrizable. Between these extremes lie, for example, the discrete spaces, as Lemma 2.2 shows. The natural numbers, N, with their usual metric, are complete; but if we view N as the set of midpoints of open intervals deleted in the construction of the Cantor set in $[0, 1]$, then $\overline{N}$ is the completion of N and has cardinality c.

LEMMA 2.2. *Let Y be a discrete space of infinite cardinality m. Then Y has a completion of cardinality $m^{\aleph_0}$.*

Proof. Let T_n be a discrete space of cardinality m, with $0, 1 \in T_n$. Let T_n^* denote $T_n - \{1\}$. Metrize $B(m) = \prod_{n=1}^{\infty} T_n$ with the (complete) "first difference metric", s ($s(x, y) = 1/j$ if $x_j \neq y_j$ and $x_i = y_i$ for all $i < j$). For each n, let $E_n = \{x \in B(m)\colon x_i \neq 1,\ i = 1, \ldots, n;\ x_{n+1} = 1;\ x_i = 0,\ i \geqslant n+2\}$ and $Y = \bigcup_{n=1}^{\infty} E_n$. Clearly, $|Y| = m$, and Y is discrete, since, if $y \neq z \in Y$ and $y \in E_n$, then $s(y, z) \geqslant 1/(n+1)$. Further, if $x \in \prod_{n=1}^{\infty} T_n^*$, then x is at distance $1/(n+1)$ from $y = (x_1, \ldots, x_n, 1, 0, 0, \ldots) \in E_n$. Hence $\overline{Y} \supseteq \prod_{n=1}^{\infty} T_n^*$, so $|\overline{Y}| = m^{\aleph_0}$. But $(\overline{Y}, s)$ is the completion of (Y, s).

We can now characterize those spaces of weight k which have a "large" completion in some metric.

THEOREM 2.3. *Suppose $w(X) = |X| = k$, where $k^{\aleph_0} > k$ and p is the least infinite cardinal such that $p^{\aleph_0} = k^{\aleph_0}$. X has a completion of cardinality $k^{\aleph_0}$ if and only if X contains a closed discrete subset of cardinality p.*

Proof. Suppose X contains such a set, D. Pick, using Lemma 2.2, a metric d' for D so that $|(\widetilde{D}, \widetilde{d}')| = p^{\aleph_0} = k^{\aleph_0}$. Extend d' to a compatible metric d on all of X (Hausdorff, [1]). Clearly, $|(\widetilde{X}, \widetilde{d})| = k^{\aleph_0}$.

Conversely, suppose X has a metric d whose completion, $\widetilde{X}$, has cardinality $k^{\aleph_0}$. By Theorem 2.1, $\widetilde{X}$ contains a closed set C, of cardinality $k^{\aleph_0}$, homeomorphic to either the Cantor set or some space $B(m)$. In the former case, since X cannot be compact, X contains a closed discrete subset of cardinal $p = \aleph_0$, and $p^{\aleph_0} = c = k^{\aleph_0}$. If C is $B(m)$, let D be a closed (in C, therefore in $\widetilde{X}$) discrete set in $B(m)$, of cardinality $m \geqslant p$. Use Hausdorff's theorem on extending metrics again to get a compatible metric d' on $\widetilde{X}$ such that $d'(x, y) = 1$ for $x, y \in D$. Then the d'-spheres of radius 1/3 about points of D are disjoint and all meet the dense set X. Thus X contains a (relatively) closed discrete subset of cardinality $m \geqslant p$.

Assume X and k satisfy the hypotheses of Theorem 2.3. If $k = \aleph_0$, we find that each non-compact countable space has a completion of cardinality c. We also remark that if k is non-sequential, then X contains a closed discrete subset of cardinality k (Stone, [4]) and therefore has a completion of cardinality $k^{\aleph_0}$. Of course the preceding theorem also provides a characterization of such spaces X with only "small" completions.

We remark that a virtually identical proof shows that, under the given hypotheses, X has a closed discrete subset of cardinality p if and only if there exists an absolute Borel (or even $\aleph_0$-analytic) set Y of cardinality $k^{\aleph_0}$ in which X is densely embedded.

If the generalized continuum hypothesis is assumed, p can be replaced by k throughout.

3. Spaces with only "large" completions. There remains the problem of characterizing those spaces of weight and cardinality k whose every completion has cardinality $k^{\aleph_0}$. This question is a bit more delicate, and some preliminary definitions and propositions are needed. The first of these occurs in [8].

DEFINITION 3.1. Given an infinite cardinal m and a space Y, let $C_m(Y) = \bigcup \{A \subseteq Y\colon A \text{ is homeomorphic to } B(m)\}$.

PROPOSITION 3.2. *$C_m(Y)$ is a closed subset of Y.*

Proof. We shall need the fact that $B(m)$ is characterized as a completely metrizable, zero-dimensional space with a dense subset of cardinality m, and such that every non-empty open set contains a closed discrete subset of cardinality m ([5]). Let d be a compatible metric on Y and assume $y \in \overline{C_m(Y)} = \overline{C}$. Since any non-empty open set in $B(m)$ contains a copy of $B(m)$, it is clear that for all $\delta > 0$, there is a copy of $B(m)$, say B_δ, in $S(y; \delta)$. If $y \in \overline{B}_\delta$ for some such B_δ, then $B_\delta \cup \{y\}$ is homeomorphic to $B(m)$, so $y \in C$. If no such $\delta > 0$ exists, pick a sequence B_{δ_i}

of copies of $B(m)$ such that each $B_{\delta i}$ is clopen in $\bigcup_{i=1}^{\infty} B_{\delta i}$ and so that $y \in \overline{\bigcup_{i=1}^{\infty} B_{\delta i}}$. Then $\{y\} \cup \bigcup_{i=1}^{\infty} B_{\delta i}$ is homeomorphic to $B(m)$, so $y \in C$.

We next describe the construction of a "kernel" in X which generalizes the ordinary perfect kernel; it is a particular case of the kernel constructions in [6].

Let m be an infinite cardinal. A point x in X is an m-limit point of X if each neighborhood of x in X has cardinality $\geqslant m$. Let $m_0(X) = X$, and assume $m_\beta(X)$ has been defined for all ordinals $\beta < \alpha$. If $\alpha = \beta+1$, let $m_\alpha(X)$ be the set of m-limit points in $m_\beta(X)$; if α is a limit ordinal, let $m_\alpha(X) = \bigcap_{\beta<\alpha} m_\beta(X)$. Note that $m_1(X)$ is the set of m-limit points of X. It is clear that each $m_\alpha(X)$ is closed in X and that, if $\alpha > \beta$, then $m_\alpha(X) \subseteq m_\beta(X)$. For some ordinal ξ (not necessarily countable) we must have $m_\xi(X) = m_{\xi+1}(X)$. We define $K_m(X) = m_\xi(X) =$ the m-kernel of X.

Every point of $K_m(X)$ is an m-limit point of $K_m(X)$, and $K_m(X)$ is clearly the largest subset of X with this property.

With the usual assumptions on k and p, we may describe $K_p(X)$ in a slightly different way. A neighborhood of x has cardinality $<p$ if and only if it has weight $m < p$, since $m^{\aleph_0} < p$. Hence $K_p(X)$ is also the largest subset of X in which every non-empty (relatively) open set has weight $\geqslant p$. In the terminology of [6], $K_p(X)$ is the "non-locally of weight $<p$" kernel of X.

It is clear that $K_p(X) = \emptyset$ if and only if every non-empty set A in X has a non-empty relatively open set of weight (cardinality) $<p$.

LEMMA 3.3. *If X is dense in Y and $w(X) = |X| = k$, then $C_p(Y) \cap X \subseteq p_1(X)$.*

Proof. Let $x \in X$ and suppose that U is an open set in Y containing x and such that $|U \cap X| = m < p$. Then $|\overline{U}| = |\overline{U \cap X}| \leqslant m^{\aleph_0} < p^{\aleph_0}$, by definition of p. Hence U contains no copy of $B(p)$, so $U \cap C_p(Y) = \emptyset$. Hence $x \notin C_p(Y)$.

We can now characterize those spaces whose every completion is "large".

THEOREM 3.4. *Suppose $w(X) = |X| = k$, $k^{\aleph_0} > k$, and let p be the least infinite cardinal such that $p^{\aleph_0} = k^{\aleph_0}$. Every completion of X has cardinality $k^{\aleph_0}$ if and only if $K_p(X) \neq \emptyset$.*

Proof. Assume every completion of X has cardinality $k^{\aleph_0}$; let $\widetilde{X}$ be some completion. Define $H_1 = C_p(\widetilde{X}) - \overline{p_1(X)}$. Assume that H_β has been defined for all ordinals β, $1 \leqslant \beta < \alpha \leqslant \xi$. If $\alpha = \beta+1$, let $H_\alpha = C_p(\overline{p_\beta(X)}) - \overline{p_\alpha(X)}$. If α is a limit ordinal, let $H_\alpha = C_p(\overline{\bigcap_{\beta<\alpha} p_\beta(X)}) - \overline{p_\alpha(X)}$. Define $O_\alpha = \widetilde{X} - \overline{p_\alpha(X)}$, $1 \leqslant \alpha \leqslant \xi$.

The O_α's form an increasing well ordered family of open sets, and, for $1 \leqslant \alpha \leqslant \xi$, $O_\alpha - \bigcup_{\beta<\alpha} O_\beta \supseteq H_\alpha$. Since each H_α is an F_σ set in $\widetilde{X}$, it follows (see, for example, Montgomery, [2]) that $H = \bigcup \{H_\alpha\colon 1 \leqslant \alpha \leqslant \xi\}$ is also an F_σ set in $\widetilde{X}$. Hence $\widetilde{X} - H$ is completely metrizable.

We note next that $H \cap X$ is empty, so $\widetilde{X} - H \supseteq X$. The proof is by transfinite induction. By Lemma 3.3, $C_p(\widetilde{X}) \cap X \subseteq p_1(X)$, so $H_1 \cap X = \emptyset$. Assume $H_\beta \cap X = \emptyset$ for all $\beta < \alpha$. Suppose $\alpha = \beta+1$. Since $p_\beta(X)$ is closed in X, $C_p(\overline{p_\beta(X)}) \cap X = C_p(\overline{p_\beta(X)}) \cap p_\beta(X)$, which, by Lemma 3.3, is a subset of $p_\alpha(X)$; so $H_\alpha \cap X = \emptyset$. If α is a limit ordinal, then, for each $\eta < \alpha$,

$$C_p(\overline{\bigcap_{\beta<\alpha} p_\beta(X)}) \cap X \subseteq C_p(\overline{p_\eta(X)}) \cap X \subseteq p_{\eta+1}(X),$$

and so

$$\bigcap_{\eta<\alpha} p_{\eta+1}(X) = \bigcap_{\eta<\alpha} p_\eta(X) = p_\alpha(X) \supseteq C_p(\overline{\bigcap_{\beta<\alpha} p_\eta(X)}) \cap X.$$

Therefore $H_\alpha \cap X = \emptyset$.

Let d be a complete metric on $\widetilde{X} - H$. Then $(\widetilde{X} - H, d)$ is a completion of X, so, by hypothesis, $|\widetilde{X} - H| = k^{\aleph_0}$. It follows from Theorem 2.1 that there is a subspace B of $\widetilde{X} - H$ homeomorphic to $B(p)$. We claim that $B \subseteq \overline{p_\alpha(X)}$ for each α. This is trivial for $\alpha = 0$; assume it for all $\beta < \alpha \leqslant \xi$. If $\alpha = \beta+1$, then $B \subseteq C_p(\overline{p_\beta(X)})$. But no point of B is in H_α, so $B \subseteq \overline{p_\alpha(X)}$. Similarly, if α is a limit ordinal, $B \subseteq C_p(\overline{\bigcap_{\beta<\alpha} p_\beta(X)})$ and, since no point of B is in H_α, $B \subseteq \overline{p_\alpha(X)}$.

Hence $\overline{p_\alpha(X)} \neq \emptyset$ for each $\alpha \leqslant \xi$; in particular $\overline{p_\xi(X)}$, and hence $p_\xi(X) = K_p(X)$, is non-empty.

Conversely, suppose $K = K_p(X) \neq \emptyset$. It clearly suffices to show that any completion of K has cardinality $k^{\aleph_0}$. Let $\widetilde{K}$ be a completion and let $m \leqslant k$ be the least cardinal of a non-empty open set in $\widetilde{K}$. By a theorem of Schmidt ([3]), $m^{\aleph_0} = m$. But every non-empty open set in $\widetilde{K}$ meets K in $\geqslant p$ points, so $k^{\aleph_0} \geqslant m \geqslant p$, and hence $m = k^{\aleph_0}$. Therefore $|\widetilde{K}| = k^{\aleph_0}$.

COROLLARY 3.5. *Every completion of a countable space X has cardinality c if and only if the perfect kernel of X is non-empty. Equivalently, some completion is countable if and only if X is scattered.*

COROLLARY 3.6. *For a completely metrizable space of weight and cardinality k $(k^{\aleph_0} > k)$, $K_p(X) = \emptyset$.*

As before, if the generalized continuum hypothesis is assumed, p may be replaced by k throughout.

We have remarked that under our assumptions on k and p, $K_p(X)$, in the terminology of [6], is the "non-locally of weight $<p$" kernel of X. In [6] (Theorem 4′), it is shown that if $K_p(X) = \emptyset$, then X is a count-

able union of closed subsets of local weight $<p$. However even if X is σ-discrete, we need not have $K_p(X) = \emptyset$, as Example 3.7 will show.

In the light of Theorem 2.1, and the remarks following Theorem 2.3, one might ask whether, with the given assumptions on X, k, and p, the preceding theorem may be extended to say that $K_p(X) \neq \emptyset$ if and only if every absolute Borel set Y in which X is densely embedded has cardinality $k^{\aleph_0}$. Example 3.7 also shows that this is false.

EXAMPLE 3.7. This space was constructed in [7] for different, but related, purposes. With the usual assumptions on k and p, let T_n be a discrete space of cardinality k, and fix $a_n \in T_n$. In $B(k) = \prod_{n=1}^{\infty} T_n$, with the "first difference metric", let $D_m = \{x \in B(k)\colon x_i = a_i \text{ if } i > m\}$. Two distinct points of D_m are at distance at least $1/m$, so D_m is a closed discrete subspace of $B(k)$. It follows that $D = \bigcup_{n=1}^{\infty} D_m$ is an absolute Borel (in fact, absolute F_σ) set of weight and cardinal k. But every point of D is a k-limit point of D, so $K_p(X) \supseteq K_k(X) \neq \emptyset$.

Added in proof. Part of Theorem 2.3 occurs in Bel'nov, *On metric extensions*, Soviet Math. Dokl. 13 (1972), pp. 220–224.

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Inducing approximations homotopic to maps between inverse limits

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Abstract. Fort, McCord, J. T. Rogers, and Tollefson have shown that maps between the limit spaces of certain types of inverse systems are ε-homotopic to maps which are induced by maps between the coordinate spaces of the inverse systems, for each $\varepsilon > 0$. This result is extended here to a much wider, but still restricted, class of inverse systems, and an example is given to show the need of the remaining restrictions.

1. Introduction. We denote an inverse system with directed set D, coordinate spaces X_d, bonding maps $f_d^e\colon X_e \to X_d$, and projection maps $f_d\colon X_\infty \to X_d$ for all d and all $e \leqslant d$ in D, by (X, f, D). If f_d^e maps X_e onto X_d for all d and all $e \leqslant d$ in D, then we call (X, f, D) a *proper* inverse sequence. The reader is referred to [3] for definitions and basic properties of inverse limits. If (P, g, N) is an inverse system such that N denotes the set of all positive integers, and for each n, P_n is a polyhedron with (finite) triangulation K_n, and g_n^{n+1} is a simplicial map relative to (K_{n+1}, K_n), then (P, g, N) is called a *uniformly simplicial inverse sequence*, and is also denoted by (P, K, g, N). Both the solenoidal sequences of [3] and the weak solenoidal sequences of [7] are very restricted special cases of uniformly simplicial inverse sequences.

If (X, f, D) and (Y, g, E) are inverse systems, and $\varrho\colon E \to D$ is order preserving, and for each e in E there is a map $\varphi_e\colon X_{\varrho(e)} \to Y_e$ such that for all $i \leqslant e$ in E, $\varphi_i f_{\varrho(i)}^{\varrho(e)} = g_i^e \varphi_e$, then the map $\varphi\colon X_\infty \to Y_\infty$ defined by the equations $g_e \varphi = \varphi_e f_{\varrho(e)}$, for all e in E, is called an *induced map*. In Theorem 4 we generalize the results of [3] and [7] by showing that if (X, f, D) is an inverse system of compact Hausdorff spaces and (P, K, g, N) is a uniformly simplicial inverse sequence, then every map $F\colon X_\infty \to P_\infty$ is ε-homotopic to an induced map for each $\varepsilon > 0$ (i.e. no point is moved more than ε during the homotopy). An example in the last section shows the theorem does not hold if the assumption that (P, K, g, N) is uniformly simplicial is dropped. Other results related to these may be found in [6] and [7].

2. Preliminary theorems. For undefined terms and notation in this section, refer to [3], Chapter II. If K is a simplicial complex, a *simple subdivision* of K is a complex K' whose vertices consist of just one point