else contains B. But each of these is impossible since $d > 4\varepsilon$ and by condition (3), $\operatorname{dia} B > d$ and by (4), $\operatorname{dia} D > d$. Hence for each i, $\gamma_i = \beta_i'$ and thus $V_i = U_i$ and $\operatorname{dia} U_i < 4\varepsilon$. For each i, let U_i' be a neighborhood of $\overline{U}_i$ in S^2 such that $\operatorname{dia} U_i' < 4\varepsilon$ and $U_i' \cap (\operatorname{Bd} A \cup \operatorname{Bd} B) = J \cup J'$.

Since there is a homeomorphism of S^2 onto itself which takes J and J' onto a pair of concentric circles and each a_i onto a radial interval joining these circles, it is easily seen that for every neighborhood V of $\overline{U}$ in S^2, there is an isotopy $f\colon S^2 \times I \to S^2$ such that $f_0 = \mathrm{id}$, $f_1(J) = J'$, and for each $t \in I$, (i) f_t is the identity on $S^2 - V$ and (ii) for each i, $f_t(U_i') \subset U_i'$. Since $\operatorname{dia} U_i' < 4\varepsilon$, this last condition implies that for each t, f_t is a 4ε-homeomorphism. Since V is an arbitrary neighborhood of $\overline{U}$, V may be chosen so that $\overline{V} \cap (\operatorname{Bd} A \cup \operatorname{Bd} B) = J \cup J'$ and $V \cap (P_1 \cup P_2 \cup \ldots \cup P_m) = \emptyset$.

Repeating the entire construction for each pair of corresponding boundary curves of A and B gives an isotopy $g\colon S^2 \times I \to S^2$ such that $g_0 = \mathrm{id}$, $g_1(\operatorname{Bd} A) = \operatorname{Bd} B$, and for each $t \in I$, g_t is a 4ε-homeomorphism which is the identity on $P_1 \cup \ldots \cup P_m$. Since $g_1(\operatorname{Bd} A) = \operatorname{Bd} B$ and $g_1(A)$ does not intersect $P_1 \cup \ldots \cup P_m$, it follows that $g_1(A) = B$. Since each g_t is a 4ε-homeomorphism, it follows from condition (5) that $g_t(A) \in \mathcal{U}$ for each $t \in I$. By Lemma 4.2 of [1], there is an arc from A to B lying in $\{g_t(A) |\ t \in I\}$, and hence there is an arc from A to B in $\mathcal{U}$, as required.

References

[1] B. J. Ball and Jo Ford, *Spaces of ANR's*, Fund. Math. 77 (1972), pp. 33–49.

[2] K. Borsuk, *On some metrizations of the hyperspace of compact sets*, Fund. Math. 41 (1954), pp. 168–202.

UNIVERSITY OF GEORGIA
Athens, Georgia
AUBURN UNIVERSITY
Auburn, Alabama

Reçu par la Rédaction le 2. 10. 1971

Embedding certain compactifications of a half-ray

by

Sam B. Nadler, Jr. (1) and **J. Quinn** (New Orleans, Louisiana)

Abstract. Two problems concerning embedding compactifications of a half-ray are stated and partially answered. In our investigations of these problems, certain types of continua are completely determined.

1. Introduction. Throughout this paper, *continuum* will mean a compact connected metric space containing more than one point. The question of what spaces can be remainders in compactifications of certain kinds of spaces has been of interest (see, for example, [1], [7], [8], [13], [15], and [16]). In [1], Aarts and Van Emde Boas showed that any continuum can be the remainder in some compactification of a given locally compact non-compact separable metric space. This implies, of course, that any continuum can be the remainder in some compactification of a half-ray (a half-ray is a topological space homeomorphic to $[0, +\infty)$). Compactifications of a half-ray have been studied by D. Bellamy [2], M. E. Rudin [14], Simon [15] (where the main aspect of a result in [16] was proved for the special case of a half-ray), and others. In [11, Lemma 5.6] we proved a result, a very special case of which is

LEMMA A. *If Σ is an arcwise connected circle-like continuum* (see [10]), *then any compactification of a half-ray with Σ as the remainder is embeddable in the plane.*

This result and others mentioned above, as well as our theorem in section 2 of this paper, raise for us the following questions.

PROBLEM 1. What continua K have the property that *there is* a compactification of a half-ray, with K as the remainder, such that the compactification is embeddable in R^n (Euclidean n-space)? Clearly, such continua are embeddable in R^n and have dimension less than n [5], p. 44.

PROBLEM 2. What continua K have the property that

(α_n) *any* compactification of a half-ray, with K as the remainder, is embeddable in R^n?

(1) The first author was partially supported by a Loyola Faculty Research Grant.

In section 2 of this paper we show that if a continuum is embeddable in R^n, then *any* compactification of a half-ray, with that continuum as the remainder, is embeddable in R^{n+1}. This shows that, for each n, the problems above remain unsolved only for those continua of dimension less than n which are not embeddable in R^{n-1}. In section 3 we answer a special case of Problem 2, namely the case when the continua are assumed to be arcwise connected and $n=2$ (see Theorem 3 below). In the process we determine (Theorem 2 below), for the first time as far as we know, precisely those continua which are arcwise connected and a-triodic (for the general definition of a-triodic, see [3]; actually, we use a generally weaker form of a-triodicity, namely that of containing no half-ray triod).

We intend, in later papers, to consider Problem 1 and more general aspects of Problem 2. In particular, we plan to investigate Problem 2 thoroughly for the case when $n=2$ without the restriction of arcwise connectedness on the continua. In this connection, note that there are continua which are not arcwise connected and which satisfy (α_2). For example, any chainable continuum satisfies (α_2) because the compactification is again chainable, hence embeddable in the plane [3].

In this paper we will use the terms *simple closed curve* and *circle* interchangeably to mean a space homeomorphic to $\{z \text{ in the plane: } |z|=1\}$. By *circle-like* we mean a metric continuum which is an inverse limit of circles with *onto* bonding maps (equivalently, a circularly chainable continuum in the sense of [4]). The symbol "$\bar{S}$" means the closure of S.

2. Embedding in R^{n+1}. In this section we prove the following

THEOREM 1. *If K is a continuum which is embeddable in R^n, then any compactification of a half-ray, with K as the remainder, is embeddable in R^{n+1}.*

Proof. Let K be a continuum which is embeddable in R^n and let $K\cup H$ be a compactification of a half-ray H with K as the remainder. Let $h\colon [0,+\infty)\to H$ be a homeomorphism of $[0,+\infty)$ onto H and, for each $i=1,2,\ldots$, let $D_i=\{h(i-1)=x_0^i, x_1^i, \ldots, x_{m(i)-1}^i=h(i)\}$ be a finite subset of $h([i-1,i])$ such that $h^{-1}(x_0^i)=i-1<h^{-1}(x_1^i)<\ldots<h^{-1}(x_{m(i)-1}^i)=i$ and such that the diameter of the arc in H with noncut points x_j^i and x_{j+1}^i, $0\leqslant j<m(i)-1$, is less than $1/2^i$. Also for each $i=1,2,\ldots$, let z_j^i, $0\leqslant j\leqslant m(i)-1$, be a point of K nearest (with respect to the metric for $K\cup H$) to x_j^i. Assume $z_0^{i+1}=z_{m(i)-1}^i$ for each $i=1,2,\ldots$ Let $e\colon K\to R^n$ be an embedding of K in R^n. For each $i=1,2,\ldots$, let y_j^i, $0\leqslant j\leqslant m(i)-1$, be the point in R^{n+1} of height $\dfrac{1}{i}-\dfrac{j}{m(i)-1}\cdot\dfrac{1}{i\cdot(i+1)}$ "above" $e(z_j^i)$ (thus, the first n coordinates of y_j^i are the same as the first n coordinates of $e(z_j^i)$ and the $(n+1)$-st coordinate of y_j^i is $\dfrac{1}{i}-\dfrac{j}{m(i)-1}\cdot\dfrac{1}{i\cdot(i+1)}$). Let $X=\{x_j^i\colon i=1,2,\ldots$ and, for each i, $0\leqslant j\leqslant m(i)-1\}$ and let $f\colon K\cup X\to R^{n+1}$ be given by

$$f(p)=\begin{cases} e(p), & p\in K,\\ y_j^i, & p=x_j^i.\end{cases}$$

We show that f is continuous. To see this (it suffices to) let $\{x_{j(k)}^{i(k)}\}$ be a sequence of points in X converging to a point $z_0\in K$. It then follows that $\{z_{j(k)}^{i(k)}\}$ also converges to z_0. Thus, $\{e(z_{j(k)}^{i(k)})\}$ converges to $e(z_0)$ which, since $y_{j(k)}^{i(k)}$ is only distance $\dfrac{1}{i(k)}-\dfrac{j(k)}{m(i(k))-1}\cdot\dfrac{1}{i(k)\cdot(i(k)+1)}$ away from $e(z_{j(k)}^{i(k)})$, implies $\{y_{j(k)}^{i(k)}\}$ converges to $e(z_0)$. This proves that $\{f(x_{j(k)}^{i(k)})\}$ converges to $f(z_0)$. Therefore, we have shown that f is continuous (hence, by compactness of $K\cup X$, uniformly continuous). Now let H' be the half-ray in R^{n+1} defined by the homeomorphism $h'\colon [0,+\infty)\to H'$, h' given by: Let $s\in[0,+\infty)$. There is a unique natural number i such that $i-1\leqslant s<i$. Also, there is a unique j, $0\leqslant j<m(i)-1$, such that $h^{-1}(x_j^i)\leqslant s<h^{-1}(x_{j+1}^i)$. Finally, there is a unique t, $0\leqslant t\leqslant 1$, such that $s=t\cdot h^{-1}(x_j^i)+(1-t)h^{-1}(x_{j+1}^i)$. Let $h'(s)=t\cdot y_j^i+(1-t)\cdot y_{j+1}^i$. Define $g\colon K\cup H\to e(K)\cup H'$ by

$$g(p)=\begin{cases} e(p), & p\in K,\\ h'(h^{-1}(p)), & p\in H.\end{cases}$$

Note that g restricted to $K\cup X$ agrees with f. It is easy to see that g is one-to-one and onto $e(K)\cup H'$. The continuity of g follows from the uniform continuity of f. To see this, note that it suffices to take a sequence $\{p_k\}_{k=1}^{\infty}$ of points in H converging to a point $z_0\in K$ and show that $\{g(p_k)\}_{k=1}^{\infty}$ converges to $g(z_0)$. But each such p_k is on an arc in H with noncut points of the form $x_{j(k)}^{i(k)}$ and $x_{j(k)+1}^{i(k)}$. Hence, $g(p_k)$ is on the convex arc $A_{j(k)}^{i(k)}$ in H' from $y_{j(k)}^{i(k)}$ to $y_{j(k)+1}^{i(k)}$. Note that the sequence $\{x_{j(k)}^{i(k)}\}$ converges to z_0 (because $x_{j(k)}^{i(k)}$ is within $1/2^{i(k)}$ of p_k, for each k). Thus, since $x_{j(k)}^{i(k)}$ is within $1/2^{i(k)}$ of $x_{j(k)+1}^{i(k)}$ and since f is uniformly continuous (so that the diameter of $A_{j(k)}^{i(k)}$ goes to zero as k goes to infinity), we have that $\{g(p_k)\}_{k=1}^{\infty}$ converges to $g(z_0)$. This completes the proof of Theorem 1.

3. Property (α_2) for arcwise connected continua. In order to determine those arcwise connected continua which satisfy (α_2), we first prove the following lemma.

LEMMA 1. *If a continuum satisfies (α_2), then the continuum does not contain a figure "T" (a figure "T" is defined to be a space homeomorphic to*

$$\{(x,y)\in R^2\colon x=0 \text{ and } 0\leqslant y\leqslant +1\}\cup\{(x,y)\in R^2\colon -1\leqslant x\leqslant +1 \text{ and } y=+1\}).$$

Proof. Let K be a continuum and assume K contains a figure "T" which we denote by T. We will construct a half-ray H in R^3 whose closure in R^3 will be a compactification of H with K as the remainder such that $K \cup H$ is not embeddable in the plane. Without loss of generality we may assume that $K \subset \{(x, y, 0): x, y \in R^1\} \subset R^3$. Furthermore (see [12], p. 170; we use this device here to simplify our notation), we may assume that $T = \beta \cup \gamma$ where β is the convex arc in R^3 with noncut points $(-1, 0, 0)$ and $(1, 0, 0)$ and γ is the convex arc in R^3 with noncut points $(0, 0, 0)$ and $(0, 1, 0)$. Let T' be the figure "T" contained in T with noncut points $(-1/2, 0, 0)$, $(1/2, 0, 0)$, and $(0, 1/2, 0)$. For each $n = 1, 2, \ldots$, let P_n be the hyperplane in R^3 defined by $P_n = \{(x, y, 1/n): x, y \in R^1\}$. Some subarcs of the half-ray H will lie in the hyperplanes and other subarcs of H will be convex arcs joining points of consecutively indexed hyperplanes. For each odd $n = 1, 3, 5, \ldots$, let T'_n be the subset of P_n given by $T'_n = \{(x, y, 1/n): (x, y, 0) \in T'\}$. For each even $n = 2, 4, 6, \ldots$, let K_n be the subset of P_n given by $K_n = \{(x, y, 1/n): (x, y, 0) \in K\}$. Now, for each $n = 1, 2, \ldots$, choose J_n to be a polygonal arc lying in P_n such that

(1) J_n has noncut points $(-1/2, 0, 1/n)$ and $(1/2, 0, 1/n)$

and

(2) if n is odd, then $\varrho(J_n, T'_n) < 1/n$ and, if n is even, then $\varrho(J_n, K_n) < 1/n$, where ϱ denotes the Hausdorff metric for the nonempty compact subsets of R^3 [6], p. 131.

For $n = 1, 2, \ldots$, let λ_n be the convex arc in R^3 with noncut points $\big((-1)^{n+1}\cdot 1/2, 0, 1/n\big)$ and $\left((-1)^{n+1}\cdot 1/2, 0, \dfrac{1}{n+1}\right)$. Let $H = \bigcup_{n=1}^{\infty} (J_n \cup \lambda_n)$. It is easy to see that H is a half-ray and that $K \cup H$ is a compactification of H with K as the remainder. We now show that $K \cup H$ is not embeddable in the plane. Suppose that $K \cup H$ is embeddable in the plane and let $h: K \cup H \to R^2$ be a planar embedding of $K \cup H$. We may assume (see [12], p. 170) that $h(T) = \beta' \cup \gamma'$, where β' is the convex arc in R^2 with noncut points $(-1, 0)$ and $(1, 0)$ and γ' is the convex arc in R^2 with noncut points $(0, 0)$ and $(0, 1)$. Note that $h(T')$ must be a figure "T" contained in $h(T)$ and having noncut points of the form $(a, 0)$, $(b, 0)$, and $(0, c)$ with $-1 < a < 0 < b < 1$ and $0 < c < 1$. Let D be the region in the plane bounded by the quadrilateral whose vertices are $\left(\dfrac{-1+a}{2}, 0\right)$, $\left(\dfrac{b+1}{2}, 0\right)$, $\left(0, \dfrac{c+1}{2}\right)$, and $(0, -1)$. Then, D is an open subset of the plane containing $h(T')$ such that $D - h(T)$ has exactly three components. Since h is uniformly continuous and since $\varrho(J_{2k-1}, T') \to 0$ as $k \to +\infty$, we have that

$$(*) \qquad \varrho\big(h(J_{2k-1}), h(T')\big) \to 0 \quad \text{as} \quad k \to +\infty .$$

Hence, there exists N such that $k \geqslant N$ implies $h(J_{2k-1}) \subset D - h(T)$. Hence, for each $k \geqslant N$, $h(J_{2k-1})$ is contained in one of the three components of $D - h(T)$. This implies

$$\varrho\big(h(J_{2k-1}), h(T')\big) > \min\{-a, b, c\} > 0 ,$$

a contradiction to $(*)$. Lemma 1 is proved.

A *half-ray triod* [11] is defined to be a continuum which is the union of a half-ray H and an arc A such that $H \cap A = \emptyset$ and $\overline{H} - H$ is a subarc or a point of A which contains neither noncut point of A. The same techniques used in the proof of Lemma 1 can be used to prove the following slightly more general lemma.

LEMMA 2. *If a continuum satisfies* (α_2), *then the continuum does not contain a half-ray triod.*

Next we determine those continua which are arcwise connected and contain no half-ray triod (as we shall see, such continua are a-triodic in the more general sense of [3]). First we prove a special case.

LEMMA 3. *A continuum which is uniquely arcwise connected and contains no half-ray triod is either*

(1) *an arc or*

(2) *an arcwise connected circle-like continuum which is not a circle.*

Proof. In what follows, X denotes a uniquely arcwise connected continuum which contains no half-ray triod. For $x_1 \neq x_2$ in X, we let $a(x_1, x_2)$ denote the unique arc in X with noncut points x_1 and x_2. We will divide the proof into several parts.

Part (1). If α and β are any two arcs in X such that $\alpha \cap \beta \neq \emptyset$, then $\alpha \cup \beta$ is an arc in X. To see this, just note that, since $\alpha \cup \beta$ is a locally connected continuum containing no simple closed curve (by unique arcwise connectivity), $\alpha \cup \beta$ must be a dendrite [17], p. 88. Hence, since $\alpha \cup \beta$ does not contain a figure "T", $\alpha \cup \beta$ must be an arc.

Part (2). If Y is any subspace of X which is arcwise connected (hence, uniquely arcwise connected) and not contained in any arc in X, then Y contains a one-to-one continuous image of $[0, +\infty)$ which is also not contained in any arc in X. To prove this, let $D = \{d_1, d_2, \ldots\}$ be a countable dense subset of Y (with $d_i \neq d_j$ for $i \neq j$). For each $n = 2, 3, \ldots$, let $\sigma_n = \bigcup_{i=2}^{n} a(d_1, d_i)$. By Part (1), each σ_n is an arc. Let $x \in \sigma_2 - \{d_1, d_2\}$. Then, for each $n \geqslant 2$, x divides σ_n into two subarcs σ'_n and σ''_n, with x a noncut point of each. The primes are chosen so that $\sigma'_n \subset \sigma'_{n+1}$ and $\sigma''_n \subset \sigma''_{n+1}$ for each $n \geqslant 2$. If $\bigcup_{n=2}^{\infty} \sigma'_n$ were contained in an arc $B' \subset X$ and $\bigcup_{n=2}^{\infty} \sigma_n''$ were contained in an arc $B'' \subset X$, then $(\bigcup_{n=2}^{\infty} \sigma_n') \cup (\bigcup_{n=2}^{\infty} \sigma_n'')$ would be contained in the arc $B' \cup B''$ (since $x \in B' \cap B''$, Part (1) can be applied to see that $B' \cup B''$ is an arc). Since D is a dense subset of Y and $D \subset (\bigcup_{n=2}^{\infty} \sigma_n') \cup (\bigcup_{n=2}^{\infty} \sigma_n'')$, we can now conclude that $Y \subset B' \cup B''$, a contradiction. Hence, without loss of generality, we may assume $\bigcup_{n=2}^{\infty} \sigma_n'$ is not contained in an arc in X. It is easy to verify that $\bigcup_{n=2}^{\infty} \sigma_n'$ is a one-to-one continuous image of $[0, +\infty)$, which completes the proof of Part (2).

Part (3). If $A = f([0, +\infty))$ is an image in X of $[0, +\infty)$, under a one-to-one continuous function f, and if $x \neq f(0)$ is a point in X, then $\alpha(x, f(0))$ and A satisfy one of the following conditions:

(a) $\alpha(x, f(0)) \subset A$,

(b) $\alpha(x, f(0)) \cap A = \{f(0)\}$, or

(c) $A \subset \alpha(x, f(0))$.

In addition: if (a) or (b) holds, then $\alpha(x, f(0)) \cup A$ is itself a one-to-one continuous image of $[0, +\infty)$. To see that (a), (b), or (c) must hold, first note that if $x \in A$ with $x = f(t)$, then the uniqueness of arcs gives us that $\alpha(x, f(0)) = f([0, t]) \subset A$, i.e., (a) holds. Thus, we may now assume that $x \notin A$. Also, assume (b) does not hold. Then there exists a $t_0 > 0$ such that $f(t_0) \in \alpha(x, f(0))$. Since $f(0)$ is a noncut point of $\alpha(x, f(0))$ and since $x \notin A$, $\{t\colon f(t) \in \alpha(x, f(0))\}$ is not bounded above (if it were bounded above, then $\alpha(x, f(0)) \cup A$ would contain a figure "T"). From the uniqueness of arcs we also have that if $f(s)$ and $f(t)$, $s \leqslant t$, are each in $\alpha(x, f(0))$, then $f([s, t]) \subset \alpha(x, f(0))$. It now follows easily that $A \subset \alpha(x, f(0))$. This completes the proof that $\alpha(x, f(0))$ and A satisfy (a), (b), or (c). The proof that if (a) or (b) holds, then $\alpha(x, f(0)) \cup A$ is itself a one-to-one continuous image of $[0, +\infty)$ is trivial and is omitted.

Part (4). If $G = g(R^1)$ is an image in X of R^1 under a one-to-one continuous function g, then $g((-\infty, 0])$ is contained in an arc in X or $g([0, +\infty))$ is contained in an arc in X. To establish this, we first note that $G \neq X$, for otherwise X would be a real curve [11] and would have to contain a half-ray triod. Thus, let $x \in X - g(R^1)$ and consider $\alpha(x, g(0))$. By virtue of Part (3) and the fact that $x \notin G$, we will be done if we can show that either $\alpha(x, g(0)) \cap g([0, +\infty)) \neq \{g(0)\}$ or $\alpha(x, g(0)) \cap g((-\infty, 0]) \neq \{g(0)\}$. But if $\alpha(x, g(0)) \cap g([0, +\infty)) = \{g(0)\}$ and $\alpha(x, g(0)) \cap g((-\infty, 0]) = \{g(0)\}$, then $T = \alpha(x, g(0)) \cup g([-1, 1])$ would be a figure "T" in X, which is a contradiction to X not containing a half-ray triod. This proves Part (4).

Part (5). Given two disjoint one-to-one continuous images of $[0, +\infty)$ in X, one or the other of them is contained in an arc in X. To prove this, let $A_1 = f_1([0, +\infty))$ and $A_2 = f_2([0, +\infty))$ be two disjoint images in X of $[0, +\infty)$ under one-to-one continuous functions f_1 and f_2 respectively. Since $A_1 \cap A_2 = \emptyset$, (a) of Part (3) can not hold for $\alpha(f_1(0), f_2(0))$ and either of A_1 or A_2. If (c) of Part (3) holds for $\alpha(f_1(0), f_2(0))$ and some A_i, $i = 1$ or $i = 2$, then we are done. Therefore, we may assume

$$\alpha(f_1(0), f_2(0)) \cap A_i = \{f_i(0)\}$$

for each $i = 1, 2$. But, under these assumptions, $\alpha(f_1(0), f_2(0)) \cup A_1 \cup A_2$ is easily seen to be a one-to-one continuous image of R^1 and the desired conclusion is a consequence of Part (4). This proves Part (5).

Part (6). We now complete the proof of Lemma 3. Assume that X is not an arc, i.e., does not satisfy (1) in the statement of Lemma 3. Then we may apply Part (2), with $Y = X$, to see that X contains an image $S = g([0, +\infty))$ of $[0, +\infty)$, under a one-to-one continuous function g, such that S is not contained in any arc in X. Using Part (3) and the fact that S is not contained in any arc in X, we can establish that if $x_1, x_2 \in (X - S)$, then $\alpha(x_1, x_2) \cap S = \emptyset$ (here we apply Part (3) to $\alpha(x_i, g(t))$, $i = 1$ and 2, and $g([t, +\infty))$ if $g(t) \in \alpha(x_1, x_2)$). Note that $X - S$ can not be a single point p; if so, then Part (3) and the fact that S is not contained in an arc would imply that $\alpha(p, g(0)) \cap S = \{g(0)\}$, a contradiction. Thus, $X - S$ is an arcwise connected subspace of X or $X - S = \emptyset$. But now we can conclude that $X - S$ is contained in an arc in X; otherwise, we could apply Part (2) to $X - S$ to obtain a one-to-one continuous image E of $[0, +\infty)$ in $X - S$ such that E is also not contained in any arc in X (since $E \cap S = \emptyset$, this would contradict Part (5)). So, let $\alpha(z_1, z_2)$ be an arc in X such that $X - S \subset \alpha(z_1, z_2)$. Note that $(*)$ $S \cup \alpha(z_1, z_2) = X$. We consider two cases involving z_1, z_2, and S. First, assume one of the points, z_1 or z_2, is not in S; without loss of generality, we assume $z_1 \notin S$. Then, applying Part (3) to $\alpha(z_1, g(0))$ and S, we obtain that $\alpha(z_1, g(0)) \cap S = \{g(0)\}$. Hence, by Part (3), $\alpha(z_1, g(0)) \cup S$ is itself a one-to-one continuous image of $[0, +\infty)$ under a function k (note that $k(0) = z_1$). We also have $\alpha(z_1, g(0)) \cup S = X$ because, if not, then (by $(*)$) $z_2 \notin (\alpha(z_1, g(0)) \cup S)$ and thus, applying Part (3) to $\alpha(z_2, k(0))$ and $k([0, +\infty))$, we see that $\alpha(z_1, g(0)) \cap \alpha(z_1, z_2) = \{z_1\}$, a contradiction to $(*)$ since $z_1 \neq g(0)$. Thus, X is a one-to-one continuous image of $[0, +\infty)$. Next, assume $z_1 \in S$ and $Z_2 \in S$. Then, $\alpha(z_1, z_2) \subset S$ which implies $X - S = \emptyset$, i.e., $X = S$ and again X is a one-to-one continuous image of $[0, +\infty)$. Therefore, we have shown (in either case) that X is a continuum which is a one-to-one continuous image of $[0, +\infty)$. Since X contains no half-ray triod, we can now conclude from the Structure Theorem in [9] that X must be an arcwise connected circle-like continuum. Since X is uniquely arcwise connected, X is not a circle. This completes the proof of Lemma 3.

THEOREM 2. *A continuum X is arcwise connected and contains no half-ray triod if and only if X is either*

(1) *an arc or*

(2) *an arcwise connected circle-like continuum.*

Proof. Let X be an arcwise connected continuum which contains no half-ray triod. Assume X contains a simple closed curve C. If there is a point $p \in (X-C)$, then let γ be an arc in X with p and some point of C as its two noncut points. Clearly, $C \cup \gamma$ contains a figure "T", which contradicts that X contains no half-ray triod. Thus, $C = X$. Next assume X does not contain a simple closed curve. Then X is uniquely arcwise connected and Lemma 3 applies. The other half of Theorem 2 is a consequence of the well-known fact that a circle-like continuum is a-triodic [4].

Remark. In particular, Theorem 2 above and a result in [10] show that an arcwise connected continuum which contains no half-ray triod is embeddable in the plane. Particularly nice embeddings of such continua can be obtained using results in [9] and [11].

Now we answer Problem 2 for the case of arcwise connected continua and $n = 2$.

THEOREM 3. *An arcwise connected continuum satisfies* (α_2) *if and only if it is an arc or an arcwise connected circle-like continuum.*

Proof. Half of the theorem follows from Lemma 2 and Theorem 2. The other half of the theorem is a consequence of Lemma A and Theorem 1 of section 2 (or the statement in the introduction that any chainable continuum, hence an arc, satisfies (α_2)).

COROLLARY 1. *A locally connected continuum satisfies* (α_2) *if and only if it is an arc or a simple closed curve.*

References

[1] J. M. Aarts and P. Van Emde Boas, *Continua as remainders in compact extensions*, Nieuw Archief voor Wiskunde 15 (1967), pp. 34–37.
[2] D. Bellamy, *Ph. D. Thesis*, Michigan State University 1968.
[3] R. H. Bing, *Snake-like continua*, Duke Math. J. 18 (1951), pp. 653–663.
[4] C. E. Burgess, *Chainable continua and indecomposability*, Pac. J. Math. 9 (1959), pp. 653–659.
[5] W. Hurewicz and H. Wallman, *Dimension Theory*, Princeton, New Jersey 1948.
[6] J. L. Kelley, *General Topology*, Princeton, New Jersey 1955.

[7] K. D. Magill, *A note on compactifications*, Math. Zeitschr. 94 (1966), pp. 322–325.
[8] — *More on remainders of spaces in compactifications*, Bull. Pol. Acad. Sci. 18 (1970), pp. 449–451.
[9] S. B. Nadler, Jr., *Continua which are a one-to-one continuous image of* $[0, \infty)$, Fund. Math. 75 (1972), pp. 123–133.
[10] — *Multicoherence techniques applied to inverse limits*, Trans. Amer. Math. Soc. 157 (1971), pp. 227–234.
[11] — and J. Quinn, *Embeddability and structure properties of real curves*, Memoirs of the Amer. Math. Soc. 125 (1972).
[12] M. H. A. Newman, *Elements of the Topology of Plane Sets of Points*, Cambridge, Mass. 1939.
[13] J. W. Rogers, Jr., *On compactifications with continua as remainders*, Notices Amer. Math. Soc. 16 (1969), p. 1045.
[14] M. E. Rudin, *Composants and βN*, Proceedings of the Washington State University Conference on General Topology, March, 1970, pp. 117–119.
[15] B. Simon, *Some pictorial compactifications of the real line*, Amer. Math. Monthly 76 (1969), pp. 536–538.
[16] A. K. Steiner and E. F. Steiner, *Compactifications as closures of graphs*, Fund. Math. 63 (1968), pp. 221–223.
[17] G. T. Whyburn, *Analytic Topology*, Amer. Math. Soc. Colloquium Publications, Providence, Rhode Island 1942.

LOYOLA UNIVERSITY
New Orleans, Louisiana
UNIVERSITY OF NORTH CAROLINA AT CHARLOTTE
Charlotte, North Carolina

Reçu par la Rédaction le 25. 11. 1971