On the integration of set-valued mappings in a Banach space

by

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Abstract. Let T be a locally compact Polish space with a finite nonatomic measure μ defined on it and X a real separable reflexive Banach space. If P is an integrably bounded set-valued mapping from T to X and $\overline{\text{co}}\, P(t)$ denotes the closed convex hull of $P(t)$, then the equality

$$\text{cl}\Big(\int_T P(t)\,d\mu(t)\Big) = \int_T \overline{\text{co}}\, P(t)\,d\mu(t)$$

holds.

Let T and X be two point sets. A mapping P from T into the subsets of X is called a *set-valued mapping* if to every t in T the mapping P associates a definite subset of X. For such mappings various notions of measurability, continuity and differentiability have been defined and a Calculus and Measure Theory have been developed. In this note a generalization of Lemma 7 from [4] and Theorem 4 from [1] will be proved.

X will denote a separable reflexive Banach space over the real numbers R and X' its topological dual. The norm in X and X' will be denoted by $\|\cdot\|$. The symbol 2^X will denote the family of all subsets of X and $K(X)$ will denote the family of all closed bounded nonempty subsets of X. If $P \subset X$ then $\text{co}\, P$ will stand for the convex hull of P, $\overline{\text{co}}\, P$ the closed convex hull of P and $\text{cl}(P)$ the closure of P in the normed topology. The symbol μ will denote a nonatomic positive Borel measure on a locally compact Polish space T with $\mu(T) < \infty$. The symbol $L(T, X)$ will stand for the Banach space of all X-valued functions Bochner integrable with respect to μ.

Definition 1. A set-valued mapping $P\colon T \to 2^X$ is said to be *measurable* if for each closed subset $A \subset X$ the set $P^-(A) = \{t \in T\colon P(t) \cap A \neq \emptyset\}$ is measurable.

Definition 2. A set-valued mapping $P\colon T \to 2^X$ is said to be *integrably bounded* if there exists $g \in L(T, R)$ such that $\sup\{\|x\|\colon x \in P(t)\} \leqslant g(t)$ a.e. on T.

DEFINITION 3. A measurable mapping $\sigma\colon T\to X$ is called a *measurable cross section of* $P\colon T\to 2^X$ if $\sigma(t)\in P(t)$ a.e. on T. Let $\mathcal{F}$ denote the family of all measurable cross sections of P. The integral of a set-valued mapping P is defined for any measurable set $A\subset T$ by the expression $\int_A P(t)\,d\mu(t) = \{\int_A \sigma(t)\,d\mu(t)\colon\ \sigma\in\mathcal{F}\}$.

Remark 1. The measurability used in Definition 3 is that with respect to the Borel subsets of X. However since X is a separable Banach space this notion coincides with the notions of strong and weak measurability. The integral used throughout is the Bochner integral.

The following lemma is a version of a result due to C. Castaing ([3], Theorem 2). However, due to the fact that Castaing's result is not generally known and is not stated in exactly the manner in which we wish to apply it, the proof will be given.

LEMMA. *For every* $g\in L(T,R)$ *the set* $\{f\in L(T,X)\colon \|f(t)\|\leqslant g(t)$ *for* $t\in T\}$ *is compact in the weak topology of* $L(T,X)$.

Proof. Let $L^\infty(T,X)$ denote the Banach space of all essentially bounded measurable mappings from T into X. Since X is reflexive and $\mu(T)<\infty$ the strong dual of $L(T,X)$ is isomorphic to $L^\infty(T,X')$ and that of $L(T,X')$ to $L^\infty(T,X)$ (see e.g. [5], p. 590).

If we consider the weak* topology $\sigma\big(L^\infty(T,X),L(T,X')\big)$ on $L^\infty(T,X)$ and the weak topology $\sigma\big(L(T,X),L^\infty(T,X')\big)$ on $L(T,X)$, then in terms of these topologies the mapping $\varphi\colon L^\infty(T,X)\to L(T,X)$ defined by $\varphi(f)=gf$ is continuous. Thus the image of the unit ball of $L^\infty(T,X)$ under φ is weakly compact in $L(T,X)$. However this set is identical with the set in the statement of the Lemma.

THEOREM. *Let* $P\colon T\to K(X)$ *be an integrably bounded measurable set-valued mapping. Then for any measurable set* $A\subset T$ *the equation* $\mathrm{cl}\big(\int_A P(t)\,d\mu(t)\big)=\int_A \overline{\mathrm{co}}\,P(t)\,d\mu(t)$ *holds. Moreover the set* $\int_A \overline{\mathrm{co}}\,P(t)\,d\mu(t)$ *is closed bounded and convex.*

Remark 2. If P is as in the Theorem, then for any measurable set A the set $\int_A P(t)\,d\mu(t)$ is nonempty (see e.g. Remark 1 in [4]). The measurability of P implies the measurability of the set-valued mapping $t\to\overline{\mathrm{co}}\,P(t)$ (see e.g. Lemma 7 in [4]).

Proof of the Theorem. Clearly $\int_A \overline{\mathrm{co}}\,P(t)\,d\mu(t)$ is convex and bounded. We shall show it is also closed. From this fact the inclusion $\mathrm{cl}\big(\int_A P(t)\,d\mu(t)\big)\subset\int_A \overline{\mathrm{co}}\,P(t)\,d\mu(t)$ follows immediately. Let $\{r_n\}\subset\int_A \overline{\mathrm{co}}\,P(t)\,d\mu(t)$ be a Cauchy sequence. Then there exists $\{\sigma_n\}\subset\overline{\mathrm{co}}\,P$ such that for each n $r_n=\int_A \sigma_n(t)\,d\mu(t)$. By a theorem due to Šmulian (see e.g. [5], Theorem 8.12.1) and the Lemma there exists a subsequence $\{\sigma_q\}\subset\{\sigma_n\}$ and a mapping $\sigma\in L(T,X)$ such that $\{\sigma_q\}\to\sigma$ weakly. Hence $\lim_n r_n = \lim_q \int_A \sigma_q(t)\,d\mu(t) = \int_A \sigma(t)\,d\mu(t)$ and $\sigma(t)\in\overline{\mathrm{co}}\,P(t)$ a.e. on T. The last is because some sequence of convex combinations of $\{\sigma_q\}$ converges strongly to σ (see e.g. [5], Theorem 2.3.1).

We next show that $\mathrm{cl}\big(\int_A P(t)\,d\mu(t)\big)$ is convex. Let $\alpha\in[0,1]$ and $\varepsilon>0$ be given. Let g be as in Definition 2. Since g is integrable there exists a measurable set $A_1\subset A$ and a positive constant M such that $\int_{A-A_1} g(t)\,d\mu(t) < \frac{1}{3}\varepsilon$ and $g(t)\leqslant M$ on A_1. Let σ_1 and σ_2 be measurable cross sections of P such that $r_i=\int_A \sigma_i(t)\,d\mu(t)$, $i=1,2$. We can apply Property 1 of [3] to the family of all measurable cross sections of P restricted to A_1 and find a measurable cross section $\hat\sigma$ of P restricted to A_1 such that

$$\Big\|\alpha\int_{A_1}\sigma_1(t)\,d\mu(t)+(1-\alpha)\int_{A_1}\sigma_2(t)\,d\mu(t)-\int_{A_1}\hat\sigma(t)\,d\mu(t)\Big\|\leqslant\tfrac{1}{3}\varepsilon.$$

Let $\sigma(t)=\hat\sigma(t)$ for $t\in A_1$ and $\sigma(t)=\sigma_1(t)$ for $t\in A-A_1$. Clearly σ is a measurable cross section of P and

$$\begin{aligned}\Big\|\alpha r_1+(1-\alpha)r_2-\int_A\sigma(t)\,d\mu(t)\Big\| &\leqslant\Big\|\alpha\int_{A_1}\sigma_1(t)\,d\mu(t)+(1-\alpha)\int_{A_1}\sigma_2(t)\,d\mu(t)-\int_{A_1}\hat\sigma(t)\,d\mu(t)\Big\|+\\ &\quad+\Big\|\alpha\int_{A-A_1}\sigma_1(t)\,d\mu(t)+(1-\alpha)\int_{A-A_1}\sigma_2(t)\,d\mu(t)-\int_{A-A_1}\sigma_1(t)\,d\mu(t)\Big\|\\ &<\tfrac{1}{3}\varepsilon+(1-\alpha)\Big[\Big\|\int_{A-A_1}\sigma_1(t)\,d\mu(t)\Big\|+\Big\|\int_{A-A_1}\sigma_2(t)\,d\mu(t)\Big\|\Big]\leqslant\varepsilon.\end{aligned}$$

The convexity of $\mathrm{cl}\int_A P(t)\,d\mu(t)$ now follows by a proof analogous to that of Theorem 1 in [3], p. 209.

In view of Remark 2 the convexity of $\mathrm{cl}\big(\int_A P(t)\,d\mu(t)\big)$ allows us to apply the same argument as in [4] (see the proof of Lemma 7, p. 233) to obtain the inclusion. This completes the proof of the theorem.

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Spaces of ANR's. II

by

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Abstract. It was shown by K. Borsuk [2] that the set of all ANR's lying in a compactum X can be metrized in such a way that the resulting space, denoted by 2_h^X, is separable and complete, and reflects the homotopy character of the ANR's in X, in the sense that if $A \in 2_h^X$, then all ANR's sufficiently close to A in 2_h^X are homotopically equivalent to A. In a previous paper [1], the authors considered a number of topological properties of these hyperspaces and proved, in particular, that every two homotopically equivalent connected ANR's in the 2-sphere S^2 can be joined by an arc in $2_h^{S^2}$. In the present note, this result is improved by showing that the space $2_h^{S^2}$ is in fact locally connected. (It was shown in [1] that in general 2_h^X need not be locally connected, even if X is an absolute retract.)

It was shown by K. Borsuk [2] that the set of all ANR's lying in a finite dimensional compactum X can be metrized in such a way that the resulting space, 2_h^X, is separable and complete, and reflects the homotopy character of the ANR's in X (in the sense that if A is any ANR in X, then all ANR's sufficiently close to A in 2_h^X are homotopically equivalent to A). In a previous paper [1], the authors studied a number of topological properties of these hyperspaces; in particular, an example was given of a 2-dimensional absolute retract X such that 2_h^X is not locally connected, and it was shown that if $\mathcal{C}$ denotes the (open and closed) subspace of $2_h^{S^2}$ consisting of all connected ANR's in S^2, then every component of $\mathcal{C}$ is arcwise connected. It is the aim of the present paper to improve this latter result by showing that $2_h^{S^2}$ is in fact locally connected.

1. Definitions and notations. If X is a compactum with metric ϱ and A and B are closed subsets of X, we will, following Borsuk, denote the Hausdorff distance between A and B by $\varrho_s(A, B)$, and will let $\varrho_c(A, B)$ denote the greatest lower bound of the set of all positive numbers ε such that each of A and B can be mapped into the other by an ε map (i.e., a continuous function which moves no point a distance ε or more). The homotopy metric, ϱ_h, which determines the space 2_h^X, is defined only in case A and B are locally contractible; in this case, $\varrho_h(A, B) = \varrho_c(A, B) + \psi(A, B)$, where $\psi(A, B)$ is a non-negative function, defined in [2], whose precise nature does not concern us here. We will, however, need