a Cauchy filter relative to d. Thus, $\mathcal{F}^x$ is real. Now $P(X) \neq P^*(X)$, and by Theorem 3.2, every member of $P(X)$ may be extended to a member of $C(T)$. However, the function $f(x, y) = \sin(y^{-1})$ belongs to $C^*(X)$, but clearly has no continuous extension to T. Thus X is not C^*-embedded in T. We also observe that there is no compatible proximity on T for which (X, δ) is a p-subspace of T. (See Example 1 of [7].)

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A remark on the independence of a basis hypothesis

by

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Abstract. In the paper we prove the independence of a basis hypothesis used by Enderton and Friedman in the proof of the existence of a minimal β_n-model for analysis. The main result is the consistency of ZFC with the axiom

$$(a)_{P(\omega)}(ER)_{P(P(\omega))}[R \in \Pi^1_2[a] \;\&\; R \cap \mathrm{HOD}[a] = 0].$$

The aim of this paper is to prove the independence of a basis hypothesis used by Enderton and Friedman [1] in the proof of the existence of a minimal β_n-model for analysis.

The hypothesis is as follows:

(BH_n): Let $a \subseteq \omega$ and R be a class of subsets of ω, defined by a Σ^1_n formula with parameter a. Then there exists a subset x of ω, defined simultaneously by the formulae Σ^1_n and Π^1_n, such that $x \in R$.

This is exactly the formulation of the fact that $\Delta^1_n[a]$ is a basis for $\Sigma^1_n[a]$. It is well known that (BH_2) is a theorem of ZF (Zermelo–Fraenkel set theory). Addison proved that the axiom of constructibility implies (BH_n) for every natural $n \geqslant 2$. Using the axiom of projective determinateness, Martin and Solovay proved that for an odd n, (BH_n) does not hold. Their conjecture is that under the same assumption (BH_n) holds for even n. Silver proved that (BH_n) is consistent with the existense of a measurable cardinal. For references see [1].

In the present paper we prove that assuming the consistency of ZF, the theory ZF with an additional axiom "(BH_3) does not hold" is consistent. Namely, our theorem is

THEOREM 1. *If M is a countable standard model for $\mathrm{ZF}+V=L$, then there exists a model $N \supsetneq M$ for ZFC, satisfying the following sentence: for every $a \subseteq \omega$ there exists a class R_a of subsets of ω, $R_a \in \Pi^1_2[a]$ such that no element of R_a is ordinal definable from a.*

In the proof we use the method of forcing, so by the well known reasoning one can obtain the following consistency results:

COROLLARY 2.

$$\mathrm{Con}(\mathrm{ZF}) \to \mathrm{Con}\big(\mathrm{ZFC} + (a)_{P(\omega)}(ER)_{P(P(\omega))}[R \in \Pi^1_2[a] \;\&\; R \cap \mathrm{HOD}[a] = 0]\big).$$

COROLLARY 3. $\mathrm{Con(ZF)} \to \mathrm{Con}(\mathrm{ZFC} + (n)[n \geq 3 \to \neg(\mathrm{BH}_n)])$.

Now we turn to the proof of theorem. Let M be a countable standard model for $\mathrm{ZF} + V = L$. We define the notion of forcing P as follows:

$$P = \{f \in M\colon \mathrm{Func}(f) \,\&\, \mathrm{dom}(f) \subseteq \omega_1^M \times \omega \,\&\, \mathrm{rg}(f) \subseteq 2 \,\&\, |\mathrm{dom}(f)|^M < \aleph_0\}.$$

The ordering of P is the inverse inclusion. For any $A \subseteq \omega_1^M$, let $P_A = \{f \in P\colon \mathrm{dom}(f) \subseteq A \times \omega\}$. If $G \subseteq P$ is P-generic over M, then $G_A = G \cap P_A$.

LEMMA 4. *If $a \subseteq \omega$ and $a \in M[G]$, then there exists a subset A of ω_1^M countable in M and such that $a \in M[G_A]$.*

For a proof see [3].

Let us denote by Q the following notion of forcing:

$$Q = \{f \in M\colon \mathrm{Func}(f) \,\&\, \mathrm{dom}(f) \subseteq \omega \,\&\, \mathrm{rg}(f) \subseteq 2 \,\&\, |\mathrm{dom}(f)|^M < \aleph_0\}.$$

LEMMA 5. *If $x, y, t \subseteq \omega$, x is Q-generic over M, y is Q-generic over $M[x]$ and $t \in M[x]$, then there exists a $z \subseteq \omega$, z being Q-generic over $M[t]$ and such that $M[x][y] = M[t][z]$.*

For a proof see [5].

Let us suppose that $G \subseteq P$, G is P-generic over M, $N = M[G]$, $a \subseteq \omega$ and $a \in N$. Let us take $A \subseteq \omega_1^M$ as in Lemma 4. We may assume that A is an ordinal, say $A = \xi$. It is easily seen that P_A and $P_{\{\xi\}}$ are isomorphic to dense subsets of Q, and so there exists a $b \subseteq \omega$, Q-generic over M and such that $M[b] = M[G_A]$. By Lemma 5 we can find a $c \subseteq \omega$, Q-generic over $M[a]$ and such that $M[a][c] = M[b][G_{\{\xi\}}]$. Hence $M[G] = M[a][c][G_{\omega_1^M - (\xi \cup \{\xi\})}]$. Note that $Q \times P_{\omega_1^M - (\xi \cup \{\xi\})}$ is a homogeneous notion of forcing in $M[a]$ (it should be observed here that the definitions of P, P_A and Q are absolute because of the finiteness of conditions). Thus we have proved that $M[G]$ can be obtained as a generic extension of $M[a]$, where the notion of forcing is homogeneous.

LEMMA 6. *If C is a homogeneous notion of forcing in M, $y \in M$ and x is hereditarily ordinal definable from y in the sense of $M[G]$ (G being C-generic over M), then $x \in M$* (Lévy [3]).

By Lemma 6 all sets which are hereditarily ordinal definable from a in the sense of $N = M[G]$ belong to $M[a]$ and hence are constructible from a in N. For every $a \subseteq \omega$, $a \in N$ define

$$R_a = \{x \in N\colon x \subseteq \omega \,\&\, N \models x \notin L[a]\}.$$

Then $R_a \in \Pi_2^1[a]$ and $R_a^N \cap \mathrm{HOD}[a] = 0$, Q.E.D.

The same result can be proved under the assumption that M is a model for ZFC^- (ZF set theory without the power set axiom but with an axiom scheme of choice). Then P may happen to be a proper class and we must use the method of forcing described in [6]. Using the methods of [7], we can obtain an analogous result for the second order arithmetics. In those cases we must formulate the basis hypothesis as a scheme. An analogous theorem is valid also for the impredicative set theory, but then Π_2^1 must be replaced by Π_1^M. For details of the method see [2] and [4].

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