a Cauchy filter relative to $d$. Thus, $F^d$ is real. Now $P(X) \neq P^d(X)$, and by Theorem 3.2, every member of $P(X)$ may be extended to a member of $\mathcal{O}(X)$. However, the function $f(x, y) = \sin(y^{-1})$ belongs to $C^\omega(X)$, but clearly has no continuous extension to $T$. Thus $X$ is not $C^\omega$-embedded in $T$. We also observe that there is no compatible proximity on $T$ for which $(X, \delta)$ is a $p$-subspace of $T$. (See Example 1 of [7].)

References


A remark on the independence of a basis hypothesis

by

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Abstract. In the paper we prove the independence of a basis hypothesis used by Enderton and Friedman in the proof of the existence of a minimal $\beta_n$-model for analysis. The main result is the consistency of ZFC with the axiom

$$\text{(BH)}: \text{Let } a \subseteq \omega \text{ and } R \text{ be a class of subsets of } \omega, \text{ defined by a } \Sigma_1^0 \text{ formula with parameter } a. \text{ Then there exists a subset } x \subseteq \omega \text{ defined simultaneously by the formulae } \Sigma_1^0 \text{ and } \Pi_1^1, \text{ such that } w \in R.$$

This is exactly the formulation of the fact that $\Delta_2^n[a]$ is a basis for $\Sigma_2^n[a]$. It is well known that $(\text{BH})$ is a theorem of ZF (Zermelo–Fraenkel set theory). Addison proved that the axiom of constructibility implies $(\text{BH})$ for every natural $n \geq 2$. Using the axiom of projective determinateness, Martin and Solovay proved that for an odd $n$, $(\text{BH})$ does not hold. Their conjecture is that under the same assumption $(\text{BH})$ holds for every $n$. Solovay proved that $(\text{BH})$ is consistent with the existence of a measurable cardinal. For references see [1].

In the present paper we prove that assuming the consistency of ZF, the theory ZF with an additional axiom "(BH) does not hold" is consistent. Namely, our theorem is

**Theorem 1.** If $M$ is a countable standard model for $\text{ZF} + \forall \alpha \exists R \in M$, then there exists a model $N \supseteq M$ for $\text{ZFC}$, satisfying the following sentence: for every $a \subseteq \omega$ there exists a class $R_a$ of subsets of $\omega$, $R_a \in H_0[\alpha]$ such that no element of $R_a$ is ordinal definable from $a$.

In the proof we use the method of forcing, so by the well known reasoning one can obtain the following consistency results:

**Corollary 2.**

$$\text{Con(}\text{ZF}\text{)} \to \text{Con(}\text{ZFC}+\text{(BH)}\text{)} \forall a \in R \in H_0[\alpha] \text{ & } R \cap \text{HOD}[\alpha] = 0\).$$
Corollary 3. $\text{Con}(\text{ZF}) \rightarrow \text{Con}(\text{ZF} + \{\forall x \exists y \exists z (y \neq z)\})$.

Now we turn to the proof of theorem. Let $M$ be a countable standard model for $\text{ZF} + V = L$. We define the notion of forcing $P$ as follows:

$$P = \{ f : M \mid \text{Func}(f) \land \text{dom}(f) \subseteq \omega^M \land \forall x \in \omega^M \land \forall y \in \text{dom}(f)^M \land x \subseteq y \}.$$ 

The ordering of $P$ is the reverse inclusion. For any $A \subseteq \omega^M$, let $P_A = \{ f : M \mid \text{dom}(f) \subseteq A \times \omega \}$. If $G \subseteq P$ is $P$-generic over $M$, then

$$G_A = G \cap P_A.$$ 

Lemma 4. If $a \subseteq \omega$ and $a \in M[G]$, then there exists a subset $A$ of $\omega^M$ countable in $M$ and such that $a \in M[G_A]$.

For a proof see [3].

Let us denote by $Q$ the following notion of forcing:

$$Q = \{ f : M \mid \text{Func}(f) \land \text{dom}(f) \subseteq \omega \land \exists x \subseteq \omega \land (\text{dom}(f) = M[x]) \}.$$ 

Lemma 5. If $x, y, t \subseteq \omega$, $x$ is $Q$-generic over $M$, $y$ is $Q$-generic over $M[x]$, and $t \in M[x]$, then there exists a $z \subseteq \omega$, $z$ being $Q$-generic over $M[t]$ such that $M[x][y] = M[t][z]$.

For a proof see [5].

Let us suppose that $G \subseteq P$, $G$ is $P$-generic over $M$, $N = M[G]$, $a \subseteq \omega$ and $a \in N$. Let us take $A \subseteq \omega^M$ as in Lemma 4. We may assume that $A$ is an ordinal, say $A = \xi$. It is easily seen that $P_A$ and $P_{\xi[A]}$ are isomorphic to dense subsets of $Q$, and so there exists a $b \subseteq \omega$, $Q$-generic over $M$ and such that $M[b] = M[G_A]$. By Lemma 5 we can find a $c \subseteq \omega$, $Q$-generic over $M[a]$ and such that $M[a][c] = M[b][G_{\xi[A]}]$. Hence $M[G] = M[a][G_{\xi[A]}]$.

Note that $Q \times P_{\xi[A]}$ is a homogeneous notion of forcing in $M[a]$ (the definition should be observed here that the definitions of $P, P_A$ and $Q$ are absolute because of the finiteness of conditions). Thus we have proved that $M[G]$ can be obtained as a generic extension of $M[a]$, where the notion of forcing is homogeneous.

Lemma 6. If $G$ is a homogeneous notion of forcing in $M$, $y \in M$ and $x$ is hereditarily ordinal definable from $y$ in the sense of $M[G]$ ($G$ being $G$-generic over $M$), then $x \in M$ (Lévy [3]).

By Lemma 6 all sets which are hereditarily ordinal definable from $a$ in the sense of $N = M[G]$ belong to $M[a]$ and hence are constructible from $a$ in $N$. For every $a \subseteq \omega$, $a \in N$ define

$$R_a = \{ x \in N \mid x \subseteq a \land N \models x \neq L[a] \}.$$ 

Then $R_a \cap P_{\xi[A]}[a]$ and $R_a \cap \text{HOD}[a] = 0$, Q.E.D.

The same result can be proved under the assumption that $M$ is a model for $\text{ZF}^-$ ($\text{ZF}$ set theory without the power set axiom but with an axiom scheme of choice). Then $P$ may happen to be a proper class and

References