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On a paper by Iqbalunnisa

by

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Abstract. It is known that if L is a complete lattice which is relatively complemented or (more generally) both section and dual section semicomplemented, then its congruence lattice is a Stone lattice. Recently, Iqbalunnisa has proved this to be true when L is a complete, weakly modular, section complemented lattice. By weakening the axioms of weak modularity and section semicomplementation, a class of lattices is produced that includes all of the above examples, and for which the above result remains valid. A second class of lattices is then introduced on which a fairly explicit formula can be given for the pseudocomplement of a congruence relation. This second class includes all section semicomplemented lattices whose dual is section semicomplemented, and the formula for pseudocomplements is a new one for these lattices also.

1. Introduction. In [3], Theorem 2, p. 316, Iqbalunnisa proves that if L is a complete, weakly modular, section complemented lattice, then the lattice of congruence relations of L forms a Stone lattice, thus generalizing a result of the author ([5], Theorem 4.8, p. 202). On the other hand, the author has shown ([6], Theorem 4.17, p. 72) that if L is a complete lattice which is both section semicomplemented and dual section semicomplemented, then its congruence lattice is a Stone lattice. Our purpose here is to provide a common generalization of these results. For convenience, our notation and terminology will follow that of [4]. Also, it will prove useful to let Axiom (X*) denote the dual of Axiom (X) throughout the paper.

2. The general case. Though all of the above lattices are weakly modular, it turns out that we can get by with a slightly weaker axiom. Accordingly, we introduce Axiom (A) in a lattice with 0:

(A) *$a/0 \to c/d$ with $c > d$ implies $c/d \to a_1/a_2$ for suitable elements a_1, a_2 such that $a \geqslant a_1 > a_2$.*

LEMMA 1. *Let L be a lattice with 0. Axiom (A) is equivalent to the assertion that for every congruence relation Θ on L, $a \equiv 0(\Theta^*)$ iff the interval $[0, a]$ contains only trivial congruence classes modulo Θ.*

Proof. Let Axiom (A) hold. If $a \equiv 0(\Theta^*)$ and $a \geqslant b \geqslant c$ with $b \equiv c(\Theta)$, then $b \equiv c(\Theta \wedge \Theta^*)$ implies $b = c$. Suppose on the other hand that $[0, a]$ contains only trivial congruence classes modulo Θ. Then if $a/0 \to b/c$ with $b \equiv c(\Theta)$ and $b > c$, we may apply Axiom (A) to deduce that $b/c \to a_1/a_2$ with $a \geqslant a_1 > a_2$. But then $a_1 \equiv a_2(\Theta)$ provides a contradiction. By [4], p. 290 we have $a \equiv 0(\Theta^*)$.

Suppose now that Axiom (A) fails. There must exist elements a, c, d such that $c > d$, $a/0 \to c/d$ but $c/d \to a_1/a_2$ with $a \geqslant a_1$ implies $a_1 = a_2$. If $a \geqslant x \geqslant y$ with $x \equiv y(\Theta_{c/d})$ we must have $x = y$. On the other hand, since $a/0 \to c/d$ we know that $a \not\equiv 0(\Theta_{c/d}{}^*)$.

LEMMA 2. *If Axiom* (A) *holds, then every dual distributive element of L is neutral.*

Proof. Let d be dual distributive. Define the congruence relation Θ by the rule $a \equiv b(\Theta) \Leftrightarrow a \wedge d = b \wedge d$. Then $x \equiv 1(\Theta)$ if and only if $x \geqslant d$, and $d \geqslant a \geqslant b$ with $a \equiv b(\Theta)$ implies $a = b$, so by Lemma 1, $d \equiv 0(\Theta^*)$. If $a \equiv b(\Theta^*)$ then $a \vee d \equiv b \vee d(\Theta \wedge \Theta^*)$ forces $a \vee d = b \vee d$. Clearly $a \vee d = b \vee d$ implies $a \equiv b(\Theta^*)$. It follows that $[0, d]$ is the kernel of Θ^*, and that d is distributive. Evidently $a \vee d = b \vee d$, $a \wedge d = b \wedge d$ together imply $a \equiv b(\Theta \wedge \Theta^*)$, whence $a = b$. By [1], Theorem 1, p. 28 and [1], Lemma 12, p. 41, d is neutral.

In a lattice L with 1, we now introduce Axiom (B):

(B) *$a > b$ implies the existence of an element t such that $t \equiv 1(\Theta_{a/q})$ and $t \ngeqslant a$.*

The significance of Axiom (B) can best be understood by considering a lattice L with 0 having the property that every congruence relation on L is the minimal one generated by a distributive ideal. If $a > b$, we must then have $a \vee t = b \vee t$ for some $t \equiv 0(\Theta_{a/b})$. This clearly implies $t \nleqslant b$, so Axiom (B*) holds. As a matter of fact, if the kernel of every congruence relation of a lattice L with 0 is a principal ideal, then by [4], Theorem 5.2, p. 295, Axiom (B*) is equivalent to the assertion that every congruence relation on L is the minimal one generated by a distributive element. Before proceeding, we consider some examples.

EXAMPLE 1. Every bounded section complemented lattice satisfies Axiom (B). To see this, let $a > b$. If c is a complement of b in the interval $[0, a]$, then $c \equiv 0(\Theta_{a/b})$. Letting t denote a complement of c in L, we have $t \equiv 1(\Theta_{a/b})$ and $t \ngeqslant a$.

EXAMPLE 2. Every dual section semicomplemented lattice satisfies Axiom (B). For if $a > b$, and if c is chosen so that $c \vee a = 1$ and $1 > c \geqslant b$, then $c \equiv 1(\Theta_{a/b})$ and $c \ngeqslant a$.

EXAMPLE 3. Let L be a bounded section semicomplemented lattice which is dual semicomplemented in the sense that $a > 0$ implies $a \vee t = 1$ for some $t < 1$. Then if $a > b$, choose c so that $0 < c \leqslant a$ and $c \wedge b = 0$. Clearly $c \equiv 0(\Theta_{a/b})$ and just as clearly if $c \vee t = 1$ with $t < 1$, then $t \equiv 1(\Theta_{a/b})$ and $t \ngeqslant a$.

Of course Example 1 is a special case of Example 3. As a matter of fact, the above examples show that the lattices in Example 3 satisfy both Axioms (B) and (B*). We now translate our axioms into a condition involving pseudocomplements of congruence relations.

THEOREM 3. *Let L be a bounded lattice. The following conditions are then equivalent:*

(1) *For each congruence relation Θ on L, $a \equiv 0(\Theta^*) \Leftrightarrow a$ is a lower bound for $\{t \in L;\ t \equiv 1(\Theta)\}$.*

(2) *L satisfies Axioms* (A) *and* (B).

Proof. (1) $\Rightarrow$ (2). Let $a > b$. Then $a \not\equiv 0(\Theta_{a/b}{}^*)$ so there must exist an element $t \equiv 1(\Theta_{a/b})$ such that $t \ngeqslant a$, thus establishing Axiom (B). To verify (A), note that for any congruence Θ, if $a \geqslant b \geqslant c$ with $b \equiv c(\Theta)$ implies $b = c$, then $a \equiv 0(\Theta^*)$. For if $t \equiv 1(\Theta)$, then $a \wedge t \equiv a(\Theta)$ forces $a = a \wedge t \leqslant t$. It is immediate that $a \equiv 0(\Theta^*) \Leftrightarrow [0, a]$ contains only trivial congruence classes modulo Θ, so by Lemma 1, Axiom (A) holds.

(2) $\Rightarrow$ (1). If $a \equiv 0(\Theta^*)$ and $t \equiv 1(\Theta)$, then $a \wedge t \equiv a(\Theta \wedge \Theta^*)$ so $a = a \wedge t \leqslant t$. Suppose on the other hand that a is a lower bound for $\{t \in L;\ t \equiv 1(\Theta)\}$. If $a \geqslant b > c$ with $b \equiv c(\Theta)$, then there is an element $t \equiv 1(\Theta_{b/c})$ such that $t \ngeqslant b$. But $t \equiv 1(\Theta_{b/c}) \Rightarrow t \equiv 1(\Theta) \Rightarrow t \geqslant a \geqslant b$. This contradiction, together with Lemma 1, shows that $a \equiv 0(\Theta^*)$.

This brings us to the theorem that provides a generalization of the results mentioned in the introduction.

THEOREM 4. *Suppose that both L and its dual satisfy Axioms* (A) *and* (B). *If L is complete, then its lattice of congruence relations is a Stone lattice.*

Proof. Let Θ be a congruence relation on L. By Theorem 3, the kernel of Θ^* is $[0, z]$, where $z = \bigwedge \{t \in L;\ t \equiv 1(\Theta)\}$. If $a > b \geqslant z$ and $a \equiv b(\Theta^*)$ there must exist an element $t \equiv 0(\Theta_{a/b})$ such that $t \nleqslant b$. But $t \equiv 0(\Theta_{a/b}) \Rightarrow t \equiv 0(\Theta^*) \Rightarrow t \leqslant z \leqslant b$, a contradiction. Thus $a \geqslant b \geqslant z$ with $a \equiv b(\Theta^*)$ implies $a = b$, so by the dual of Lemma 1, $z \equiv 1(\Theta^{**})$. This shows that Θ^* and Θ^{**} are complements in the lattice of congruence relations of L.

In the presence of lower continuity we are able to improve Theorem 4 as follows:

THEOREM 5. *Let L be a lower continuous lattice satisfying Axioms* (A) *and* (B). *The lattice of congruence relations of L is then a Stone lattice.*

Proof. Let Θ be a congruence relation on L. By the dual of [4], Theorem 2.1, p. 291, $\{t \in L;\ t \equiv 1(\Theta^*)\}$ is a principal filter of L. By Theorem 3, $a \equiv 0(\Theta^{**}) \Leftrightarrow a$ is a lower bound for this set. Thus if $[z, 1] = \{t \in L;\ t \equiv 1(\Theta^*)\}$ then $z \equiv 0(\Theta^{**})$ and $z \equiv 1(\Theta^*)$, thereby completing the proof.

It should be mentioned that the above proof shows the theorem to be valid for any lattice L satisfying (A) and (B), provided that for each congruence relation Θ on L, $\{t \in L;\ t \equiv 1(\Theta^*)\}$ is a principal filter.

3. The incomplete case. In this section we shall generalize [6], Theorem 4.13, p. 71. In connection with this, we shall need the following strengthened version of Axiom (B) in a lattice L with 1:

(C) *$a > b$ implies the existence of an element t such that $t \equiv 1(\Theta_{a/b})$, $\geqslant b$, but $t \not\geqslant a$.*

This axiom is clearly satisfied by any dual section semicomplemented lattice. It turns out that Axiom (C) is related to standard ideals in much the same way as Axiom (B) is related to distributive ideals. In order to see this, we consider a lattice L with 0 having the property that every congruence relation on L is the minimal one generated by a standard ideal. If $a > b$, we then have $a = b \vee t$ for some $t \equiv 0(\Theta_{a/b})$. Evidently $t \leqslant a$ and $t \not\leqslant b$, so (C*) holds. Suppose in fact that L is a lattice with 0 having the property that the kernel of every congruence relation on L is a principal ideal. By [4], Theorem 5.3, p. 296, Axiom (C*) is equivalent to the assertion that every congruence relation on L is the minimal one generated by a standard element.

In the case of a distributive lattice L with 1, Axiom (C) is easily seen to be equivalent to L being dual section semicomplemented. It is also worth noting that every bounded simple lattice satisfies both Axioms (C) and (C*). In the next theorem we translate Axiom (C) to a condition involving pseudocomplements of congruence relations.

THEOREM 6. *For a bounded lattice L, the following conditions are equivalent*:

(1) *For each congruence relation Θ on L,*

$$a \equiv b(\Theta^*) \Leftrightarrow [0, a] \cap \ker\Theta = [0, b] \cap \ker\Theta .$$

(2) *L is weakly modular and satisfies* (C*).

Proof. (1) $\Rightarrow$ (2). Let $a > b$ in L. Then $a \not\equiv b({\Theta_{a/b}}^*)$ so $[0, a] \cap \ker\Theta_{a/b} \supset [0, b] \cap \ker\Theta_{a/b}$. Thus there must exist an element $t \equiv 0(\Theta_{a/b})$ such that $t \leqslant a$ but $t \not\leqslant b$. This establishes Axiom (C*).

In order to verify weak modularity, we next consider an arbitrary congruence relation Θ on L. If $a \equiv b(\Theta^*)$, the interval $[a\wedge b, a\vee b]$ clearly contains only trivial congruence classes modulo Θ. Suppose conversely that $[a\wedge b, a\vee b]$ contains only trivial congruence classes with respect to Θ. If $t \in [0, a] \cap \ker\Theta$, we claim that $t \leqslant b$. Otherwise, $a\wedge b < (a\wedge b)\vee t \leqslant a$ with $a\wedge b \equiv (a\wedge b)\vee t(\Theta)$ produces a contradiction. Thus $[0, a] \cap \ker\Theta \subseteq [0, b] \cap \ker\Theta$, and a similar argument produces the reverse inclusion. Consequently $a \equiv b(\Theta^*)$, and by [2], Theorem 4, p. 230, L is weakly modular.

(2) $\Rightarrow$ (1). Suppose now that L is weakly modular and satisfies Axiom (C*). If $a \equiv b(\Theta^*)$ and if $t \in [0, a] \cap \ker\Theta$, then $t = t\wedge a \equiv t\wedge b(\Theta^*)$ and $t\wedge b \equiv t \equiv 0(\Theta)$ together show that $t = t\wedge b \leqslant b$. By the symmetry of a and b, we deduce that

$$(*) \qquad [0, a] \cap \ker\Theta = [0, b] \cap \ker\Theta .$$

Suppose on the other hand that (*) is true. We are to show that $a \equiv b(\Theta^*)$. It clearly suffices to show that $a \equiv a\wedge b(\Theta^*)$, and by weak modularity, this will follow if we can just show that $[a\wedge b, a]$ contains no nontrivial congruence classes modulo Θ. If this were false, we could find x, y such that $a \geqslant x > y \geqslant a\wedge b$ with $x \equiv y(\Theta)$. Choosing $t \in L$ so that $t \leqslant x$, $t \not\leqslant y$ and $t \equiv 0(\Theta_{x/y})$, then $t \equiv 0(\Theta)$ and $t \leqslant a$ implies $t \leqslant b$, so $t \leqslant a\wedge b \leqslant y$. From this contradiction we deduce that $a \equiv b(\Theta^*)$, as desired.

THEOREM 7. *Let L be a bounded weakly modular lattice satisfying Axioms* (C) *and* (C*). *Let Θ be a congruence relation on L and let J be the kernel of Θ^*. Then*:

(1) *J is a central element of $\bar{L}$, the completion by cuts of L.*

(2) *$a \equiv b(\Theta^*) \Leftrightarrow [0, a]\vee J = [0, b]\vee J$ in $\bar{L}$.*

Proof. (1) The fact that $J \in \bar{L}$ follows from Theorem 3. Let J' denote the kernel of Θ^{**}. Our plan of attack will be to show that for arbitrary $K \in \bar{L}$,

$$K = (K \cap J)\vee(K \cap J') = (K\vee J) \cap (K\vee J') .$$

By [4], Theorem 2.7, p. 299 it will follow that J is central with J' its unique complement.

To begin with, let a be an upper bound for $(K \cap J)\vee(K \cap J')$, and let $k \in K$. Then $[0, a\wedge k] \cap J' = [0, k] \cap J'$ so by Theorem 6, $k \equiv a\wedge k(\Theta^*)$. Similarly, $[0, a\wedge k] \cap J = [0, k] \cap J$ implies $k \equiv a\wedge k(\Theta^{**})$. We conclude that $k = a\wedge k \leqslant a$, so that a is an upper bound for K. Thus

$$K = (K \cap J)\vee(K \cap J') .$$

If K^* denotes the set of upper bounds of K, then by the dual of Theorem 3, $J^* = \{t \in L;\ t \equiv 1(\Theta^{**})\}$ and $J'^* = \{t \in L;\ t \equiv 1(\Theta^*)\}$. An argument dual to the one given above shows that

$$K^* = (K^*\vee J^*) \cap (K^*\vee J'^*)$$

in the completion by cuts of the dual of L. As in the proof of [7], Lemma 7, p. 5 we see that $K = (K\vee J) \cap (K\vee J')$ in $\bar{L}$.

(2) By Theorem 6, $a \equiv b(\Theta^*) \Leftrightarrow [0, a] \cap J' = [0, b] \cap J'$. Since J is central with J' its unique complement in L, this is equivalent to saying that $[0, a]\vee J = [0, b]\vee J$.

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Real maximal round filters in proximity spaces

by

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Abstract. Given a proximity space (X, δ), where $P(X)$ is the collection of real-valued proximity functions on (X, δ), a maximal round filter is called *real* whenever the corresponding maximal p-ideal is real. The maximal p-ideals in $P(X)$ which are not real are characterized in terms of their corresponding maximal round filters. From this follow results concerning the realcompletion of (X, δ). The realcompletion is distinguished from the completion of X relative to the total structure associated with δ and from the completion by local clusters.

If (X, δ) is a dense (topological) subspace of T, conditions are obtained which characterize when every member of $P(X)$ can be continuously extended to T. Examples concerning these results are also provided.

1. Introduction. Let (X, δ) be a proximity space with Smirnov compactification δX. The points x of δX may then serve as indices which make explicit the one-one correspondence between the maximal round filters $\mathcal{F}^x$ on (X, δ) and the maximal "p-ideals" I^x in the collection $P(X)$ of real-valued proximity functions on (X, δ). A maximal round filter $\mathcal{F}^x$ is called *real* if the corresponding maximal p-ideal I^x is real. In this paper we characterize the maximal p-ideals I^x which are not real in terms of maximal round filters. It then follows that the realcompletion of (X, δ) is the completion of the generalized uniform space $(X, \mathfrak{U})$, where $\mathfrak{U}$ is the weak generalized uniformity determined by $P(X)$. It is also shown that the realcompletion of (X, δ) is not, in general, coincidental with the completion of X relative to the total structure $\mathfrak{U}_\alpha$ associated with δ, nor with the completion of (X, δ) by clusters.

When (X, δ) is a dense (topological) subspace of T, conditions are obtained which characterize the property that every member of $P(X)$ can be continuously extended to T. This supplements the results of [6], [7] and [8]. An example is provided to show that this property can hold when X is not C-embedded in T and when there is no compatible proximity on T for which (X, δ) is a p-subspace.

2. Real maximal round filters. We note that the collection $P(X)$ need not be a group nor a lattice (cf. [2], p. 135). The theory of p-ideals (or p-systems) in $P(X)$ is developed in [8] and [9]. Appropriate definitions