$f_n\colon E_1 \to E_1$ such that $\{f_n\}_{n=0}^{\infty}$ converges pointwise to u. Put $u_0 = f_0$, $u_n = f_n - (f_0 + f_1 + \ldots + f_{n-1})$ for $1 \leqslant n < \omega$ and $u_\xi = 0$ for $\omega \leqslant \xi < \Omega$. Then

$$\sum_{\xi<\Omega} u_\xi = u\,, \quad u_\xi \in U(E_1, E_1) \text{ and } u \notin U(E_1, E_1)\,.$$

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Property Z and Property Y sets in F-manifolds

by

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Abstract. Let M be a manifold modelled on a Fréchet space F such that $F \cong F^\omega$. K, a closed subset of M, will have *Property Y* if given an open neighborhood U of K and an open cover of M, there exists a set N which is a closed neighborhood of K contained in U and a homeomorphism $h\colon N \Rightarrow \mathrm{Bd}(N) \times [0, 1)$ such that for $x \in \mathrm{Bd}(N)$, $h(x) = (x, 0)$ and $h^{-1}(\{x\} \times [0, 1))$ is contained in some element of the cover. It is shown that (1) Property Y implies infinite deficiency, and (2) Property Z implies Property Y for separable M. The combination gives an alternative proof to the proof of Anderson's that Property Z implies infinite deficiency.

Key words and phrases. Infinite-dimensional manifold, F-manifold, deficiency, Property Z, negligibility, variable product.

1. Introduction. An *F-manifold* is a manifold modelled on a Fréchet space F such that $F \cong F^\omega$. A closed subset K of an F-manifold M has *Property Z* if for every open, non-empty, homotopically trivial set U in M, then $U - K$ is non-empty and homotopically trivial. K has *F-deficiency* if there is a homeomorphism $h\colon M \to M \times F$ such that for $x \in K$, $h(x) = (x, 0)$.

Anderson was the first to show that Property Z implies F-deficiency for separable F-manifolds [1]. More recent results due to Chapman [2] have established this for non-separable F-manifolds. This paper gives a new approach to the problem, one which avoids use of the Hilbert cube, the useful compactification of l_2 (separable Hilbert space), which has no good generalization for other Fréchet spaces. We will also define a new type of deficient subset, which will be used as a stepping stone in the proof that Property Z implies F-deficiency.

Let K be a closed subset of a space X. Then K has *Property Y* if given an open neighborhood U of K and an open cover of X, there exists

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a set N which is a closed neighborhood of K contained in U, and such that there exists a homeomorphism $h\colon N\to \mathrm{Bd}(N)\times[0,1)$ such that for $x\in \mathrm{Bd}(N)$, $h(x)=(x,0)$ and $h^{-1}(\{x\}\times[0,1))$ is contained in some element of the cover.

The following theorems will be proved:

THEOREM 1. *Let M be an F-manifold and let K be a subset of M having Property Y. Then K is F-deficient.*

THEOREM 2. *Let M be a separable F-manifold and let K be a closed subset of M having Property Z. Then K has Property Y.*

Combining the theorems, we get an alternative proof to Anderson's that Property Z implies F-deficiency for separable F-manifolds.

2. Further definitions. For a metric space X, define $C(X)$ to be the open metric cone of X, that is, the space $[0,\infty)\times X$ with $\{0\}\times X$ identified, with the following topology: At the vertex, a basis for the open sets will be the sets $\{(t,x)\in[0,\infty)\times X\mid t<s\}$ for each $s\in(0,\infty)$, and at other points the product topology will be used. By Lemma 2 of [5], $F\cong C(E)$ for some metric space E. Let $h\colon F\to C(E)$ be such a homeomorphism. For $y\in F$, define $\|y\|=\pi_1 h(y)$ where π_1 is projection onto the first factor.

Let X be a topological space, F as above and $r\colon X\to I$ a continuous function. Then the *variable product of X by F with respect to r* is defined to be

$$X\underset{r}{\times}F=\{(x,y)\in X\times F\mid\ y=0\ \text{ or }\ \|y\|<r(x)/(1-r(x))\}.$$

We say that this variable product is *zero over* $r^{-1}(0)$. For U a subset of X, the *fiber over* U is the set $(U\times F)\cap(X\underset{r}{\times}F)$, and may also be called the *variable product restricted to* U.

3. Proof of Theorem 1. Let $G_1, G_2, \ldots$ be a sequence of open covers of M whose mesh approach 0. Let N_1 be a closed neighborhood of K and $h_1\colon N_1\to \mathrm{Bd}(N_1)\times[0,1)$ be a homeomorphism satisfying the conditions in the definition of Property Y with G_1 as the cover. In addition, by reparametrization of h_1, we may assume that $h_1^{-1}(\mathrm{Bd}(N_1)\times[0,\tfrac12])\cap K=\emptyset$.

For $i=1,2,3,\ldots$ inductively define N_i and h_i as above with the additional conditions that $N_i\subset N_{i-1}-h_{i-1}^{-1}(\mathrm{Bd}(N_{i-1})\times[0,\tfrac12])$ and $d(K,X-N_i)<1/2^i$. Let $p_i\colon[1/i,1/(i-1)]\to[\tfrac14,\tfrac12]$ be a homeomorphism such that $p_i(1/i)=\tfrac12$. (For the case $i=1$, use the extended reals.)

Define L_i inductively as follows:

$$L_0=M\times F,$$

$$L_i=L_{i-1}-\{(x,y)\in N_i\times F\mid 1/i\leqslant\|y\|\leqslant 1/(i-1)\ \text{ and }\ \pi_2h_i(x)\geqslant p_i(\|y\|)\}.$$

Let $r_i\colon [0,1)\times[2/(2i+1),1/(i-1))\to[0,1)\times[2/(2i+1),1/(i-1))$ be an embedding such that:

(a) $r_i=\mathrm{id}$ on $([0,1)\times(2/(2i+1)))\cup(\{0\}\times[2/(2i+1),1/(i-1)))$,

(b) r_i is onto $\{(s,t)\in[0,1)\times[2/(2i+1),1/(i-1))\mid t<1/i\ \text{ or }\ s<p_i(t)\}$,

(c) if $t\geqslant 2/(2i-1)$, then $\pi_2r_i((s,t))\geqslant 2/(2i-1)$.

Now define $f_i\colon L_{i-1}\to L_i$ by

$$f_i((x,y))=\Big(h_i^{-1}\big(\pi_1h_i(x),\pi_1r_i(\pi_2h_i(x),\|y\|)\big),\ \pi_2r_i(\pi_2h_i(x),\|y\|)y/\|y\|\Big)$$
$$\text{for } x\in N_i \text{ and } 2/(2i+1)<\|y\|<1/(i-1)$$
$$=\mathrm{id}\quad\text{elsewhere}.$$

f_i is a homeomorphism which satisfies the following conditions:

(1) $f_i=\mathrm{id}$ outside $N_i\times\{y\in F\mid 2/(2i+1)<\|y\|<1/(i-1)\}$,

(2) if $\|y\|\geqslant 2/(2i-1)$, then $\|\pi_2f_i((x,y))\|\geqslant 2/(2i-1)$,

(3) for $(x,y)\in L_{i-1}$, $\pi_1((x,y))$ and $\pi_1f_i((x,y))$ are contained in some element of G_i.

Let $f=\lim_{i\to\infty}\{f_i\circ f_{i-1}\circ\ldots\circ f_2\circ f_1\}$. By conditions (1) and (2), no point of $M\times F$ is moved more than twice. By condition (3), f approaches the identity on $M\times\{0\}$. Hence, we see that f is a homeomorphism onto a variable product of M by F which is zero on K and such that the homeomorphism is the identity on $M\times\{0\}$. Applying the techniques of [4], we can "squash down" the rest of the variable product by the use of Schori's stability theorem (Corollary 2.3 of [7]).

4. Proof of Theorem 2. The method of proof is due to Henderson and Burghelea and makes use of the following Lemma:

LEMMA. *Let U be an open subset of l_2 and let G be a cover of l_2. Then there exists a countable locally finite simplicial complex K and a homeomorphism $h\colon |K|\times l_2\to U$ such that for any simplex $\sigma\in K$, $h(|\sigma|\times l_2)$ is contained in some element of the cover.*

The proof is contained in [3].

Let U be an open neighborhood of K and let G be an open cover of M. Let G' be a refinement of G such that there do not exist sets $g,g'\in G'$ (possibly the same) such that $g\cap K\neq\emptyset$, $g\cap g'\neq\emptyset$ and $g'\cap(M-U)\neq\emptyset$.

Since all separable F-manifolds are open subsets of l_2 (see [6]), we may apply the Lemma to get a countably locally finite simplicial complex L and a homeomorphism $f\colon |L|\times l_2\to U$ such that for each $\sigma\in L$, $f(|\sigma|\times l_2)$ is contained in some element of G'. Let $K'=f^{-1}(K)$.

Let L' be the full subcomplex of L generated by simplexes $\{\sigma\in L\mid (|\sigma|\times l_2)\cap K'\neq\emptyset\}$. Let A be the set of simplexes σ such that

$|\sigma| \cap |L'| \neq \emptyset$ but σ is not in L'. Let $v_1, \ldots, v_n$ be the vertices of σ which are in L', and $u_1, \ldots, u_m$ be those which are not. Let

$$N_\sigma = \Big\{x \in |\sigma| \,\Big|\, \text{if } x = \sum_1^n a_i v_i + \sum_1^m b_i u_i \text{ then } \sum_1^n a_i \geqslant \tfrac{1}{2}\Big\}$$

and let $N' = |L'| \cup \bigcup_{\sigma \in A} N_\sigma$. Let $\partial N'$ be the topological boundary of N' in $|L|$. It can easily be seen that L can be subdivided to L^* so that N' and $\partial N'$ are realizations of subcomplexes of L^*.

Define $c\colon \partial N' \times [0, 1) \to N'$ as follows: If $x \in N_\sigma \subset N'$ and $x = \sum_1^n a_i v_i + \sum_1^m b_i u_i$ with the v_i and u_i as above, then

$$c\big((x, s)\big) = \sum_1^n (1+s) a_i v_i + \sum_1^m (1-s) b_i u_i\,.$$

c is an embedding onto a collar of $\partial N'$ in N'. Let $N^* = N' \times l_2$ and then $\mathrm{Bd}(N^*) = \partial N' \times l_2$.

By a simple argument using the definition of Property Z and general position, it can be shown that a set with Property Z can be moved off a finite-dimensional cell by a small motion. For details see Lemma 6.1 of [1].

Using the above and induction, there is a homeomorphism $g\colon |L^*| \times l_2 \to |L^*| \times l_2$ such that

(1) $g = \mathrm{id}$ outside N^*,

(2) $g(K') \cap (|L^*| \times \{0\}) = \emptyset$,

(3) for $\sigma \in L$, $fg^{-1}(|\sigma| \times l_2)$ is contained in some element of G'.

Let $r\colon |L'| \to (0, 1]$ be a continuous function such that for $x \in N'$, $\big(\{x\} \times \{y \in F|\ \|y\| \leqslant r(x)\}\big) \cap g(K') = \emptyset$. Define

$$N = fg^{-1}\big(N^* - \{(x, y) \in N' \times l_2|\ \|y\| < r(x)\}\big)\,.$$

We will define $h\colon \mathrm{Bd}(N) \times [0, 1) \to N$ in the following way: Let $z \in \mathrm{Bd}(N)$. Suppose $(x, y) \in N' \times l_2$ and $z = fg^{-1}\big((x, y)\big)$. If x is not in $\mathrm{im}(c)$ (the collar of $\partial N'$), then define

$$h(z, s) = fg^{-1}\big(x, y/(1-s)\big)\,.$$

If x is in $\mathrm{im}(c)$, then let $t = \pi_2 c^{-1}(x)$, $v = \pi_1 c^{-1}(x)$ and $x' = c\big(v, t + s(1-t)\big)$. Now we define

$$h(z, s) = fg^{-1}\Big(x', y\cdot\big(r(x')/r(x)(1-s)\big)\Big)\,.$$

Then h satisfies the requirements in the definition of Property Y.

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