On a plane compactum with the maximal shape

by

S. Spież (Warszawa)

Abstract. In this note a proof is given that there exists a compactum $Y_0 \subset E^2$ such that every compactum $X \subset E^2$ has the same shape as a retract of Y_0.

§ 1. Preliminaries. The definitions of a fundamental sequence, of shape $\mathrm{Sh}(X)$, of the relation $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(Y)$ and of related concepts may be found in [1]. It is known ([2], p. 236) that there exists in the plane E^2 a continuum X_0 (actually, any continuum decomposing E^2 into an infinite number of regions) such that $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(X_0)$ for every plane continuum X. We say that X_0 is a *majorant* for the shapes of plane continua. On the other hand one knows ([3], p. 108) that for continua lying in the space E^3 the situation is different, because already for the family γ of all selenoids, no compactum X_0 satisfies the condition $\mathrm{Sh}(X) \leqslant \mathrm{Sh}(X_0)$ for every $X \in \gamma$.

The purpose of this note is to prove that there exists a compactum $Y_0 \subset E^2$ which is a majorant for shapes of all plane compacta.

For a subset X of E^2 by $\overline{X}$, $\mathring{X}$ and $\dot{X}$ we always understand respectively, the closure, interior and boundary of the set X in E^2.

§ 2. Construction of Y_0. By a *k-perforated geometric disk* (where k is a non negative integer) we understand a 2-dimensional continuum $Q \subset E^2$ with the boundary $\dot{Q}$ which is the union of $k+1$ disjoint geometric circles. In particular, a 0-perforated geometric disk is any geometric disk lying in E^2. The 1-perforated geometric disks will be called also *geometric annula.*

A k-perforated geometric disk Q' is said to be *inscribed* into an m-perforated geometric disk Q (where $k \geqslant m$) if Q' is contained in the interior $\mathring{Q}$ of Q and if the closure $\overline{Q \setminus Q'}$ of the set $Q \setminus Q'$ is the union of $k+1$ mutually disjoint sets, among them $m+1$ geometric annula and $k-m$ geometric disks (see Fig. 1).

By a set of *type* I inscribed into an m-perforated geometric disk Q we understand a set $Z \subset \mathring{Q}$ satisfying the following two conditions, (1_{I}) and (2):

(1_{I}) *The family of components of Z consists of one m-perforated geometric disk Q' inscribed in Q (called the main component of Z), of $\aleph_0$ geometric disks and of $\aleph_0$ geometric annula.*

Observe that (1_{I}) implies that each component of $\overline{Q\setminus Q'}$ is geometric annulus.

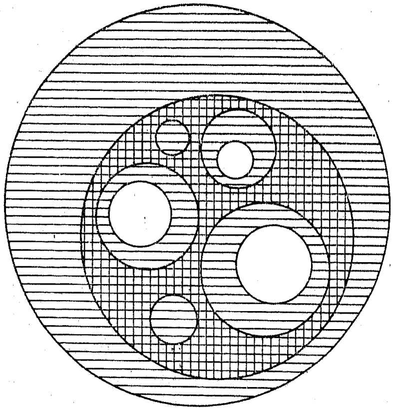

Fig. 1

(2) *Every geometric annulus A which is a component of $\overline{Q\setminus Q'}$ contains $\aleph_0$ geometric annula which are components of Z, the union of those annula is compact and none of them is contractible in A. Moreover, A contains $\aleph_0$ geometric disks which are components of Z, and the union of those disks is compact.*

By a set of *type* II, inscribed into an m-perforated geometric disk Q we understand a set $Z\subset \mathring{Q}$ satisfying conditions (1_{II}), (2) and (3):

(1_{II}) *The family of components of Z consists of one $(m+1)$-perforated geometric disk Q' (called the main component of Z) inscribed into Q, of $\aleph_0$ geometric disks and of $\aleph_0$ geometric annula.*

Observe that (1_{II}) implies that $\overline{Q\setminus Q'}$ contains $m+2$ components of which one is a geometric disk and the other are geometric annula.

(3) *The component R of $\overline{Q\setminus Q'}$, which is a geometric disk, is disjoint to Z.*

Now let us define by induction a decreasing sequence $\{Y_n\}$ of compacta lying in E^2 and a sequence of functions

$$a_n\colon\ \square(Y_n)\to\mathfrak{N}\cup(0)\cup(\infty),$$

where $\square(Y_n)$ denotes the set of components of Y_n and $\mathfrak{N}$ denotes the set of all natural numbers.

1) Let Y_1 be a compactum lying in E^2 and having $\aleph_0$ components each of which is a geometric disk. Let a_1 be any one-to-one functions mapping $\square(Y_1)$ onto $\mathfrak{N}\cup(0)\cup(\infty)$.

2) Assume that for an index n a compactum $Y_n\subset E^2$ is defined such that every component of Y_n is a perforated geometric disk or a circle. Moreover, assume that we already have a function $a_n\colon\ \square(Y_n)\to\mathfrak{N}\cup(0)\cup(\infty)$ such that if $Y_{n,i}\in\square(Y_n)$ is a k-perforated disk, then $a_n(Y_{n,i})\geqslant k$ (we observe that the function a_1 satisfies this condition). Let $Y_{n,i}$ be a k-perforated geometric disk which is a component of Y_n. If $a_n(Y_{n,i})=k$ then $\psi_n(Y_{n,i})$ denotes a set of type I inscribed into $Y_{n,i}$ and if $a_n(Y_{n,i})>k$, then $\psi_n(Y_{n,i})$ denotes a set of type II inscribed into $Y_{n,i}$. Let $\varphi_n(Y_{n,i})$ denotes the main component of $\psi_n(Y_{n,i})$. If $Y_{n,i}$ is a circle which is a component of Y_n, then $\psi_n(Y_{n,i})$ denotes $Y_{n,i}$.

We define Y_{n+1} by the formula

$$Y_{n+1}=\bigcup_{Y_{n,i}\in\square(Y_n)}\psi_n(Y_{n,i})\cup\dot{Y}_n$$

and we define a_{n+1} as a function mapping $\square(Y_{n+1})$ onto $\mathfrak{N}\cup(0)\cup(\infty)$ and satisfying the following two conditions:

(2.1) $a_{n+1}(\varphi_n(Y_{n,i}))=a_n(Y_{n,i})$ for every perforated geometric disk $Y_{n,i}$ which is component of Y_n.

(2.2) If $Y_{n,i}$ is a component of Y_n which is a perforated geometric disk and if A is a geometric annulus which is a component of $\overline{Y_{n,i}\setminus\varphi_n(Y_{n,i})}$, then a_{n+1} assigns in a one-to-one manner to all geometric disks which are components of Y_{n+1} lying in A the elements of the set $\mathfrak{N}\cup(0)\cup(\infty)$, and to all geometric annula which are components of Y_{n+1} lying in A the elements of the set $\mathfrak{N}\cup(\infty)$. We observe that if $Y_{n+1,i}\in\square(Y_{n+1})$ is a k-perforated disk, then $a_{n+1}(Y_{n+1,i})\geqslant k$.

We observe that

(2.3) If $Y_{n,i}\in\square(Y_n)$ is a k-perforated disk and $a_n(Y_{n,i})=k$ ($a_n(Y_{n,i})>k$), then the set $\varphi_n(Y_{n,i})$ is a k-perforated disk (resp. a $(k+1)$-perforated disk).

(2.4) If a perforated disk $Y_{n+1,j}$ which is a component of Y_{n+1} is contained in a component $Y_{n,i}$ of Y_n, then $Y_{n+1,j}$ is contained in the interior $\mathring{Y}_{n,i}$ of $Y_{n,i}$.

(2.5) If an annulus $Y_{n+1,j}$ which is a component of Y_{n+1} is contained in a component $Y_{n,i}$ of Y_n and is different from $\varphi_n(Y_{n,i})$, then $Y_{n+1,j}$ is not contractible in the set $Y_{n,i}\setminus\varphi_n(Y_{n,i})$.

For a perforated disk $Y_{n,i}$ which is a component of the set Y_n we define:

$$\varphi_\infty(Y_{n,i}) = \bigcap_{j\in\mathfrak{N}} \varphi_{n+j}\varphi_{n+j-1}\dots\varphi_n(Y_{n,i}) .$$

It is obvious that:

(2.6) For every $n\in\mathfrak{N}$ and for every perforated disk $Y_{n,i}\in\square(Y_n)$ the set $\varphi_\infty(Y_{n,i})$ is non-empty, and if it contains more than one point then the boundary of every component of the set $E^2\setminus\varphi_\infty(Y_{n,i})$ is a geometric circle.

We observe that the sequence $\{Y_n\}$ can be constructed in such a manner that the following condition is satisfying:

(2.7) There exist a sequence of positive numbers $\{\varepsilon_n\}$ converging to 0 and such that for every $n\in\mathfrak{N}$ and for every perforated disk $Y_{n,i}$ which is a component of Y_n we have:

$$\varrho\big(x, \varphi_\infty(Y_{n,i})\big) < \varepsilon_n \quad \text{for every } x\in Y_{n,i} .$$

We will show in §§ 4–7 that setting

(2.8) $$Y_0 = \bigcap_{n\in\mathfrak{N}} Y_n ,$$

we obtain a majorant for the shapes of all plane compacta. It simply follows from the definition that each set Y_n is compact, then the set Y_0 is also compact.

§ 3. A special system of neighbourhoods of a compactum in E^2. A subset of E^2 homeomorphic to a k-perforated geometric disk is said to be a *k-perforated disk*. In particular, 0-perforated disks are disks, and 1-perforated disks are annula. A k-perforated disk Q' is said to be *inscribed* into an m-perforated disk Q (where $k\geqslant m$) if $Q'\subset\mathring{Q}$ and if the set $\overline{Q\setminus Q'}$ is the union of $k+1$ mutually disjoint sets, among them $m+1$ annula and $k-m$ disks.

Let X be a compactum lying in E^2. One can easily see that there exists a sequence $\{P_n\}$ of neighbourhoods of X in E^2 satisfying the following conditions:

(3.1) P_1 is a disk.

(3.2) P_n has only a finite number of components and all these components are perforated disks.

(3.3) $$X = \bigcap_{n\in\mathfrak{N}} P_n .$$

(3.4) $$P_{n+1}\subset\mathring{P}_n \quad \text{for every } n\in\mathfrak{N}.$$

(3.5) Let $P_{n,i}$ be a k-perforated disk which is a component of P_n. Then $P_{n,i}$ contains one or two components of P_{n+1}. One of those components is a k-perforated or $(k+1)$-perforated disk inscribed into $P_{n,i}$, we denote it by $\hat{\varphi}_n(P_{n,i})$. The second of these components (if it exists) is a disk or an annulus lying in an annulus A which is a component of the set $\overline{P_{n,i}\setminus\hat{\varphi}_n(P_{n,i})}$ and if this component is an annulus, then it is not contractible in A.

Consider now a component $P_{n,i}$ of P_n and let us set

$$\hat{\varphi}_\infty(P_{n,i}) = \bigcap_{j\in\mathfrak{N}} \hat{\varphi}_{n+j}\dots\varphi_n(P_{n,i}) ,$$

and let $\hat{a}_n(P_{n,i})$ denote the number of bounded components of the set $E^2\setminus\hat{\varphi}_\infty(P_{n,i})$. Then

$$\hat{a}_n\colon\ \square(P_n)\to\mathfrak{N}\cup(0)\cup(\infty)$$

and it is clear that one can select the sequence $\{P_n\}$ so that, additionally, the following condition is satisfied:

(3.6) If $P_{n,i}\in\square(P_n)$ is a k-perforated disk, where $k<\hat{a}_n(P_{n,i})$, then $\hat{\varphi}_n(P_{n,i})$ is a $(k+1)$-perforated disk.

We observe that

(3.7) If $P_{n,i}\in\square(P_n)$ is a k-perforated disk then $\hat{a}_n(P_{n,i})\geqslant k$.

(3.8) For each $P_{n,i}\in\square(P_n)$ we have $\hat{a}_{n+1}\big(\hat{\varphi}_n(P_{n,i})\big) = \hat{a}_n(P_{n,i})$.

(3.9) If $P_{n,i}\in\square(P_n)$ is a k-perforated disk and $\hat{a}_n(P_{n,i}) = k$, then $\hat{\varphi}_n(P_{n,i})$ is also a k-perforated disk.

From conditions (2.3), (3.6), (3.7) and (3.9) it follows that:

(3.10) If the components $P_{n,i}\in\square(P_n)$ and $Y_{n,i'}\in\square(Y_n)$ are k-perforated disks such that $\hat{a}_n(P_{n,i}) = a_n(Y_{n,i'})$ then either the sets $\hat{\varphi}_n(P_{n,i})$ and $\varphi_n(Y_{n,i})$ are both k-perforated disks or they are both $(k+1)$-perforated disks.

§ 4. The sets $Y_0(X)$. Keeping the notation used in §§ 2 and 3, let us show that we can assign to every non-empty compactum $X\subset E^2$ a subset $Y_0(X)$ satisfying the following conditions:

(4.1) $Y_0(X)$ is a retract of Y_0.

(4.2) $\mathrm{Sh}\big(Y_0(X)\big) = \mathrm{Sh}(X)$.

Let $Y_0 = \bigcap_{n\in\mathfrak{N}} Y_n$ where $\{Y_n\}$ is the sequence (satisfying condition (2.7)) which we constructed in § 2. Let $\{P_n\}$ be a special system of neighbourhoods of X in E^2 satisfying the conditions (3.1)–(3.6). Let us assign to every perforated disk $Y_{n,i}$ which is a component of the set Y_n a function

$$\beta_{Y_{n,i}}\colon\ \square(E^2\setminus Y_{n,i})\to\mathfrak{N}\cup(0)$$

in the following manner:

If $n=1$, then $Y_{1,i}$ is a disk and $E^2\setminus Y_{1,i}$ is a connected set. Then we set

$$\beta_{Y_{1,i}}(E^2\setminus Y_{1,i})=0\,.$$

Assume that for a natural number n the function $\beta_{Y_{n,i}}$ is already defined for every perforated disk $Y_{n,i}$ which is a component of the set Y_n. Let $Y_{n+1,j}$ be a perforated disk which is a component of Y_{n+1}. If $Y_{n+1,j}$ is a disk, then we set

$$\beta_{Y_{n+1,j}}(E^2\setminus Y_{n+1,j})=0\,.$$

If $Y_{n+1,j}$ is an annulus, then $E^2\setminus Y_{n+1,j}$ has two components C', C'' from which one, say C', is unbounded and the other, C'', is bounded. Then we set

$$\beta_{Y_{n+1,j}}(C')=0 \quad\text{ and }\quad \beta_{Y_{n+1,j}}(C'')=1\,.$$

If $Y_{n+1,j}$ is a k-perforated disk with $k\geqslant 2$, then $Y_{n+1,j}$ is inscribed into a certain component $Y_{n,i}$ of the set Y_n. The set $Y_{n,i}$ is either a k-perforated or a $(k-1)$-perforated disk. If C is a component of $E^2\setminus Y_{n+1,j}$ containing a component C' of the set $E^2\setminus Y_{n,i}$, then we set

$$\beta_{Y_{n+1,j}}(C)=\beta_{Y_{n,i}}(C')\,.$$

If C is a component of $E^2\setminus Y_{n+1,j}$ which does not contain any component of $E^2\setminus Y_{n,i}$ (there exists at most one such component C), then we set

$$\beta_{Y_{n+1,j}}(C)=k\,.$$

In an analogous manner (replacing Y_n by P_n) we define for every component $P_{n,i}$ of the set P_n the map

$$\hat\beta_{P_{n,i}}\colon\ \square(E^2\setminus P_{n,i})\to\mathfrak{N}\cup(0)\,.$$

(4.3) Lemma. *There exists a sequence of functions*

$$\omega_n\colon\ \square(P_n)\to\square(Y_n)$$

satisfying the following conditions for every natural number n and for every component $P_{n,i}$ of P_n:

$(4.4)_{n,P_{n,i}}$ *If $P_{n,i}$ is a k-perforated disk, then $\omega_n(P_{n,i})$ is a k-perforated disk.*

$(4.5)_{n,P_{n,i}}$ $$\omega_{n+1}\big(\hat\varphi_n(P_{n,i})\big)=\varphi_n\big(\omega_n(P_{n,i})\big)\,.$$

$(4.6)_{n,P_{n,i}}$ *If $P_{n+1,j}$ is a component of P_{n+1} contained in $P_{n,i}$ and different from $\hat\varphi_n(P_{n,i})$, then*

(a) $\omega_{n+1}(P_{n+1,j})\subset\omega_n(P_{n,i})$,

(b) *if $P_{n+1,j}$ lies in a component $\hat C$ of the set $E^2\setminus\hat\varphi_n(P_{n,i})$ then $\omega_{n+1}(P_{n+1,j})$ lies in a component C of the set $E^2\setminus\varphi_n\big(\omega_n(P_{n,i})\big)$ satisfying condition:*

$$\hat\beta_{\hat\varphi_n(P_{n,i})}(\hat C)=\beta_{\varphi_n(\omega_n(P_{n,i}))}(C)\,.$$

Proof. Let us define function $\omega_1\colon\ \square(P_1)\to\square(Y_1)$. We have $\square(P_1)=\{P_1\}$ because P_1 is a disk (see (3.1)). We define $\omega_1(P_1)$ as a component of Y_1 satisfying condition $a_1(\omega_1(P_1))=\hat a_1(P_1)$ (since the a_1 is a one-to-one function mapping $\square(Y_1)$ onto $\mathfrak{N}\cup(0)\cup(\infty)$ and $\hat a_1(P_1)\in\mathfrak{N}\cup(0)\cup(\infty)$, there exists such a component $\omega_1(P_1)$ — only one — of Y_1).

Suppose that we already have a function $\omega_n\colon\ \square(P_n)\to\square(Y_n)$ satisfying for every component $P_{n,i}$ of P_n the condition $(4.4)_{n,P_{n,i}}$ and a condition

$(4.7)_{n,P_{n,i}}$ $$a_n\big(\omega_n(P_{n,i})\big)=\hat a_n(P_{n,i})\,.$$

We observe that the conditions $(4.4)_{1,P_1}$ and $(4.7)_{1,P_1}$ are satisfied.

Let $P_{n+1,j}\in\square(P_{n+1})$. If $P_{n+1,j}=\hat\varphi_n(P_{n,i})$ for a some component $P_{n,i}$ of P_n then we define $\omega_{n+1}(P_{n+1,j})$ by condition $(4.5)_{n,P_{n,i}}$ i.e. $\omega_{n+1}(P_{n+1,j})=\varphi_n\big(\omega_n(P_{n,i})\big)$. From (3.10), $(4.4)_{n,P_{n,i}}$ and $(4.7)_{n,P_{n,i}}$ it follows that the condition $(4.4)_{n+1,P_{n+1,j}}$ is satisfied. From (2.1), (3.8) and $(4.7)_{n,P_{n,i}}$ it follows that the condition $(4.7)_{n+1,P_{n+1,j}}$ is satisfied. If $P_{n+1,j}$ is contained in a component $P_{n,i}$ of P_n and $P_{n+1,j}$ is different from $\hat\varphi_n(P_{n,i})$ then we define $\omega_{n+1}(P_{n+1,j})$ as a component of Y_{n+1} satisfying the conditions $(4.4)_{n+1,P_{n+1,j}}$, $(4.6)_{n,P_{n,i}}$ and such that $a_n\big(\omega_{n+1}(P_{n+1,j})\big)=\hat a_n(P_{n+1,j})$. We must prove that there exists such a component $\omega_{n+1}(P_{n+1,j})$ of Y_{n+1}. Let $\hat C$ and C be components of $E^2\setminus\hat\varphi_n(P_{n,i})$ and $E^2\setminus\varphi_n\big(\omega_n(P_{n,i})\big)$, respectively, such that $\hat\beta_{\hat\varphi_n(P_{n,i})}(\hat C)=\beta_{\varphi_n(\omega_n(P_{n,i}))}(C)$ and $\hat C$ contains $P_{n+1,j}$. From (3.5) it follows that $\hat C$ is not contained in $P_{n,i}$. We know that if $P_{n,i}$ is a k-perforated disk, then $\omega_n(P_{n,i})$ is a k-perforated disk and either the sets $\hat\varphi_n(P_{n,i})$ and $\varphi_n\big(\omega_n(P_{n,i})\big)$ are both k-perforated disks or they are both $(k+1)$-perforated disks. It follows that C is not contained in $\omega_n(P_{n,i})$, hence C contains an annulus A which is a component of $\overline{\omega_n(P_{n,i})\setminus\varphi_n\big(\omega_n(P_{n,i})\big)}$. The set $P_{n+1,j}$ is a disk or an annulus (see (3.5)). Then (see (2.2) and (3.7)) it follows that there exist such a component $\omega_{n+1}(P_{n+1,j})$ (only one) of Y_{n+1}. The proof of Lemma (4.3) is finished.

For $l=4,5,6$, by condition $(4.l)$ we understand the following condition: for each $n\in\mathfrak{N}$ and for each component $P_{n,i}$ of P_n the condition $(4.l)_{n,P_{n,i}}$ is satisfied.

From (2.4), (4.5) and (4.6) (a) it follows that

(4.7) If a component $P_{n+1,j}$ of P_{n+1} is contained in a component $P_{n,i}$ of P_n, then the set $\omega_{n+1}(P_{n+1,j})$ is contained in the interior $\big(\omega_n(P_{n,i})\big)^\circ$ of $\omega_n(P_{n,i})$.

From (2.5), (4.4) and (4.6) it follows that

(4.8) If an annulus $P_{n+1,j}$ which is a component of P_{n+1} is contained in a component $P_{n,i}$ of P_n and is different from $\hat{\varphi}(P_{n,i})$, then $\omega_{n+1}(P_{n+1,j})$ is an annulus not contractible in $\omega_n(P_{n,i})\backslash\varphi_n(\omega_n(P_{n,i}))$

Since each component $P_{n,i}$ of P_n contains at most one component of P_{n+1} different from $\hat{\varphi}_n(P_{n,i})$ (see (3.5)), it follows from the conditions (4.5) and (4.6) that

(4.9) For each $n \in \mathfrak{N}$ the function $\omega_n\colon \square(P_n)\to\square(Y_n)$ is single-valued.

Now let us set

$$Y_n(X) = \bigcup_{P_{n,i}\in\square(P_n)} \omega_n(P_{n,i}) \quad \text{and} \quad Y_0(X) = \bigcap_{n\in\mathfrak{N}} Y_n(X).$$

For each number $n \in \mathfrak{N}$ the set $Y_n(X)$ is compact, because it is the union of a finite number of perforated disks. It follows that the set $Y_0(X)$ is also compact.

From the conditions (4.7), (4.9) and definition of $Y_n(X)$ it follows that:

(4.10) If a component $P_{n,i}$ of P_n contains exactly one component (exactly two components) of P_{n+1} then $\omega_n(P_{n,i})$ contains exactly one component (resp. exactly two components) of $Y_{n+1}(X)$.

§ 5. Proof of (4.1). We observe that if a perforated disk $Y_{n,i}$ is a component of the set $Y_n(X)$, then $\varphi_\infty(Y_{n,i}) \subset Y_0(X)$.

For every $n \in \mathfrak{N}$ the set $Y_0 \cap (Y_n(X))^\circ$ is a compactum, because it is the intersection of the compact set Y_0 with the union of all the set $\varphi_n(Y_{n,i})$, where $Y_{n,i} \in \square(Y_n(X))$ (we know that the number of components of $Y_n(X)$ is finite and that each of the sets $\varphi_n(Y_{n,i})$ is compact).

Consider a map

$$r_0\colon Y_0 \to Y_0 \cap (Y_1(X))^\circ$$

which asigns to all points $y \in Y_0\backslash(Y_1(X))^\circ$ one point of the set $Y_0(X)$ and which satisfies the condition $r_0(y) = y$ for every point $y \in Y_0 \cap (Y_1(X))^\circ$.

Now let us construct a sequence of maps

$$r_n\colon Y_0 \cap (Y_n(X))^\circ \to Y_0 \cap (Y_{n+1}(X))^\circ \quad \text{for} \quad n \in \mathfrak{N}$$

as follows:

If $y \in Y_0 \cap (Y_{n+1}(X))^\circ$ then $r_n(y) = y$.

If $Y_{n,i}$ is a component of the set $Y_n(X)$ then:

If $\varphi_\infty(Y_{n,i})$ consists of only one point then the restriction $r_n|[Y_0 \cap \mathring{Y}_{n,i})\backslash(Y_{n+1}(X))^\circ]$ is a map with values in $\varphi_\infty(Y_{n,i})$.

If $\varphi_\infty(Y_{n,i})$ contains many points then the boundary of every component of the set $E^2\backslash\varphi_\infty(Y_{n,i})$ is a geometric circle (see (2.6)). Let C be a component of the set $E^2\backslash\varphi_n(Y_{n,i})$ and let C' denote the component of the set $E^2\backslash\varphi_\infty(Y_{n,i})$ containing C. If C is bounded and contains a component C'' of the set $E^2\backslash Y_{n,i}$, then the restriction

$$r_n|[(Y_0 \cap \mathring{Y}_{n,i} \cap C')\backslash(Y_{n+1}(X))^\circ]$$

coincides with the projection from the centre of the open disk C'' onto the circle forming the boundary of C'. If C is bounded and contains no component of the set $E^2\backslash Y_{n,i}$, then the restriction

$$r_n|[(Y_0 \cap \mathring{Y}_{n,1} \cap C')\backslash(Y_{n+1}(X))^\circ]$$

coincides with the projection from the centre of the open disk C onto the circle forming the boundary of C'. We observe that from the definition of $\{Y_n\}$ and from condition (3) in the definition of the set of type II (§ 2) it follows that if C contains no component of $E^2\backslash Y_{n,i}$ then C is disjoint with Y_0. However, if C is unbounded, then

$$r_n|[(Y_0 \cap \mathring{Y}_{n,i} \cap C')\backslash(Y_{n+1}(X))^\circ]$$

coincides with the projection onto the circle forming the boundary of C'.

Let us observe that the maps

$$r_0\colon Y_0 \to Y_0 \cap (Y_1(X))^\circ,$$

$$r_n\colon Y_0 \cap (Y_n(X))^\circ \to Y_0 \cap (Y_{n+1}(X))^\circ \quad \text{for} \quad n \in \mathfrak{N}$$

are retractions such that

1) If $\varepsilon_n' = \sup_{y\in Y_0\cap(Y_n(X))^\circ} \varrho(r_n(y), y)$ then $\lim_{n=\infty} \varepsilon_n' = 0$,

2) $r_0(Y_0\backslash(Y_1(X))^\circ) \subset Y_0(X)$,

3) $r_n((Y_0 \cap (Y_n(X))^\circ)\backslash(Y_0 \cup (Y_{n+1}(X))^\circ)) \subset Y_0(X)$ for $n \in \mathfrak{N}$.

(The above Condition 1) follows from (2.7)).

Moreover, one can easily see that $Y_0 \cap (Y_n(X))^\circ \supset Y_0 \cap (Y_{n+1}(X))^\circ$ and that $\bigcap_{n\in\mathfrak{N}} Y_0 \cap (Y_n(X))^\circ = Y_0(X)$. It follows that setting

$$r(y) = r_0(y) \quad \text{for every point } y \in (Y_0\backslash(Y_1(X))^\circ) \cup Y_0(X),$$

$$r(y) = r_n(y) \quad \text{for every point } y \in Y_0 \cap ((Y_n(X))^\circ\backslash(Y_{n+1}(X))^\circ)$$

we get a retraction $r\colon Y_0 \to Y_0(X)$.

§ 6. Proof of (4.2). Let us construct for every $n = 1, 2, \ldots$ a homeomorphism

$$h_n\colon E^2 \xrightarrow[\text{onto}]{} E^2$$

so that

$$(6.1)_n \qquad h_n(P_n) = Y_n(X)$$

and

$$(6.2)_n \qquad h_{n+1}|(E^2\setminus\mathring{P}_n) = h_n|(E^2\setminus\mathring{P}_n) .$$

Since P_1 and $Y_1(X)$ are disks, there exists a homeomorphism $h_1\colon E^2 \xrightarrow[\text{onto}]{} E^2$ such that $h_1(P_1) = Y_1(X)$.

Suppose that we already have a homeomorphism $h_n\colon E^2 \xrightarrow[\text{onto}]{} E^2$ satisfying the condition $(6.1)_n$ and such that

$(6.3)_n$ If $P_{n,i}$ is a component of P_n, then $h_n(P_{n,i}) = \omega_n(P_{n,i})$

and

$(6.4)_n$ If $P_{n,i}$ is a component of P_n and $\hat{C}$ is a component of $E^2\setminus P_{n,i}$, then $\hat{\beta}_{P_{n,i}}(\hat{C}) = \beta_{\omega_n(P_{n,i})}(h_n(\hat{C}))$.

One can easily see that the conditions $(6.1)_1$, $(6.3)_1$, $(6.4)_1$ are satisfied.

As we know, each component $P_{n,i}$ of the set P_n contains one or two components of the set P_{n+1}, a k-perforated disk $\hat{\varphi}_n(P_{n,i})$ inscribed into $P_{n,i}$ and perhaps also a component $P_{n+1,j}$ which is a disk or an annulus (see (3.2) and (3.5)). The component $\omega_n(P_{n,i}) = h_n(P_{n,i})$ contains one or two components of the set $Y_{n+1}(X)$, one (which is the set $\omega_{n+1}(\hat{\varphi}_n(P_{n,i}))$ if $P_{n,1}$ contains only one component of P_{n+1}, and two (of which one is the set $\omega_{n+1}(\hat{\varphi}_n(P_{n,i}))$ and the second is the set $\omega_{n+1}(P_{n+1,j})$) if $P_{n,i}$ contains two components of P_{n+1} (see (4.10)). The set $\omega_{n+1}(\hat{\varphi}_n(P_{n,i}))$ $= \varphi_n(\omega_n(P_{n,i}))$ is a k-perforated disk inscribed into $\omega_n(P_{n,i})$ (this follows from (4.4) and (4.5)). The sets $P_{n+1,j}$ and $\omega_{n+1}(P_{n+1,j})$ are contained, respectively, in $\mathring{P}_{n,i}$ and $[\omega_n(P_{n,i})]^\circ$ (this follows from (3.4) and (4.7)). From (4.4) it follows that either the components $P_{n+1,j}$ and $\omega_{n+1}(P_{n+1,j})$ are both disks, or they are both annula. In this last case these annula are not contractible in the sets $P_{n,i}\setminus\hat{\varphi}_n(P_{n,i})$ and $\omega_n(P_{n,i})\setminus\omega_{n+1}(\hat{\varphi}_n(P_{n,i}))$, respectively, (see (3.5) and (4.8)). From (4.6) (b) and $(6.4)_n$ and the definition of the sequences of functions $\{\beta_n\}$ and $\{\hat{\beta}_n\}$ it follows that if a circle S which is a component of $\dot{P}_{n,i}$ and the set $P_{n+1,j}$ are contained in the same component $\hat{C}$ of the set $E^2\setminus\hat{\varphi}_n(P_{n,i})$ then the circle $h_n(S)$ (it is a component of $[\omega_n(P_{n,i})]^\bullet$) and the set $\omega_{n+1}(P_{n+1,j})$ are contained in the same component C of the set $E^2\setminus\omega_{n+1}(\hat{\varphi}_n(P_{n,i}))$.

From the above facts (see Fig. 2) it follows that the homeomorphism

$$h_{P_{n,i}}\colon \dot{P}_{n,i} \to [\omega_n(P_{n,i})]^\bullet$$

defined by the formula

$$h_{P_{n,i}}(x) = h_n(x) \quad \text{for every point } x \in \dot{P}_{n,i}$$

can be extended to a homeomorphism $\bar{h}_{P_{n,i}}\colon P_{n,i} \to \omega_n(P_{n,i})$ satisfying the conditions:

$$\bar{h}_{P_{n,i}}(\hat{\varphi}_n(P_{n,i})) = \omega_{n+1}(\hat{\varphi}_n(P_{n,i}))$$

and

$$\bar{h}_{P_{n,i}}(P_{n+1,j}) = \omega_{n+1}(P_{n+1,j})$$

(this last condition only in the case where $P_{n,i}$ contains two components of the set P_{n+1}).

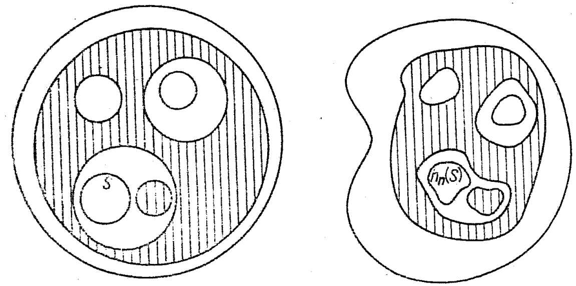

Fig. 2

Setting

$$h_{n+1}(x) = h_n(x) \quad \text{for every point } x \in E^2\setminus P_n ,$$
$$h_{n+1}(x) = \bar{h}_{P_{n,i}}(x) \quad \text{for every point } x \in P_{n,i} , \quad \text{where } P_{n,i} \in \Box(P_n) ,$$

one gets a homeomorphism $h_{n+1}\colon E^2 \xrightarrow[\text{onto}]{} E^2$ satisfying the conditions $(6.1)_{n+1}$, $(6.2)_n$, $(6.3)_{n+1}$ and $(6.4)_{n+1}$. The conditions $(6.1)_n$ and $(6.2)_n$ imply that $\boldsymbol{h} = \{h_n, X, Y_0(X)\}$ and $\boldsymbol{g} = \{h_n^{-1}, Y_0(X), X\}$ are fundamental sequences. It is clear that

$$(6.5) \qquad \boldsymbol{g}\circ\boldsymbol{h} = \boldsymbol{i}_X , \quad \boldsymbol{h}\circ\boldsymbol{g} = \boldsymbol{i}_{Y_0(X)} ,$$

where $\boldsymbol{i}_A$ denotes, for every compactum $A \subset E^2$, the fundamental identity sequence. It is obvious that (6.5) implies that $\mathrm{Sh}(X) = \mathrm{Sh}(Y_0(X))$.

§ 7. Main theorem. As we have shown, there exists for every compactum $X \subset E^2$ a compactum $Y_0(X) \subset Y_0$ satisfying the conditions (4.1) and (4.2). Thus we have the following

(7.1) THEOREM. *There is in the plane E^2 a compactum Y_0 such that every compactum $X \subset E^2$ has the same shape as a retract of Y_0.*

It is known ([2], p. 234) that for every retract A_0 of any compactum A the relation $\mathrm{Sh}(A_0) \leqslant \mathrm{Sh}(A)$ holds true. Thus one gets from theorem (7.1) the following

(7.2) COROLLARY. *There is in the plane E^2 a compactum which is a majorant for the shapes of all plane compacta.*

Let us add that by a slight modification of the construction of the compactum Y_0 one can obtain a plane compactum Y'_0 of dimension 1 such that every plane compactum X has the same shape as a retract of Y'_0.

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On transfinite sequences of *B*-measurable functions

by

Tibor Šalát (Bratislava)

Abstract. The notion of the convergence of transfinite sequences of real numbers and functions was introduced by Professor W. Sierpiński (Fund. Math. 1 (1920), pp. 132–141). In this paper that notion is extended for metric spaces. A part of results of the paper generalizes some earlier results of W. Sierpiński and H. Malchair, further the transfinite sequences of functions with closed graphs are investigated.

In paper [10] the notion of the limit of the transfinite sequence of real numbers and the notion of the limit function of the transfinite sequence of real functions were introduced. The idea and some results of paper [10] were developed in some further papers by H. Malchair and M. M. Lavrentieff (see e.g. [3]–[7]).

We shall generalize these notions and some results of the above-mentioned papers to metric spaces and we shall prove one theorem on limit functions of transfinite sequences of functions with closed graphs (see Theorem 4).

The following definitions generalize the above-mentioned notions.

Definition 1. Let (X, ϱ) be a metric space and let Ω denote the first uncountable ordinal number. The transfinite sequence

$$\{a_\xi\}_{\xi<\Omega} \tag{1}$$

of elements of the space X is said to be *convergent* and have a limit $a \in X$ if for each $\varepsilon > 0$ there exist an ordinal number $\alpha < \Omega$ such that for each ξ, $\alpha \leqslant \xi < \Omega$ the inequality $\varrho(a_\xi, a) < \varepsilon$ holds. If (1) has the limit a, we write $\lim_{\xi\to\Omega} a_\xi = a$ (or briefly $a_\xi \to a$).

Definition 2. Let X be a set and let (Y, ϱ_1) be a metric space. The transfinite sequence

$$\{f_\xi\}_{\xi<\Omega} \tag{2}$$

of functions $f_\xi\colon X\to Y$ is said to be *convergent* and have a limit function $f\colon X\to Y$ if for each $x \in X$ we have $\lim_{\xi\to\Omega} f_\xi(x) = f(x)$. If (2) has the limit function f, we write $\lim_{\xi\to\Omega} f_\xi = f$ (or briefly $f_\xi \to f$).