On the Liaponov type fixed point theorem

by

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Abstract. J. George, V. M. Sehgal and R. Smithson recently introduced the concept of a Liaponov function to prove a general fixed point theorem. In this note we show that this Liaponov type fixed point theorem is actually a special case of the more general fixed point theorem for φ-coherent mappings which was introduced by M. Furi and A. Vignoli.

Introduction. In [3] George, Sehgal and Smithson introduced a Liaponov function to prove a general fixed point theorem which included as special cases a large number of classical fixed point theorems. The purpose of this paper is to show that with a slight strengthening of the hypothesis the Liaponov method is, in fact, a special case of the concept of φ-coherent mappings developed by Furi and Vignoli [2]. The fixed point theorem for φ-coherent mappings was done in a very general setting.

Though the fixed point theorem in [3] is proved for multifunctions, we shall work with only single-valued mappings. It is not hard to see that the work done in this paper can be extended to multifunctions and hence Theorem 2.2 of [3] is actually a special case (with the strengthened hypotheses) of a slight variation of the fixed point theorem for φ-coherent multifunctions proved by the author, [4].

Preliminaries. Let $T\colon X\to X$ be a continuous mapping from the topological space X into itself. For $x\in X$ we shall consider the sequence $\{T^n(x)\}$, $T^0(x)=x$, $T^n(x)=T\big(T^{n-1}(x)\big)$ of iterates of x. Let $L(x)$ denote the set of subsequential limit points of $\{T^n(x)\}$.

Suppose now that $\varphi\colon X\to Y$ is a continuous mapping of X into the Hausdorff space Y. We then have the following definition.

DEFINITION 1. The mapping T is said to be *φ-coherent on* $\{T^n(x)\}$ if $x=T(x)$ is a necessary and sufficient condition for $\varphi\big(T^n(x)\big)$ to be

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constant for all n. If T is φ-coherent on $\{T^n(x)\}$ for all $x \in X$, we say that T is *φ-coherent on X*.

The following lemma is proved in [2]

LEMMA 2. *If for some $x \in X$ the sequence $\{\varphi(T^n(x))\}$ converges to $y \in Y$, then $\varphi(z) = y$ for all $z \in L(x)$.*

We can then prove the following variation of Theorem 1 of [2].

THEOREM 3. *Let T: $X \to X$ be a continuous mapping on a topological space X such that for some integer k, T^k is φ-coherent on X. If for some $x \in X$ the sequence $\{\varphi(T^n(x))\}$ converges and $z \in L(x)$, then $z = T^k(z)$, i.e. z is a periodic point of T.*

Proof. The continuity of T implies that if $z \in L(x)$, then $T^n(z) \in L(x)$ for all n. The convergence of $\varphi\{(T^n(x))\}$ and Lemma 2 then implies that $\varphi(T^n(z)) =$ constant for all n. Since $\{\varphi(T^{nk}(z))\}_{n=0}^{\infty}$ is a subsequence of $\{\varphi(T^n(z))\}$, then $\{\varphi(T^{nk}(z))\}_{n=0}^{\infty} =$ constant also. The φ-coherency of T^k then implies that $T^k(z) = z$.

We remark that Theorem 1 of [2] is the case of Theorem 3 when $k = 1$. The proof of Theorem 3 is essentially the same as Furi and Vignoli's and is included for completeness and so that we can note that φ-coherency on all of X is not needed. It is sufficient to assume only that T^k is φ-coherent on $\{T^{nk}(z)\}_{n=0}^{\infty}$. It should also be noticed that this form of the φ-coherent fixed point theorem also makes it applicable to Edelstein's theorem for periodic points. (See [1].)

Now let us suppose that (X, d) is a metric space and T a function from X into itself. We then define a Liaponov function as follows.

DEFINITION 4. A *Liaponov function* for T on $\{T^n(x)\}$ is a continuous function V: $X \to R$ (the reals) such that (1) $V(T^{n+1}(x)) \leqslant V(T^n(x))$ for all n and (2) V is bounded below.

The main theorem of [3] is then the following.

THEOREM 5. *Let T: $X \to X$ and suppose that $\{T^n(x)\}$ has convergent subsequences $T^{m_i}(x) \to y_0$ and $T^{m_i+k}(x) \to y_k$ for some k, $k = 1, \ldots, m$. If there is a Liaponov function V for T on $\{T^n(x)\}$ such that $y_0 \neq y_k$ implies that $V(y_0) \neq V(y_k)$, then $y_0 = y_k$. If $k = 1$ and if $y_1 = T(y_0)$, then y_0 is a fixed point of T.*

Main results. We begin by noting that the setting under which Theorem 3 is proved is much more general than that of Theorem 5. Thus it is possible that Theorem 5 is a special case of Theorem 3 while the converse is impossible.

The relaxation of the hypothesis of Theorem 5 that we shall make is that we assume that T is continuous. This condition is satisfied by all of the classical fixed point theorems that were shown to be special cases of Theorem 5 in [3]. Also we should note that the assumption that T is continuous allows us to omit (1) obviously the assumption that $T^{m_i+k}(x) \to y_k$ and $y_1 = T(y_0)$ (two very difficult facts to establish without the continuity of T) and (2) that V is bounded below.

We have the following theorem.

THEOREM 6. *Let T: $X \to X$ be a continuous mapping from the metric space X into itself. Suppose that for some $x \in X$, $y_0 \in L(x)$. Suppose also that there exists a continuous function V: $X \to R$ for which* (i) *$V(T^{n+1}(x)) \leqslant V(T^n(x))$ for all n and* (ii) *$V(y_0) = V(T^k(y_0))$ implies that $y_0 = T^k(y_0)$ for some fixed integer k. Then*

(1) *T^k is φ-coherent on $\{T^{nk}(y_0)\}_{n=0}^{\infty}$ where $\varphi = V$.*

(2) *The sequence $\{\varphi(T^n(x))\}$ is convergent. And*

(3) *y_0 is a periodic point of T.*

Proof. (1) Suppose that $T^k(y_0) = y_0$. Then $T^{nk}(y_0) = y_0$ also, and hence $\varphi(T^{nk}(y_0)) = \varphi(y_0) =$ constant for all n.

Now if $\varphi(T^{nk}(y_0)) =$ constant for all n we note that $\varphi(T^0(y_0)) = V(y_0) = \varphi(T^k(y_0)) = V(T^k(y_0))$. Condition (ii) above then implies that $T^k(y_0) = y_0$.

(2) Since $y_0 \in L(x)$, we know that there exists a subsequence $\{T^{m_i}(x)\}$ such that $T^{m_i}(x) \to y_0$. The continuity of V (and hence φ) implies that $\varphi(T^{m_i}(x)) \to \varphi(y_0)$. Condition (i) above shows that $\varphi(T^{n+1}(x)) \leqslant \varphi(T^n(x))$. Thus $\varphi(T^n(x)) \to \varphi(y_0)$ also.

(3) The fact that y_0 is a periodic point of T then follows easily from Theorem 3.

Thus we see that the Liaponov type fixed point theorem is a special case the Furi and Vignoli's fixed point for φ-coherent mappings. As was stated previously, the multifunction form of the theorem will follow analogously.

References

[1] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. Lond. Math. Soc. 37 (1962), pp. 74–79.

[2] M. Furi and A. Vignoli, *A remark on some fixed point theorems*, Boll., UMI, S VI, 3 (1970), pp. 197–200.

[3] J. H. George, V. M. Sehgal, and R. E. Smithson, *Application of Liaponov's direct method to fixed point theorems*, AMS Proc. 28 (2) (1971), pp. 613–620.

[4] J. W. Thomas, *φ-coherent multifunctions*, to appear Boll. UMI. 4 (4) (1971), pp. 232–238.

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