In the first case we get the categories $\mathfrak{R}^*_{\simeq}$ and $\hat{\mathfrak{R}}^*_{\simeq}$ introduced by S. Mardešić and J. Segal in [3]. In the second case we get $\mathcal{G}^*$ and $\hat{\mathcal{G}}^*$ used here in § 5.

References

[1] K. Borsuk, *Concerning homotopy properties of compacta*, Fund. Math. 62 (1968), pp. 223–254.

[2] — *Theory of retracts*, Warszawa 1967.

[3] S. Mardešić and J. Segal, *Shapes of compacta and* ANR-*systems*, Fund. Math. 72 (1971), pp. 41–59.

[4] — — *Equivalence of the Borsuk and the* ANR-*system approach to shapes*, Fund. Math. 72 (1971), pp. 61–68.

[5] — *Retracts in Shape Theory*, Glasnik Mat. 6 (26) (1971), pp. 153–163.

[6] B. Mitchell, *Theory of Categories*, New-York–London 1965.

[7] M. Moszyńska, *Uniformly movable compact spaces and their algebraic properties*, Fund. Math. 77 (1972), pp. 125–144.

Reçu par la Rédaction le 29. 9. 1971

On free pseudo-complemented and relatively pseudo-complemented semi-lattices (*)

by

Raymond Balbes (St. Louis, Missouri)

Abstract. The first part of this paper is concerned with properties of the free algebras in the class of pseudo-complemented semi-lattices. In particular, an explicit construction is given for the free pseudo-complemented semi-lattice $P(n)$ with n free generators ($n < \infty$). As a result of this construction, the word problem is solved for the free algebras in this class and it is shown that the number of elements in $P(n)$ is $1+\sum_{k=0}^{n}\binom{n}{k}(2^{2^{n-k}}-1)$. In the last section we develop some properties of free relatively pseudo-complemented semi-lattices with n free generators ($n < \infty$). It is shown that these algebras are all (distributive) lattices and that for $n = 2$ the free algebra is isomorphic with $2\times 3\times 3$.

1. Preliminaries. The notation $\Pi_L S$ (or simply ΠS) will be used to denote the greatest lower bound of a non-empty subset S of a meet semi-lattice L; the greatest element of L, if it exists, is denoted by $1_L(1)$. If $S = \{x, y\}$ then $\Pi S = xy$; it is convenient to define $\Pi\phi = 1$ when L has a greatest element. The symbols ΣS, 0, $x+y$, and $\Sigma\phi$ are defined dually. We will identify each integer $n \geqslant 0$ with the set $\{0, \ldots, n-1\}$. In Sections 2 and 3, the topic is pseudo-complemented semi-lattices and so the terms "homomorphism", "subalgebra", etc. should be regarded in this context. However, the meaning of these terms is suitably altered in Section 4, where we discuss relatively pseudo-complemented semi-lattices.

2. Pseudo-complemented semi-lattices. A pseudo-complemented semi-lattice is an algebra $\langle L; \cdot, 0, *\rangle$ in which $\langle L; \cdot, 0\rangle$ is a meet semi-lattice with 0 and such that for each $x \in L$, there exists a largest y (denoted by x^*) such that $xy = 0$. It is well known that these algebras form an equational class. Some of the elementary properties of these algebras are listed below for easy reference (cf. [4]).

(*) This research was supported, in part, by a Summer Fellowship from the University of Missouri–St. Louis.

(i) $xy = 0 \Leftrightarrow x \leqslant y^*$,

(ii) $x \leqslant x^{**}$,

(iii) $x^{***} = x^*$,

(iv) $x \leqslant y \Rightarrow y^* \leqslant x^*$,

(v) $x^{**}y^{**} = (xy)^{**}$,

(vi) $x(xy)^* = xy^*$.

The elements of L which are of the form x^* are called *regular* (also known as *skeletal* or *closed*). We denote the set of regular elements by $R(L)$. More generally, for any $S \subseteq L$, write $S^* = \{s^* \mid s \in S\}$.

(vii) $R(L) = L^* = \{x \in L \mid x = x^{**}\}$.

We will need the following extension, due to Frink [4], of a theorem of Glivenko on regular elements of a pseudo-complemented distributive lattice.

THEOREM 2.1. *Let L be a pseudo-complemented semi-lattice. Then $R(L)$ is a Boolean algebra in which the meet operation in L and $R(L)$ are the same, the zero and unit of L and $R(L)$ are the same, the join operation on $R(L)$ is defined by $u \oplus v = (u^*v^*)^*$ and the complement operation in $R(L)$ is defined by $x' = x^*$. Furthermore $x \to x^{**}$ is a homomorphism of L onto $R(L)$.*

COROLLARY 2.2. *If X is a finite non-empty subset of a pseudo-complemented semi-lattice L then*

$$\prod_{S \subseteq X} [(\Pi S)(\Pi(X \sim S)^*)]^* = 0 .$$

Proof. It is routine to prove that if X is a finite non-empty subset of a Boolean algebra then

$$\prod_{S \subseteq X} \Sigma(S' \cup (X \sim S)) = 0 \quad \text{where} \quad S' = \{s' \mid s \in S\} .$$

In the present case Theorem 2.1 implies that

$$\prod_{S \subseteq X} [(\Pi S)(\Pi(X \sim S)^*)]^* = \prod_{S \subseteq X} [(\Pi S^{**})(\Pi(X \sim S)^*)]^*$$

$$= \prod_{S \subseteq X^{R(L)}} \Sigma_{R(L)}[S^* \cup (X \sim S)^{**}] = 0 .$$

COROLLARY 2.3. *Suppose X freely generates a pseudo-complemented semi-lattice L. Then X^{**} freely generates (as a Boolean algebra) the Boolean algebra $R(L)$.*

Proof. Since X generates L, X^{**} generates $R(L)$ as a pseudo-complemented semi-lattice, and therefore also as a Boolean algebra.

To conclude the proof, let B be a Boolean algebra and $\{a_x\}_{x \in X}$ a sequence in B. Since B is a pseudo-complemented semi-lattice, there is a homomorphism $f\colon L \to B$ such that $f(x) = a_x$. The required Boolean homomorphism from $R(L)$ to B is obtained by restricting f to $R(L)$.

3. Construction of $P(n)$. Let n be a non-negative integer. For each $S \subseteq n$, let B_S denote the lattice 2^S of all subsets of S together with a new zero, 0_S; that is, $0_S < T$ for every $T \in 2^S$. It is easy to see that B_S is a pseudo-complemented semi-lattice and so $L = \bigtimes_{S \subseteq n} B_S$ has the same property.

For a subset R of n, define x_R and b_R in L by

$$x_R(S) = \begin{cases} S \sim R & \text{if} \quad R \subseteq S \\ 0_S & \text{if} \quad R \not\subseteq S \end{cases}$$

and

$$b_R = \Big[\Big(\prod_{i \in R} x_{\{i\}}\Big)\Big(\prod_{j \notin R} x^*_{\{j\}}\Big)\Big]^* .$$

LEMMA 3.1. *Suppose $p \in n$, $T \subseteq n$ and $\mathfrak{R}$ is a family of subsets of n. Then*

(i) $\Pi\{x_R \mid R \in \mathfrak{R}\} = x_{\bigcup \mathfrak{R}}$,

(ii) $x_\phi = 1_L$,

(iii) $b_R(S) = \begin{cases} 1 & \text{if } R \neq S \\ 0 & \text{if } R = S \end{cases}$ *for each* $S \subseteq n$,

(iv) $(x_T \Pi\{b_R \mid R \in \mathfrak{R}\})^* = \Pi\{b_V \mid T \subseteq V \text{ and } V \notin \mathfrak{R}\}$,

(v) $x_T \Pi\{b_R \mid R \in \mathfrak{R}\} \leqslant b_V \Leftrightarrow V \in \mathfrak{R}$ *or* $T \not\subseteq V$,

(vi) $x_T \Pi\{b_R \mid R \in \mathfrak{R}\} \leqslant x_{\{p\}} \Leftrightarrow p \in T$ *or* $\{S \subseteq n \mid T \subseteq S\} \subseteq \mathfrak{R}$.

Proof. (i) and (ii) are easily verified.

(iii) Let $S \subseteq n$. Since 0_S is meet irreducible in B_S, $b_R(S)$ is either 0 or 1. Moreover,

$$b_R(S) = 1 \Leftrightarrow \Big(\prod_{i \in R} x_{\{i\}}(S)\Big)\Big(\prod_{j \notin R} x^*_{\{j\}}(S)\Big) = 0_S \Leftrightarrow x_{\{i\}}(S) = 0_S$$

for some $i \in R$ or $(x_{\{j\}}(S))^* = 0_S$ for some $j \notin R \Leftrightarrow i \notin S$ for some $i \in R$ or $j \in S$ for some $j \notin R \Leftrightarrow S \neq R$.

(iv) For each $S \subseteq n$, both sides of the stated equality, evaluated at S, are in $\{0_S, 1\}$. Moreover,

$$\Pi\{b_V \mid T \subseteq V \text{ and } V \notin \mathfrak{R}\}(S) = 1 \Leftrightarrow b_V(S) = 1$$

for all $V \subseteq n$ such that $T \subseteq V$ and $V \notin \mathfrak{R} \Leftrightarrow V \neq S$ for all $V \subseteq n$ such that $T \subseteq V$ and $V \notin \mathfrak{R} \Leftrightarrow T \nsubseteq S$ or $S \in \mathfrak{R} \Leftrightarrow x_T(S) = 0$ or $b_R(S) = 0$ for some $R \in \mathfrak{R}$

$$\Leftrightarrow x_T(S)\Pi\{b_R|\ R \in \mathfrak{R}\}(S) = 0 \Leftrightarrow x_T(S)\Pi\{b_R|\ R \in \mathfrak{R}\}^*(S) = 1 .$$

(v) Since $b_V(S) = 1$ exactly when $V \neq S$,

$$x_T\Pi\{b_R|\ R \in \mathfrak{R}\} \leqslant b_V \Leftrightarrow x_T(V)\Pi\{b_R|\ R \in \mathfrak{R}\}(V) = 0 .$$

But, again the meet irreducibility of 0_V implies the result.

(vi) First suppose $x_T\Pi\{b_R|\ R \in \mathfrak{R}\} \leqslant x_{\{p\}}$ but that there exists $S \subseteq n$ such that $T \subseteq S$ and $S \notin \mathfrak{R}$. Then $b_R(S) = 1$ for all $R \in \mathfrak{R}$ so $S \sim T = x_T(S) \leqslant x_{\{p\}}(S)$. Since $x_{\{p\}}(S) \geqslant \phi$, we infer that $p \in S$ and that $p \in S \subseteq x_{\{p\}}(S) \cup T = (S \sim \{p\}) \cup T$; thus $p \in T$. Conversely, (i) implies the result if $p \in T$ so assume $\{S \subseteq n|\ T \subseteq S\} \subseteq \mathfrak{R}$. Let $V \subseteq n$. If $T \subseteq V$ then $V \in \mathfrak{R}$ and so

$$x_T\Pi\{b_R|\ R \in \mathfrak{R}\}(V) \leqslant b_V(V) = 0 \leqslant x_{\{p\}}(V) ,$$

and if $T \nsubseteq V$ then

$$x_T\Pi\{b_R|\ R \in \mathfrak{R}\}(V) \leqslant x_T(V) = 0 \leqslant x_{\{p\}}(V) .$$

We now combine these results to prove the main theorem:

THEOREM 3.2. *Let $0 \leqslant n < \infty$ and L_1 the subalgebra of all elements $x \in \bigtimes_{S \subseteq n} B_S$ for which there exists a set $T \subseteq n$ such that for each $S \subseteq n$:*

$$x(S) \in \{0_S, S \sim T\} \tag{1}$$

and

$$T \nsubseteq S \text{ implies } x(S) = 0_S . \tag{2}$$

Then L_1 is the free pseudo-complemented semi-lattice with n free generators.

Proof. For $n = 0$, $L_1 \cong 2$ which is the free pseudo-complemented semi-lattice on 0 free generators. Now assume $n \geqslant 1$.

To show that L_1 is a subalgebra of $\bigtimes_{S \subseteq n} B_S$ suppose $T_1, T_2 \subseteq n$ and x_1, x_2 are members of $\bigtimes_{S \subseteq n} B_S$ which, together with T_1 and T_2, respectively, satisfy (1) and (2). It is easy to verify that x_1x_2 satisfies (1) and (2) with respect to the set $T_1 \cup T_2$ and that x_1^* satisfies (1) and (2) with respect to ϕ.

We will show that $\{x_{\{i\}}|\ i \in n\}$ freely generates L_1. First, to prove that this set generates L_1, let $x \in L_1$ and suppose T is a subset of n such that x and T satisfy (1) and (2). Since $x_T = \Pi\{x_{\{t\}}|\ t \in T\}$ it is sufficient to prove:

$$x = x_T\Pi\{b_R|\ x(R) = 0\} .$$

Indeed, let $S \subseteq n$. If $T \nsubseteq S$ then by (2), $x(S) = 0_S$ and

$$x_T\Pi\{b_R|\ x(R) = 0\}(S) \leqslant x_T(S) = 0 .$$

Now suppose $T \subseteq S$. If $x(S) = 0_S$ then also $x_T\Pi\{b_R|\ x(R) = 0\}(S) \leqslant b_S(S) = 0$ and if $x(S) \neq 0_S$ then $b_R(S) = 1$ for all R such that $x(R) = 0_R$ so $x_T\Pi\{b_R|\ x(R) = 0\}(S) = x_T(S) = S \sim T = x(S)$.

In order to prove that $\{x_{\{i\}}|\ i \in n\}$ freely generates L_1, let M be a pseudo-complemented semi-lattice and $\{a_i\}_{i \in n}$ a sequence in M. To simplify notation, let $A(T) = \{a_t|\ t \in T\}$ and $A^*(T) = \{a_t^*|\ t \in T\}$ whenever $T \subseteq n$. Now define $f\colon L_1 \to M$ as follows. For $x \in L_1$ there exists $T \subseteq n$ and $\mathfrak{R} \subseteq 2^n$ such that $x = x_T\Pi\{b_R|\ R \in \mathfrak{R}\}$. Define

$$f(x) = \Pi A(T) \prod_{R \in \mathfrak{R}} [\Pi(A(R) \cup A^*(n \sim R))]^* .$$

Our first step is to demonstrate that f is actually well defined. For this it suffices to prove that if $T_1, T_2 \subseteq n$; $\mathfrak{R}_1, \mathfrak{R}_2 \subseteq 2^n$ and

$$x_{T_1}\Pi\{b_R|\ R \in \mathfrak{R}_1\} \leqslant x_{T_2}\Pi\{b_R|\ R \in \mathfrak{R}_2\} \tag{3}$$

then

$$\Pi A(T_1) \prod_{R \in \mathfrak{R}_1} [\Pi(A(R) \cup A^*(n \sim R))]^* \leqslant \Pi A(T_2) \tag{4}$$

and

$$\Pi A(T_1) \prod_{R \in \mathfrak{R}_1} [\Pi(A(R) \cup A^*(n \sim R))]^* \leqslant \prod_{R \in \mathfrak{R}_2} [\Pi(A(R) \cup A^*(n \sim R))]^* . \tag{5}$$

For (4), let $t \in T_2$. Then by (3), $x_{T_1}\Pi\{b_R|\ R \in \mathfrak{R}_1\} \leqslant x_{\{t\}}$ and so by Lemma 3.1 (vi) either $t \in T_1$ or $\{S \subseteq n|\ T_1 \subseteq S\} \subseteq \mathfrak{R}_1$. In the first case $a_t \in A(T_1)$ so obviously

$$\Pi A(T_1) \prod_{R \in \mathfrak{R}_1} [\Pi(A(R) \cup A^*(n \sim R))]^* \leqslant a_t . \tag{6}$$

In the second case, we can assume $T_1 \neq n$:

$$\begin{aligned} &\Pi A(T_1) \prod_{R \in \mathfrak{R}_1} [\Pi(A(R) \cup A^*(n \sim R))]^* \\ &\quad \leqslant \Pi A(T_1) \prod_{T_1 \subseteq S \subseteq n} [\Pi(A(S) \cup A^*(n \sim S))]^* \\ &\quad = \Pi A(T_1) \prod_{V \subseteq n \sim T_1} [\Pi(A(T_1 \cup V)) \cup A^*(n \sim (T_1 \cup V))]^* \\ &\quad = \Pi A(T_1) \prod_{V \subseteq n \sim T_1} \Big[\Pi\Big(A(V) \cup A^*\big(((n \sim T_1) \sim V)\big)\Big)\Big]^* \\ &\quad = 0 \leqslant a_t \quad \text{by Corollary 2.2.} \end{aligned}$$

Hence (6) holds for all $t \in T_2$ and establishes the validity of (4).

To dispose of (5) let R_2 be any member of $\mathfrak{R}_2$. Then $x_{T_1}\Pi\{b_R|\ R\in\mathfrak{R}_1\} \leqslant b_{R_2}$ implies $R_2\in\mathfrak{R}_1$ or $T_1\not\subseteq R_2$. The first case clearly implies

(7) $$\Pi A(T_1)\prod_{R\in\mathfrak{R}_1}[\Pi(A(R)\cup A^*(n\sim R))]^* \leqslant [\Pi(A(R_2)\cup A^*(n\sim R_2))]^*$$

and in the second case, selecting $t_1\in T_1\sim R_2$, we have

$$\Pi A(T_1)[\Pi(A(R_2)\cup A^*(n\sim R_2))] \leqslant a_{t_1}\cdot a_{t_1}^* = 0$$

and so

$$\Pi A(T_1)\prod_{R\in\mathfrak{R}_1}[\Pi(A(R)\cup A^*(n\sim R))]^* \leqslant \Pi A(T_1) \leqslant [\Pi(A(R_2)\cup A^*(n\sim R_2))]^* .$$

Thus, (7) holds in both cases. This proves (5) and so f is well defined.

From the definition of f, it is trivial to check that $f(x_{\{i\}}) = a_i$ for all $i<n$ and that f preserves products. f also preserves 0 since

$$x_n b_n = (\Pi\{x_{\{i\}}|\ i\in n\})(\Pi\{x_{\{i\}}|\ i\in n\})^* = 0$$

implies

$$f(0) = f(x_n b_n) = (\Pi A(n))(\Pi A(n))^* = 0 .$$

So it remains to prove that f preserves *. Let $T\subseteq n$ and $\mathfrak{R}\subseteq 2^n$. Since f preserves products and 0, we need only show that

$$\big(f(x_T\Pi\{b_R|\ R\in\mathfrak{R}\})\big)^* \leqslant f\big((x_T\Pi\{b_R|\ R\in\mathfrak{R}\})^*\big).$$

In view of Lemma 3.1 (iv) this is equivalent to

(8) $$\Big\{\Pi A(T)\prod_{R\in\mathfrak{R}}[\Pi(A(R)\cup A^*(n\sim R))]^*\Big\}^* \leqslant \prod_{\substack{T\subseteq V\\ V\notin\mathfrak{R}}}[\Pi(A(V)\cup A^*(n\sim V))]^* .$$

Suppose $T\subseteq V$ and $V\notin\mathfrak{R}$. Then for each $R\in\mathfrak{R}$, either $V\not\subseteq R$ or $R\not\subseteq V$. In the first case there exists $v\in V\sim R$ so $a_v\in A(V)$ and $a_v^*\in A^*(n\sim R)$ and therefore

(9) $$\Pi(A(V)\cup A^*(n\sim V)\cup A(R)\cup A^*(n\sim R)) \leqslant a_v a_v^* = 0 .$$

In the second case there exists $r\in R\sim V$ and so $a_r\in A(R)$ and $a_r^*\in A^*(n\sim V)$ and again (9) holds.

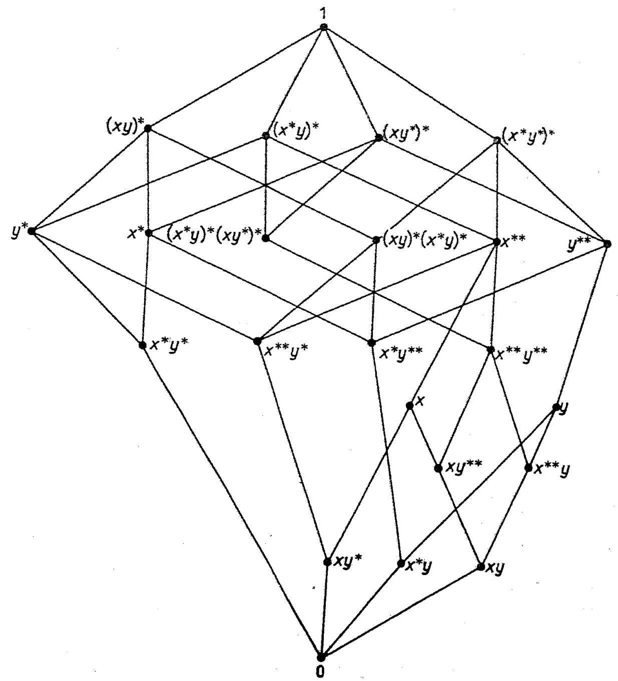
From (9),

$$\Pi(A(V)\cup A^*(n\sim V))\Big\{\Pi A(T)\prod_{R\in\mathfrak{R}}[\Pi(A(R)\cup A^*(n\sim R))]^*\Big\}^*$$
$$= \Pi(A(V)\cup A^*(n\sim V))\times$$
$$\times\Big\{\Pi A(T)\prod_{R\in\mathfrak{R}}[\Pi(A(V)\cup A^*(n\sim V)\cup A(R)\cup A^*(n\sim R))]^*\Big\}^*$$
$$= \Pi(A(V)\cup A^*(n\sim V))\Big\{\Pi A(T)\prod_{R\in\mathfrak{R}}[0]^*\Big\}^* = \Pi(A(V)\cup A^*(n\sim V))(\Pi A(T))^*$$
$$\leqslant \Pi A(V)(\Pi A(T))^* \leqslant \Pi A(T)(\Pi A(T))^* = 0 .$$

So for each $V\subseteq n$ such that $T\subseteq V$ and $V\notin\mathfrak{R}$:

$$\Big\{\Pi A(T)\prod_{R\in\mathfrak{R}}[\Pi(A(R)\cup A^*(n\sim R))]^*\Big\}^* \leqslant \big(\Pi(A(V)\cup A^*(n\sim V))\big)^*,$$

which verifies (8) and completes the proof.



The free pseudo-complemented semi-lattice on two free generators x, y

Corollary 3.3. *The number of elements in the free pseudo-complemented semi-lattice with n free generators is*

$$1+\sum_{k=0}^{n}{}'\binom{n}{k}(2^{2^{n-k}}-1) .$$

Proof. Let $T\subseteq n$ and let L_T be the members of $\bigtimes_{S\subseteq n} B_S$ which, together with T, satisfy conditions (1) and (2). For a given member x of L_T, there are two possible values for $x(S)$ if $T\subseteq S$ and only one possible value if $T \not\subseteq S$. Since there are $2^{n-|T|}$ subsets S of n for which $T \subseteq S$ we have $|L_T| = 2^{2^{n-|T|}}$.

Now for $T_1 \neq T_2$, $L_{T_1} \cap L_{T_2} = \{0\}$. Indeed, it is obvious that $0 \in L_T$ for all T and if $x \in L_{T_1} \cap L_{T_2}$ but $x \neq 0$ then there is a set S such that $x(S) \neq 0$. Since $x \in L_{T_i}$ we have $x(S) = S \sim T_i$ so $S \sim T_1 = S \sim T_2$ and also $T_1 \subseteq S$, $T_2 \subseteq S$. This implies the contradiction $T_1 = T_2$.

Thus, L_T has $2^{2^{n-|T|}}-1$ non zero members, and since there are $\binom{n}{|T|}$ subsets $T \subseteq n$ with $|T|$ elements, L has $\sum_{k=0}^{n} \binom{n}{k} (2^{2^{n-k}}-1)$ non-zero members.

A solution to the word problem for pseudo-complemented semi-lattices is given by the following finite procedure for determining whether or not $p(x_0, \ldots, x_{n-1}) \leqslant q(x_0, \ldots, x_{n-1})$ for arbitrary n-ary polynomials p, q and for free generators $x_0, \ldots, x_{n-1}$.

Now $p(x_0, \ldots, x_{n-1}) = x_{i_1} \cdot \ldots \cdot x_{i_r} p_1^*(x_0, \ldots, x_{n-1}) \cdot \ldots \cdot p_s^*(x_0, \ldots, x_{n-1})$ where $p_1, \ldots, p_s$ are n-ary polynomials. Let $X = \{x_0, \ldots, x_{n-1}\}$ and for each $R \subseteq X$, set $B_R = [(\Pi R)(\Pi(X-R))]^*$. Denote by L, the free pseudo-complemented semi-lattice generated by X.

Since the word problem is solved for Boolean algebras (c.f. [2], p. 61) and $\{p_j^*(x_0, \ldots, x_n) |\ j = 1, \ldots, s\} \cup \{B_R |\ R \subseteq n\}$ is a subset of the Boolean algebra $B(L)$, Corollary 2.3 implies that there is a finite procedure for determining the set

$$\mathfrak{R}_1 = \{R |\ p_1^*(x_0 \cdot \ldots \cdot x_{n-1}) \cdot \ldots \cdot p_s^*(x_0, \ldots, x_{n-1}) \leqslant B_R\}.$$

Note that $p_1^*(x_0, \ldots, x_{n-1}) \cdot \ldots \cdot p_s^*(x_0, \ldots, x_{n-1}) = \Pi\{B_R |\ R \in \mathfrak{R}_1\}$. Similarly, the set

$$\mathfrak{R}_2 = \{R |\ q_1^*(x_0, \ldots, x_{n-1}) \cdot \ldots \cdot q_u^*(x_0, \ldots, x_{n-1}) \leqslant B_R\}$$

can also be determined where $q_1, \ldots, q_u$ are n-ary polynomials and

$$q(x_0, \ldots, x_{n-1}) = x_{j_1} \cdot \ldots \cdot x_{j_t} q_1^*(x_0, \ldots, x_{n-1}) \cdot \ldots \cdot q_u^*(x_0, \ldots, x_{n-1}).$$

It follows from Lemma 3.1 (v), (vi) that $p(x_0, \ldots, x_{n-1}) \leqslant q(x_0, \ldots, x_{n-1})$ if and only if the following conditions are both satisfied

(c1) Either $\{x_{j_1}, \ldots, x_{j_t}\} \subseteq \{x_{i_1}, \ldots, x_{i_r}\}$ or

$$\{T \subseteq \{x_0, \ldots, x_{n-1}\} | \{x_{i_1}, \ldots, x_{i_r}\} \subseteq T\} \subseteq \mathfrak{R}_1.$$

(c2) If $R \in \mathfrak{R}_2 \sim \mathfrak{R}_1$ then $\{x_{i_1}, \ldots, x_{i_r}\} \not\subseteq R$.

4. Free relatively pseudo-complemented semi-lattices. There are many equational classes, closely related to pseudo-complemented semi-lattices, in which the free algebras have not been determined. The pseudo-complemented distributive lattices is an example. In fact, the lattice of equational subclasses of pseudo-complemented distributive lattices form a chain $\mathcal{B}_{-1} \subset \mathcal{B}_0 \subset \mathcal{B}_1 \subset \ldots \subset \mathcal{B}_\omega$ and only in the first three classes has the problem been solved (c.f. [2], [1]).

In another direction, there are the Heyting algebras (relatively pseudo-complemented lattices) for which the problem is solved only for $n = 1$, [9]. Along these lines, we conclude with some results on relatively pseudo-complemented semi-lattices.

Theorem 3.2 provides a simple method for constructing free pseudo-complemented semi-lattices but neither the statement nor the proof indicate how the construction was motivated. In this section however, we take the point of view that since we are dealing with an equational class, the free algebras already exist and we are merely trying to determine their properties.

A *relatively pseudo-complemented semi-lattice* (for brevity, *implicative semi-lattice*) is an algebra $\langle L; \cdot, \to\rangle$ in which $\langle L; \cdot\rangle$ is a meet semi-lattice and for any pair of elements $x, y \in L$ there is a largest z (denoted by $x \to y$) such that $xz \leqslant y$. As stated above, these algebras form an equational class. Most of the following identities can be found in [3].

(i) $xy \leqslant z \Leftrightarrow x \leqslant y \to z$,

(ii) $x \to x = 1$,

(iii) $1 \to x = x$,

(iv) $x(x \to y) = xy$,

(v) $x(y \to z) = x(xy \to xz) = x(xy \to z) = x(y \to xz)$,

(vi) $x \leqslant y \Rightarrow z \to x \leqslant z \to y$ and $y \to z \leqslant x \to z$,

(vii) $x \to (y \to z) = xy \to z$,

(viii) $x \to yz = (x \to y)(x \to z)$,

(ix) $(x \to y) \to x \leqslant (x \to y) \to y$,

(x) $((x \to y) \to x) \to y = x \to y$,

(xi) $((x \to y) \to x)(y \to x) \to y) = xy$.

We will restrict our attention to the proof of (xi):

$$x((y \to x) \to y) = x(x(y \to x) \to y) = x(x \to y) = xy$$

and so by (v)

$$((x \to y) \to x)((y \to x) \to y) = ((xy \to y) \to x)((y \to x) \to y)$$
$$= (1 \to x)((y \to x) \to y) = x((y \to x) \to y) = xy.$$

In what follows, $\mathfrak{I}(S)$ is the free implicative semi-lattice with S as a finite non-empty set of free generators.

LEMMA 4.1. *Each element of $\mathfrak{I}(S)$ is a finite product of elements of the form $u \to s$ where $u \in \mathfrak{I}(S)$ and $s \in S$.*

Proof. The set L of elements of the form described is obviously closed under products. Also $S \subseteq L$ since $1 \to s = s$ for each $s \in S$. Suppose

$x, y \in L$ and $y = \prod_{i<n} u_i \to s_i$ where $u_i \in \mathfrak{I}(S)$ and $s_i \in S$. Then

$$x \to y = x \to \Big(\prod_{i<n} (u_i \to s_i)\Big) = \prod_{i<n} \big(x \to (u_i \to s_i)\big) = \prod_{i<n} (xu_i \to s_i) \in L\,.$$

Thus, since S generates $\mathfrak{I}(S)$, $L = \mathfrak{I}(S)$.

THEOREM 4.2. *$\mathfrak{I}(S)$ is a bounded distributive lattice. For $p, q \in \mathfrak{I}(S)$*

$$p + q = \prod_{s \in S} \big((p \to s)(q \to s) \to s\big)\,.$$

Proof. By Lemma 4.1, it is evident that ΠS is the least member of $\mathfrak{I}(S)$. By (i) and (iv) $p, q \leqslant (p \to s)(q \to s) \to s$ for each $s \in S$ so $u = \prod_{s \in S} \big((p \to s)(q \to s) \to s\big)$ is an upper bound of p, q. Now assume $p \leqslant x$, $q \leqslant x$ and $x = \prod_{i<n} (u_i \to s_i)$ where $u_i \in \mathfrak{I}(S)$, $s_i \in S$. For each $i < n$, $p \leqslant x \leqslant u_i \to s_i$ so $u_i \leqslant p \to s_i$. Similarly $u_i \leqslant q \to s_i$. Thus $u_i \leqslant (p \to s_i)(q \to s_i)$ and $(p \to s_i)(q \to s_i) \to s_i \leqslant u_i \to s_i$. Hence

$$u = \prod_{s \in S} \big((p \to s)(q \to s) \to s\big) \leqslant \prod_{i<n} \big((p \to s_i)(q \to s_i) \to s_i\big) \leqslant \prod_{i<n} u_i \to s_i = x\,.$$

Finally any implicative lattice is distributive.

Note that in $\mathfrak{I}(S)$,

$$p_1 + \ldots + p_m = \prod_{s \in S} \Big[\Big(\prod_{i<m} (p_i \to s)\Big) \to s\Big]\,.$$

LEMMA 4.3. *Let $n \geqslant 1$ and $p, q \leqslant n$. Then in any implicative semi-lattic*

$$[(a_n \to a_0) \prod_{i<n} (a_i \to a_{i+1})] \to a_p = [(a_n \to a_0) \prod_{i<n} (a_i \to a_{i+1})] \to a_q\,.$$

Proof. From (iv) it is readily seen that $x_0 \prod_{i<n} (x_i \to x_{i+1}) = \prod_{i \leqslant n} x_i$. Thus

$$\{(a_n \to a_0) \prod_{i<n} (a_i \to a_{i+1})\}\{[(a_n \to a_0) \prod_{i<n} (a_i \to a_{i+1}) \to a_p\}$$
$$= [(a_n \to a_0) \prod_{i<n} (a_i \to a_{i+1})] a_p = \prod_{j \leqslant n} a_j \leqslant a_q$$

which implies the result.

It is well known that if S is a finite non-empty subset of a distributive lattice with 0, 1 and S has the property that $\Sigma S = 1$ and $st = 0$ when s, t are distinct in S, then L is isomorphic with $\bigtimes_{s \in S} (s]$.

Returning to $\mathfrak{I}(S)$, we find it convenient to label the member of S. Thus, let $S = \{s_i |\ i \leqslant n\}$. The element $a = (s_n \to s_0) \prod_{i<n} (s_i \to s_{i+1})$ plays a special roll in what follows. Lemma 4.3 asserts that $a \to s_p = a \to s_q$ for $p, q \leqslant n$.

THEOREM 4.4. $\mathfrak{I}(S) \cong (a] \times (a \to s_0]$.

Proof. Since $a \to s_0 \leqslant a \to s_p$ for all $p \leqslant n$, $a(a \to s_0) \leqslant s_p$ for all p. Thus, $a(a \to s_0) \leqslant \prod_{p \leqslant n} s_p = 0$. Next, we verify

(i) $(s_n \to s_0) + (a \to s_0) = 1$ and

(ii) $(s_i \to s_{i+1}) + (a \to s_0) = 1$ for $i \leqslant n-1$.

For (i), we start with the inequality $a \leqslant s_n \to s_0$. For each $j \leqslant n$, $(s_n \to s_0) \to s_j \leqslant a \to s_j = a \to s_0$. So,

$$\big((s_n \to s_0) \to s_j\big)\big((a \to s_0) \to s_j\big) \leqslant (a \to s_0)\big((a \to s_0) \to s_j\big) \leqslant s_j\,.$$

Thus

$$s_n \to s_0 + a \to s_0 = \prod_{i \leqslant n} \big[\big((s_n \to s_0) \to s_j\big)\big((a \to s_0) \to s_j\big) \to s_j\big] = 1\,.$$

The proof of (ii) is essentially the same as that of (i).

To complete the proof:

$$a + a \to s_0 = \big((s_n \to s_0) \prod_{i<n} (s_i \to s_{i+1})\big) + a \to s_0$$
$$= \big((s_n \to s_0) + a \to s_0\big) \prod_{i<n} \big((s_i \to s_{i+1}) + (a \to s_0)\big) = 1\,.$$

The principal ideal $(a]$ obviously contains the elements 0 and a and these elements are distinct since this would be the case if S were a set of free generators of a Boolean algebra. The following lemma is a useful tool in determining whether one has been successful in finding all of the elements of a principal ideal.

LEMMA 4.5. *Suppose $x \in \mathfrak{I}(S)$ and T is a subset of $(x]$ that satisfies*

(i) *$x \in T$,*

(ii) *$xs \in T$ for each $s \in S$,*

(iii) *If $t_1, t_2 \in T$ and $t_1 \nleqslant t_2$ then $t_1 t_2 \in T$ and $x(t_1 \to t_2) \in T$.*

Then $T = (x]$.

Proof. $(x]$ is an implicative semi-lattice with the same meet as that of $\mathfrak{I}(S)$, but relative complement $\Rightarrow$ defined by $u \Rightarrow v = x(u \to v)$; also the map $u \to ux$ is a homomorphism of $\mathfrak{I}(S)$ onto $(x]$. Now (i)–(iii) imply that T is a subalgebra of $(x]$ that contains $\{xs |\ s \in S\}$. But, since S generates $\mathfrak{I}(S)$, $\{xs |\ s \in S\}$ generates $(x]$ so $T = (x]$.

THEOREM 4.6. $\mathfrak{I}(S) \cong 2 \times (a \to s_0]$.

Proof. Let $T = \{0, a\}$ and verify the hypothesis of Lemma 4.4.

We have been unable to determine $(a \to s_0]$ when $n \geqslant 3$. However, when $n = 2$, the results are surprising:

THEOREM 4.7. *The free relatively pseudo-complemented semi-lattice on two free generators is $2 \times 3 \times 3$.*

Proof. Let $S=\{x, y\}$.

By Lemma 4.3, $u=(x\to y)\to x$ and $v=(y\to x)\to y$ are members of $(a\to x]$.

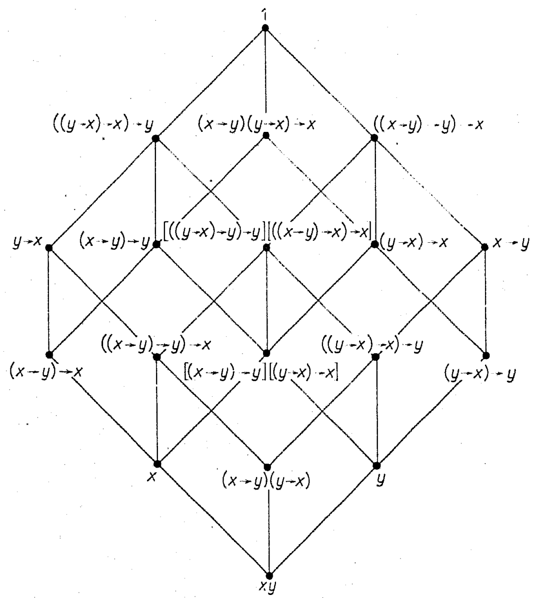
Now,

$$
\begin{aligned}
(u\to x)(v\to x)(a\to x) &= [((x\to y)\to x)\to x][((y\to x)\to y)\to x][(x\to y)(y\to x)\to x] \\
&= [((x\to y)\to x)\to x][y\to x][(x\to y)(y\to x)\to x] && \text{by (x)} \\
&= [((x\to y)\to x)\to x][y\to x][(x\to y)\to x] && \text{by (v)} \\
&\leqslant x && \text{by (xi).}
\end{aligned}
$$

So, $a\to x\leqslant (u\to x)(v\to x)\to x$ and by symmetry $a\to x = a\to y \leqslant (u\to y)(v\to y)\to y$ and hence

$$u+v\leqslant a\to x\leqslant [(u\to x)(v\to x)\to x][(u\to y)(v\to y)\to y]=u+v\,.$$

Combining this with $uv=0$ (see (xi)) we obtain $(a\to x]=(u]\times(v]$. Finally an application of Lemma 4.3 and the observation that $0<x<(x\to y)\to x$, $0<y<(y\to x)\to y$ yields $(a\to x]=3\times 3$.



The free implicative semi-lattice on two free generators x, y

References

[1] R. Balbes and A. Horn, *Stone lattices*, Duke Math. J. 38 (1970), pp. 537–545.

[2] G. Birkhoff, *Lattice theory* (third edition), Amer. Math. Soc. Publ., Vol. 25, Providence, R. I. (1967).

[3] H. B. Curry, *Foundations of Mathematical Logic*, New York 1963.

[4] O. Frink, *Pseudo-complements in semi-lattices*, Duke Math. J. 29 (1962), pp. 505–514.

[5] V. Glivenko, *Sur quelques points de la logique de M. Brouwer*, Bull. Acad. Sci. Belg. 15 (1929), pp. 183–188.

[6] H. Lakser, *The structure of pseudo-complemented distributive lattices, I, Subdirect products*, Trans. Amer. Math. Soc. 156 (1971).

[7] K. B. Lee, *Equational classes of distributive pseudo-complemented lattices*, Canad. J. Math. 22 (1970), pp. 881–891.

[8] W. C. Nemitz, *Implicative semi-lattices*, Trans. Amer. Math. Soc. 117 (1965), pp. 128–142.

[9] I. Nishimura, *On formulas of one variable in intuitionistic propositional calculus*, J. Symbolic Logic 25 (1960), pp. 327–331.

UNIVERSITY OF MISSOURI
St. Louis

Reçu par la Rédaction le 25. 11. 1971