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Models in which all long indiscernible sequences are indiscernible sets

by

Wilfrid Hodges (London)

Abstract. If T is any first-order theory with infinite models, and φ a formula, we construct models of T of all regular cardinalities $\varkappa$ such that $\varkappa\not\to(\varkappa)^2_2$, in which all φ-indiscernible sequences of order-type $\varkappa$ are in fact φ-indiscernible sets, i.e. φ-indiscernible under all orderings. The restriction to regular $\varkappa$ is essential, since if $\varkappa$ is a singular strong limit number, we show that every model of Peano arithmetic of cardinality $\varkappa$ contains an increasing sequence of order-type $\varkappa$. We show finally that if $\varkappa$ is regular, $\varkappa\not\to(\varkappa)^2_2$, $\varkappa>|T|$ and there is a model of T of cardinality $\varkappa$ which is elementarily embeddable in every model of T of cardinality $\varkappa$, then all models of T of cardinality T have indiscernible sets of cardinality $\varkappa$.

We extend a theorem of Ehrenfeucht, so as to find models of all regular non-weakly-compact cardinalities $\varkappa$ in which all indiscernible sequences of order-type $\varkappa$ are in fact indiscernible sets. We show an obstacle to extending the theorem to singular cardinals. Finally we apply the theorem to prove that prime structures of certain cardinalities contain large indiscernible sets [1].

§ 1. Let $\mathfrak{A}$ be an L-structure with domain A, $\varphi(v_0, \ldots, v_{n-1})$ a formula of L, and $(X, <)$ a linearly ordered set with $X\subseteq A$. We call $(X, <)$ a *φ-indiscernible sequence* if for any two n-tuples $\bar{a}=(a_0, \ldots, a_{n-1})$, $\bar{b}=(b_0, \ldots, b_{n-1})$ of distinct elements of X in increasing order, and any permutation π of $\{0, \ldots, n-1\}$,

$$\mathfrak{A}\models\varphi[a_{\pi(0)}, \ldots, a_{\pi(n-1)}] \quad \text{iff} \quad \mathfrak{A}\models\varphi[b_{\pi(0)}, \ldots, b_{\pi(n-1)}]\,.$$

We call $(X, <)$ a *φ-indiscernible set* if for every linear ordering $\lessdot$ of X, $(X, \lessdot)$ is a φ-indiscernible sequence. By *indiscernible sequence* (*set*) we mean φ-indiscernible sequence (set) for all formulae φ. (See Shelah [4] Defs. 5.1, 5.2 for this terminology.)

Let T be a complete first-order theory with infinite models. We seek those models $\mathfrak{A}$ of T in which for every formula φ, there is no φ-indis-

(1) This paper is related to work in the author's D. Phil. thesis, written at Oxford University under the kind supervision of John N. Crossley.

cernible sequence of order-type $|\mathfrak{A}|$ (= cardinality of $\mathfrak{A}$) which is not also a φ-indiscernible set. Let $E(T)$ be the class of all cardinals $\varkappa$ such that there is such a model of cardinality $\varkappa$. A. Ehrenfeucht showed in his classic paper [1] that if some model of T has an infinite φ-indiscernible sequence which is not also a φ-indiscernible set, then $E(T)$ contains all regular cardinals $\varkappa > |T|$ which are not strong limit numbers. (This is what Ehrenfeucht actually proved, though not what he said he proved.) On the other hand if $\varkappa \to (\varkappa)^2_2$, then clearly $\varkappa \notin E(T)$. This leaves a gap to be bridged. Ehrenfeucht's argument will not generalise to bridge it; we present another argument which will (at least for regular cardinals) [2].

THEOREM 1. *Let T be a theory with infinite models, and $\varkappa$ a regular cardinal $> |T|$ such that $\varkappa \not\to (\varkappa)^2_2$. Then T has a model $\mathfrak{A}$ of cardinality $\varkappa$ such that if φ is a formula and $(X, <)$ is (in $\mathfrak{A}$) a φ-indiscernible sequence of order-type $\varkappa$, then $(X, <)$ is in fact a φ-indiscernible set.*

Proof. Skolemise T to get T^* with $|T^*| = |T|$. Since $\varkappa \not\to (\varkappa)^2_2$, a theorem of Hanf [3] says that there is a linearly ordered set $(Y, \precsim)$ of cardinality $\varkappa$ in which neither $\varkappa$ nor $\varkappa^*$ can be order-embedded. Construct a model of T^* in which $(Y, \precsim)$ is an indiscernible sequence. Let $\mathfrak{A}^*$ be the Skolem hull of Y in this model, and let $\mathfrak{A}$ be the reduct of $\mathfrak{A}^*$ to T's language. We claim $\mathfrak{A}$ works for the theorem. Clearly $\mathfrak{A}$ has cardinality $\varkappa$.

Suppose then that $(X, <)$ is, in $\mathfrak{A}^*$, a $\varphi(v_0, \ldots, v_{n-1})$-indiscernible sequence of order-type $\varkappa$. By regularity of $\varkappa$ we can assume without loss that there is a term $\tau(v_0, \ldots, v_{m-1})$ such that each element of X has form $\tau^{\mathfrak{A}^*}(\bar{a})$, where $\bar{a}$ is in the set $[Y]^m$ of strictly increasing m-tuples from $(Y, \precsim)$. Choose $B \subseteq [Y]^m$ so that $\tau^{\mathfrak{A}^*}$ maps B bijectively onto X. Since $\varkappa$ is regular, we can use standard tricks (cf. [1]) to find in B a subset B', also of cardinality $\varkappa$, such that if $\bar{a} \in B'$ and $i < m$, then

$$\begin{array}{lll} \textit{either} & a_i = b_i & \text{for all } \bar{b} \in B' \\ \textit{or} & a_i \neq b_j & \text{for all } \bar{b} \in B' - \{\bar{a}\} \text{ and all } j < m. \end{array}$$

LEMMA 2. *Every set $Z \subseteq Y$ of cardinality $\varkappa$ is split into two sets Z_1, Z_2 both of cardinality $\varkappa$ by some element z of Z; i.e. if $z_1 \in Z_1$ and $z_2 \in Z_2$ then $z_1 \precsim z \precsim z_2$.*

Proof of lemma. Suppose not; then there is a set $Z \subseteq Y$ of cardinality $\varkappa$ such that every element z of Z has either $<\varkappa$ predecessors or $<\varkappa$ successors in Z. Put

$$P = \{z \in Z\colon z \text{ has } <\varkappa \text{ predecessors in } Z\},$$
$$S = \{z \in Z\colon z \text{ has } <\varkappa \text{ successors in } Z\}.$$

(2) A. Ehrenfeucht was aiming to show in [1] that if $\varkappa$ is a transfinite cardinal and T a countable complete theory categorical in power $2^\varkappa$, then any infinite φ-indiscernible sequence in a model of T is a φ-indiscernible set. This is still true, as Morley showed by a different proof in Trans. Amer. Math. Soc. 114 (1965), pp. 514–538.

Since $P \cup S = Z$, either P or S has cardinality $\varkappa$. If P has, then $\varkappa$ is order-embeddable in P; if S has, then $\varkappa^*$ is order-embeddable in S. Either possibility contradicts the choice of $(Y, \precsim)$. This proves the lemma.

The following lemma is easy.

LEMMA 3. *If M, N are disjoint sets of cardinality $\varkappa$, and P, Q are disjoint sets of cardinality $\varkappa$ such that $M \cup N = P \cup Q$, then either $M \cap P$ and $N \cap Q$ both have cardinality $\varkappa$, or $M \cap Q$ and $N \cap P$ both have cardinality $\varkappa$.*

Suppose now that $i_0, \ldots, i_{j-1}$ are the $i < m$ such that if $\bar{a} \neq \bar{b}$ in B', then $a_i \neq b_i$. Enumerate $\{i_0, \ldots, i_{j-1}\} \times \{i_0, \ldots, i_{j-1}\}$, say as $(k_0, k_0'), \ldots, (k_{j^2-1}, k'_{j^2-1})$. Split B' into two disjoint parts D_0, D_0' both of cardinality $\varkappa$. By Lemmas 2, 3 there are sets $D_1 \subseteq D_0$, $D_1' \subseteq D_0'$, both of cardinality $\varkappa$, and $z_0 \in Y$ such that

$$\begin{array}{ll} \textit{either} & \bar{a} \in D_1,\ \bar{b} \in D_1' \text{ implies } a_{k_0} \precsim z_0 \precsim b_{k_0'}, \\ \textit{or} & \bar{a} \in D_1,\ \bar{b} \in D_1' \text{ implies } b_{k_0'} \precsim z_0 \precsim a_{k_0}. \end{array}$$

Repeat j^2 times, to find $D_0 \supseteq \ldots \supseteq D_{j^2}$ and $D_0' \supseteq \ldots \supseteq D'_{j^2}$ all of cardinality $\varkappa$, and $z_0, \ldots, z_{j^2-1} \in Y$, such that for all $p < j^2$,

$$(*)\qquad \begin{array}{ll} \textit{either} & \bar{a} \in D_{j^2},\ \bar{b} \in D'_{j^2} \text{ implies } a_{k_p} \precsim z_p \precsim b_{k_p'}, \\ \textit{or} & \bar{a} \in D_{j^2},\ \bar{b} \in D'_{j^2} \text{ implies } b_{k_p'} \precsim z_n \precsim a_{k_p}. \end{array}$$

This done, apply exactly the same process to D'_{j^2} as we have just applied to B', to get disjoint sets $\bar{D}_{j^2}, \bar{D}'_{j^2}$, both of cardinality $\varkappa$, both $\subseteq D'_{j^2}$, and $\bar{z}_0, \ldots, \bar{z}_{j^2-1} \in Y$, such that the equivalent of $(*)$ holds. Repeat the process with $\bar{D}'_{j^2}$ etc., until eventually we reach disjoint sets E_0 $(= D_{j^2})$, E_1 $(= \bar{D}_{j^2}), \ldots, E_{n-1} \subseteq B'$, all of cardinality $\varkappa$, such that if $\bar{a}^i, \bar{b}^i \in E_i$ for each $i < n$, then the two concatenated sequences $\bar{a}^{0\frown} \ldots {}^\frown\bar{a}^{n-1}$, $\bar{b}^{0\frown} \ldots {}^\frown\bar{b}^{n-1}$ have their terms in the same relative order in $(Y, \precsim)$.

Suppose now that $\mathfrak{A}^* \models \varphi(\tau[\bar{a}^0], \ldots, \tau[\bar{a}^{n-1}])$ for some $\bar{a}^0, \ldots, \bar{a}^{n-1}$ in B, and π is a permutation of $\{0, \ldots, n-1\}$. If we can show that $\mathfrak{A}^* \models \varphi(\tau[\bar{a}^{\pi(0)}], \ldots, \tau[\bar{a}^{\pi(n-1)}])$, then we have proved the theorem.

$E_0, \ldots, E_{n-1}$ all have cardinality $\varkappa$, so we can find $\bar{b}^0 \in E_0, \ldots, \bar{b}^{n-1} \in E_{n-1}$ such that $\tau^{\mathfrak{A}^*}(\bar{b}^0), \ldots, \tau^{\mathfrak{A}^*}(\bar{b}^{n-1})$ stand in the same relative order in $(X, <)$ as do $\tau^{\mathfrak{A}^*}(\bar{a}^0), \ldots, \tau^{\mathfrak{A}^*}(\bar{a}^{n-1})$. By φ-indiscernibility of $(X, <)$, $\mathfrak{A}^* \models \varphi(\tau[\bar{b}^0], \ldots, \tau[\bar{b}^{n-1}])$. But similarly we can find $\bar{c}^0 \in E_0, \ldots, \bar{c}^{n-1} \in E_{n-1}$ so that $\tau^{\mathfrak{A}^*}(\bar{c}^0), \ldots, \tau^{\mathfrak{A}^*}(\bar{c}^{n-1})$ stand in the same relative order in $(X, <)$ as do $\tau^{\mathfrak{A}^*}(\bar{a}^{\pi(0)}), \ldots, \tau^{\mathfrak{A}^*}(\bar{a}^{\pi(n-1)})$. By choice of $E_0, \ldots, E_{n-1}$ and indiscernibility of $(Y, \precsim)$ we have $\mathfrak{A}^* \models \varphi(\tau[\bar{c}^0], \ldots, \tau[\bar{c}^{n-1}])$. Hence by φ-indiscernibility of $(X, <)$ we have $\mathfrak{A}^* \models \varphi(\tau[\bar{a}^{\pi(0)}], \ldots, \tau[\bar{a}^{\pi(n-1)}])$, and we are home.

Theorem 4 below will show that Theorem 1 fails to generalise to singular cardinals.

THEOREM 4. *Let T be any countable complete extension of Peano arithmetic, let $\varkappa$ be a singular strong limit number, and let $\mathfrak{A}$ be a model of T with cardinality $\varkappa$. Then $\mathfrak{A}$ contains an increasing sequence of order-type $\varkappa$ (in the natural ordering of the model).*

Proof. Write A for the domain of $\mathfrak{A}$. Let μ be the cofinality of $\varkappa$, and let $\varkappa = \sum_{a<\mu} \lambda_a$, where $\langle\lambda_a\rangle_{a<\mu}$ is a strictly increasing sequence of cardinals $<\varkappa$. By Erdös and Rado [2] Theorem 39, we can find for each $a<\mu$ a map $h_a\colon \lambda_a \to A$ which is either order-preserving or order-reversing. Using subtraction, we can suppose all the h_a are order-preserving. Put $B = \bigcup_{a<\mu} \operatorname{im} h_a$.

Case 1. For each $c \in A$, $|\{b \in B\colon b < c\}| < \varkappa$. A then has cofinality μ, and by taking appropriate pieces of the sets $\operatorname{im} h_a$ we can put together a set of order-type $\varkappa$.

Case 2. For some $c \in A$, $|\{b \in B\colon b < c\}| = \varkappa$. We can then suppose without loss that for all $b \in B$, $b < c$. Find an order-preserving map $f\colon \mu \to A$. Define $B = \{c.f(a)+h_a(\beta)\colon a<\mu, \beta<\lambda_a\}$, taking $+$, $.$ in the sense of $\mathfrak{A}$. If $a < a'$, $\beta < \lambda_a$ and $\beta' < \lambda_{a'}$, then

$$c.f(a)+h_a(\beta) < c.\big(f(a)+1\big) \leqslant c.f(a') \leqslant c.f(a')+h_{a'}(\beta') .$$

If $\beta, \beta' < \lambda_a$, then

$$c.f(a)+h_a(\beta) < c.f(a)+h_a(\beta') .$$

Hence B has order-type $\varkappa$. This concludes the proof.

We do not know what happens to Theorem 1 if we replace $\varkappa$ by a singular cardinal which is not a strong limit number. Ehrenfeuch claims it for all cardinals of form 2^μ, but in fact he uses regularity ([1 p. 244 top).

§ 2. Let T be a complete theory with infinite models. Let $\mathfrak{A}$ be a model of T. We call $\mathfrak{A}$ a *$\varkappa$-prime* model if $|\mathfrak{A}| = \varkappa$ and $\mathfrak{A}$ is elementarily embeddable in every model of T of cardinality $\varkappa$. We write $\operatorname{Spec}(T)$ for the class of cardinals $\varkappa$ such that T has a $\varkappa$-prime model.

THEOREM 5. *Suppose $\varkappa$ is regular, $\varkappa \not\to (\varkappa)^2_2$, $\varkappa > |T|$ and $\varkappa \in \operatorname{Spec}(T)$. Then every model of T of cardinality $\varkappa$ has an indiscernible set of cardinality $\varkappa$.*

Proof. Skolemise T to get T^* with $|T^*| = |T|$. Let $\mathfrak{A}$ be a model of T of cardinality $\varkappa$.

Now we have structure $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}^*$, $\mathfrak{D}$, all of cardinality $\varkappa$, as follows. $\mathfrak{B}$ is a $\varkappa$-prime model of T. $\mathfrak{C}$ is a model of T, guaranteed by Theorem 1, such that for every formula φ, every φ-indiscernible sequence in $\mathfrak{C}$ of order-type $\varkappa$ is a φ-indiscernible set. $\mathfrak{D}^*$ is a model of T^* which is the Skolem hull of an indiscernible sequence $(Y, \precsim)$ of order-type $\varkappa$. $\mathfrak{D}$ is the reduct of $\mathfrak{D}^*$ to T's language. We have then three elementary embeddings, $f\colon \mathfrak{B}\to\mathfrak{A}$, $g\colon \mathfrak{B}\to\mathfrak{C}$, $h\colon \mathfrak{B}\to\mathfrak{D}$. Using the embedding h first, we shall show that $\mathfrak{B}$ has an indiscernible sequence of order-type $\varkappa$. Using g, we shall show that this sequence is an indiscernible set. Finally we shall use f to inject the set into $\mathfrak{A}$.

Let $h\mathfrak{B}$ be the image of $\mathfrak{B}$ in $\mathfrak{D}^*$ under h. $h\mathfrak{B}$ has cardinality $\varkappa$, which is regular and $>|T^*|$. Therefore there is a term $\tau(v_0, \ldots, v_{m-1})$ such that $h\mathfrak{B}$ contains $\varkappa$ elements of form $\tau^{\mathfrak{D}^*}(\bar{a})$, $\bar{a} \in [Y]^m$. Find a subset B of $[Y]^m$ so that $|B| = \varkappa$ and $\tau^{\mathfrak{D}^*}$ maps B injectively into $h\mathfrak{B}$. Proceeding much as in the proof of Theorem 1, find a subset B' of B, also of cardinality $\varkappa$, and a number $j<m$, so that if $\bar{a}$, $\bar{b}$ are distinct elements of B' then $a_i = b_i$ for all $i<j$, and

$$\text{either } a_{m-1} \precnsim b_j \text{ or } b_{m-1} \precnsim a_j .$$

Let X be the set $\{h^{-1}\tau^{\mathfrak{D}^*}(\bar{a})\colon \bar{a} \in B'\}$, and put an ordering $<$ on X by setting $h^{-1}\tau^{\mathfrak{D}^*}(\bar{a}) < h^{-1}\tau^{\mathfrak{D}^*}(\bar{b})$ iff $a_j \precnsim b_j$. Then $(X, <)$ is an indiscernible sequence of order-type $\varkappa$ in $\mathfrak{B}$.

Now $g(X, <)$, the image of $(X, <)$ in $\mathfrak{C}$, is an ordered set with order-type $\varkappa$. Let φ be any formula of T's language. Then $g(X, <)$ is a φ-indiscernible sequence, because g is elementary. By choice of $\mathfrak{C}$, $g(X, <)$ must then be a φ-indiscernible set. Hence $(X, <)$ is an indiscernible set in $\mathfrak{B}$. Finally f injects $(X, <)$ into $\mathfrak{A}$, and we are done.

We give a couple of examples. Let T_0 be the theory of infinite atomic boolean algebras; let T_1 be the theory of the structure $\langle \boldsymbol{Z}, R\rangle$ where $\boldsymbol{Z}$ is the integers and $R = \{\langle a, b, c\rangle\colon b \text{ lies strictly between } a \text{ and } c \text{ in } \boldsymbol{Z}\}$. Both T_0 and T_1 are complete and have infinite models.

Now $\operatorname{Spec}(T_0)$ consists of all strong limit numbers, since these are precisely the cardinals $\varkappa$ such that every model of T_0 has $\varkappa$ atoms. (The prime model in cardinality $\varkappa$ is the finite-cofinite algebra with $\varkappa$ atoms.) It follows by Theorem 3 that if $\varkappa$ is a strong limit number and $\varkappa \not\to (\varkappa)^2_2$, then every model of T_0 of cardinality $\varkappa$ has an indiscernible set of cardinality $\varkappa$. The set of atoms is such a set.

On the other hand it's plain that no model of T_1 contains an infinite indiscernible set. Hence by Theorem 3, no regular cardinal $\varkappa$ such that $\varkappa \not\to (\varkappa)^2_2$ is in $\operatorname{Spec}(T_1)$. Singular cardinals apart, this result is best possible; it is easy to show that if $\varkappa \not\to (\varkappa)^2_2$ then $\varkappa \in \operatorname{Spec}(T_1)$. As a matter of fact no singular cardinal is in $\operatorname{Spec}(T_1)$, but this may only be because T_1 is so simple.

Finally we mention a quantifier-free version of Theorem 5, which gives a little new information. The proof is virtually the same.

THEOREM 6. *Let T be a theory (not necessarily complete) with infinite models. Suppose $\varkappa$ is regular, $\varkappa \not\to (\varkappa)^2_2$, $\varkappa > |T|$, and there is a model of T of cardinality $\varkappa$ is embeddable in every model of T of cardinality $\varkappa$. Then every* *model of T of cardinality $\varkappa$ has a set of cardinality $\varkappa$ which is φ-indiscernible for all quantifier-free formulae φ.*

It follows for instance that if $\varkappa$ is regular and $\varkappa \nrightarrow (\varkappa)^2_2$, then there is no model of Peano arithmetic of cardinality $\varkappa$ which is embeddable in all models of Peano arithmetic of cardinality $\varkappa$. One presumes the same is true for all uncountable $\varkappa$, but for $\varkappa$ singular or weakly compact the proof must be different.

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On successors in cardinal arithmetic

by

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Abstract. Properties of the three kinds of successor of a cardinal number defined by Tarski (Indagationes Mathematicae 16 (1954), pp. 26–32) are discussed. Let them be 1, 2, 3-successors respectively. A Fraenkel-Mostowski model is given in which the axiom of choice fails, but every cardinal has a unique 1-successor. It is proved that if every cardinal has a 3-successor, then x infinite implies $x = 2x$. Models are given containing cardinals x, y such that $2x$ is a successor of x, and y^2 a successor of y, respectively, and various other properties and characterizations of 3-successors are mentioned. The positive results are based mainly on Tarski's methods in cardinal arithmetic (see Lindenbaum-Tarski, *Communication sur les recherches de la Théorie des Ensembles*, C. R. Soc. Sc. Varsovie, Cl. III 19 (1926), pp. 299–330), together with some cofinality arguments.

§ 1. Introduction (1). In [8] Tarski defined three types of successor of a cardinal number (henceforth called 1, 2, 3-successors respectively) and proved that "for all x (x has a 2-successor)" implies the axiom of choice. (If x has a 2-successor, it is necessarily unique). We show in § 3 that "for all x (x has a unique 1-successor)" does not imply the axiom of choice (at least in a Fraenkel-Mostowski setting) nor even that every Dedekind finite cardinal is finite. In § 4 we show that "for all x (x has a 3-successor)" implies that for all infinite x, $x = 2x$. We feel that probably neither of these assertions, nor even the former with "unique" inserted, implies the axiom of choice, but no proofs of any of these have yet been announced. For completeness we begin § 4 with a proof, pointed out to the author by Prof. A. Levy, that "for all well-ordered x (x has a 2-successor)" implies the axiom of choice, and conclude it with one or two characterizations of cardinals which can or cannot be 3-successors.

§ 5 is devoted to a few special cases. Models are given in which there are cardinals x, y such that $2x$ is a 3-successor of x and y^2 is a 3-successor of y. Of course it is known that 2^x can be a 1-successor of x. We show that whenever this happens, 2^x is also a 3-successor of x. The same is

(1) In a letter, Professor Tarski informed the author that he had proved Theorem 3 independently some time ago. Lemma 2 and Theorem 7 (ii) were first announced in Lindenbaum–Tarski, *Communication sur les Recherches de la Théorie des Ensembles*, C. R. Soc. Sc. Varsovie, Cl. III 19 (1926), pp. 299–330.