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T-sequential topological spaces

by

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Let $(X, \mathfrak{J})$ be a topological space. If $A \subset X$, the *sequential adherence* of A, written $\mathrm{Ad}_s(A)$, is the union of A and the set of all points in X which are limits of sequences in A. If we define a set function $\mathrm{Ad}_s\colon P(X) \to P(X)$ such that $\mathrm{Ad}_s(A)$ is the sequential adherence of A for each A in $P(X)$, then (X, Ad_s) is a closure space ([1], p. 237). We shall call (X, Ad_s) the *sequential closure space* generated by the topological space $(X, \mathfrak{J})$. In general, the closure space (X, Ad_s) is not a topological closure space ([1], p. 250), i.e., it is not the case that $\mathrm{Ad}_s(\mathrm{Ad}_s(A)) = \mathrm{Ad}_s(A)$ for each subset A of X ([5], p. 109). A topological space $(X, \mathfrak{J})$ will be called *topological-sequential* or *T-sequential* if the sequential closure space (X, Ad_s) generated by it is a topological closure space.

Before giving a number of equivalent characterizations of T-sequential topological spaces, we need the following definitions which are special cases of those which occur in the study of closure spaces. Let $A \subseteq X$. The *sequential interior* of A, written $\mathrm{Int}_s(A)$, is the set $\mathrm{Int}_s(A) = A \setminus \mathrm{Ad}_s(X \setminus A)$. Thus, if $x \in X$, then $x \in \mathrm{Int}_s(A)$ if and only if $x \in A$ and there is no sequence (x_n) in $X \setminus A$ such that (x_n) is convergent to x. The set A is *sequentially closed* if $\mathrm{Ad}_s(A) = A$. Thus A is sequentially closed if and only if A contains all the points of X which are limits of sequences in A. The set A is *sequentially open* if its complement is sequentially closed. Thus A is sequentially open if and only if every sequence in X which converges to a point in A is ultimately in A. The set A is a *sequential neighborhood* of a point a in X if $a \in \mathrm{Int}_s(A)$. Thus A is a sequential neighborhood of a if and only if every sequence in X which converges to a is ultimately in A. Theorem 1 is an easy consequence of these definitions or of a listing of some necessary and sufficient conditions for a closure space to be a topological closure space.

THEOREM 1. *Let $(X, \mathfrak{J})$ be a topological space. Then the following statements are equivalent:*

(1) *$(X, \mathfrak{J})$ is T-sequential.*

(2) *The sequential adherence of every subset of X is sequentially closed.*

(3) *The sequential interior of every subset of X is sequentially open.*

(4) *The sequential adherence of each subset A of X is the intersection of all the sequentially closed subsets of X containing A.*

(5) *The sequential interior of each subset A of X is the union of all the sequentially open subsets of X contained in A.*

(6) *At each point x of X, the collection of all sequentially open sequential neighborhoods of x is a local base, i.e., given any point x of X and given any sequential neighborhood U of x, there exists a sequentially open subset G of X such that $x \in G \subseteq U$.*

(7) *Every sequential neighborhood of a point x in X is a sequential neighborhood of a sequential neighborhood of x, i.e., given any point x in X and given any sequential neighborhood U of x, there exists a sequential neighborhood V of x such that U is a sequential neighborhood of every point of V.*

A topological space $(X, \mathfrak{J})$ is said to be *sequential* if every sequentially open subset of X is open. It is said to be *neighborhood-sequential* or *N-sequential* if every sequential neighborhood of a point is a neighborhood of that point. Sequential and N-sequential topological spaces were first introduced and studied by S. P. Franklin, [3] and [4]. Incidentally, Franklin calls N-sequential topological spaces Fréchet spaces. A. Wilansky calls then closure-sequential spaces ([8], p. 30). Franklin ([4], p. 54) has asked the question: when is a sequential space N-sequential? An answer is given in the following theorem.

THEOREM 2. *Let $(X, \mathfrak{J})$ be a topological space. Then $(X, \mathfrak{J})$ is N-sequential if and only if $(X, \mathfrak{J})$ is both sequential and T-sequential.*

Proof. Assume $(X, \mathfrak{J})$ is N-sequential. Since every sequentially open set is a sequential neighborhood of each of its points, $(X, \mathfrak{J})$ is sequential. From Theorem 1, Part 6, we see that $(X, \mathfrak{J})$ is T-sequential. Conversely, if $(X, \mathfrak{J})$ is both sequential and T-sequential, every sequential neighborhood of a point is, by Theorem 1, Part 6, and the definition of a sequential space, a neighborhood of that point. Thus $(X, \mathfrak{J})$ is N-sequential.

Every first countable topological space is N-sequential and hence both sequential and T-sequential. S. P. Franklin ([3], p. 113) and J. H. Webb ([7], p. 362) have given examples of topological spaces (and topological vector spaces) which are sequential but not N-sequential. Clearly, these examples are examples of spaces which are sequential but not T-sequential. We now give two examples of topological spaces (one a topological vector space) which are T-sequential but not sequential.

EXAMPLE 1. Consider the real line R (or any uncountable set) with the cocountable topology $\mathfrak{J}$. Then $\mathfrak{J}$ consists of R, $\emptyset$, and the complements of countable sets. A sequence (a_n) in R is $\mathfrak{J}$-convergent to a point a in R if and only if ultimately $a_n = a$. Since every subset of R is sequentially closed, $(R, \mathfrak{J})$ is T-sequential by Theorem 1, Part 2. Given any point a in R, the singleton set $\{a\}$ is sequentially open but not open. Thus $(R, \mathfrak{J})$ is not sequential.

EXAMPLE 2. Consider the sequence space $l^1 = \{x = (\xi_n): \sum_{n=1}^{+\infty} |\xi_n| < +\infty\}$ with the weak topology $\sigma(l^1, l^\infty)$. Of course, $(l^1, \|\cdot\|)$ is a Banach space with norm $\|\cdot\|: l^1 \to R$ defined by $\|x\| = \sum_{n=1}^{+\infty} |\xi_n|$ for all $x = (\xi_n)$ in l^1. The topological dual of l^1 with the norm topology $\mathfrak{J}$ is the space of all bounded sequences $l^\infty = \{y = (\varphi_n): \sup_n |\varphi_n| < +\infty\}$, i.e., every $\mathfrak{J}$-continuous linear functional u on l^1 can be represented by a bounded sequence $y = (\varphi_n)$ in l^∞ and in fact $u(x) = \sum_{n=1}^{+\infty} \xi_n \varphi_n$ for all $x = (\xi_n)$ in l^1. The space $(l^1, \sigma(l^1, l^\infty))$ is a locally convex topological vector space with $\sigma(l^1, l^\infty)$ being the vector topology on l^1 generated by the family of semi-norms $\{P_y: y \in l^\infty\}$ where $P_y: l^1 \to R$ is defined by the correspondence $P_y(x) = |\sum_{n=1}^{+\infty} \xi_n \varphi_n|$ for all $x = (\xi_n)$ in l^1 with $y = (\varphi_n)$ in l^∞. $\sigma(l^1, l^\infty)$ is the weakest topology on l^1 for which l^∞ is its topological dual. Our example depends upon the following properties of these two spaces:

(1) Since $(l^1, \|\cdot\|)$ is an infinite dimensional normed linear space, the weak topology $\sigma(l^1, l^\infty)$ is strictly weaker than the norm topology $\mathfrak{J}$ ([6], p. 235; or [2]). Thus the norm $\|\cdot\|: l^1 \to R$ is $\mathfrak{J}$-continuous but not $\sigma(l^1, l^\infty)$-continuous.

(2) Weak convergence and norm convergence of sequences in l^1 are the same ([6], p. 281; or [2]), i.e., if $x \in l^1$ and if (x_n) is a sequence in l^1, then (x_n) is $\mathfrak{J}$-convergent to x if and only if (x_n) is $\sigma(l^1, l^\infty)$-convergent to x.

Using these facts, we can now show that $(l^1, \sigma(l^1, l^\infty))$ is T-sequential but not sequential. In order to prove that $(l^1, \sigma(l^1, l^\infty))$ is T-sequential, we need only to show (see Theorem 1, Part 6) that every $\sigma(l^1, l^\infty)$-sequential neighborhood of the zero vector 0 in l^1 contains a $\sigma(l^1, l^\infty)$-sequentially open $\sigma(l^1, l^\infty)$-sequential neighborhood of 0. Let U be a $\sigma(l^1, l^\infty)$-sequential neighborhood of 0. Then U is a $\mathfrak{J}$-sequential neighborhood of 0. Since $(l^1, \mathfrak{J})$ is a normable locally convex topological vector space, U is a $\mathfrak{J}$-neighborhood of 0. There exists a $\mathfrak{J}$-open ball $B_\varepsilon(0) = \{x \in l^1: \|x\| < \varepsilon\}$ such that $0 \in B_\varepsilon(0) \subset U$. Since $B_\varepsilon(0)$ is $\mathfrak{J}$-open, it is $\mathfrak{J}$-sequentially open and hence $\sigma(l^1, l^\infty)$-sequentially open. Of course, $B_\varepsilon(0)$ is a $\sigma(l^1, l^\infty)$-sequential neighborhood of 0. This proves that $(l^1, \sigma(l^1, l^\infty))$ is T-sequential. In order to show that $(l^1, \sigma(l^1, l^\infty))$ is not sequential, we must find a $\sigma(l^1, l^\infty)$-sequentially open subset of l^1 which is not $\sigma(l^1, l^\infty)$-open. Consider the unit ball $B_1(0) = \{x \in l^1: \|x\| < 1\}$. Since the norm $\|\cdot\|: l^1 \to R$ is $\mathfrak{J}$-continuous, $B_1(0)$ is $\mathfrak{J}$-open and hence $\sigma(l^1, l^\infty)$-sequentially open. However, the norm $\|\cdot\|: l^1 \to R$ is not $\sigma(l^1, l^\infty)$-continuous. Consequently, $B_1(0)$ is not $\sigma(l^1, l^\infty)$-open. Thus $(l^1, \sigma(l^1, l^\infty))$ is not sequential.

References

[1] E. Čech, *Topological Spaces*, New York 1966.
[2] J. B. Conway, *The inadequacy of sequences*, Amer. Math. Monthly 76 (1969), pp. 68–69.
[3] S. P. Franklin, *Spaces in which sequences suffice*, Fund. Math. 57 (1965), pp. 107–115.
[4] — *Spaces in which sequences suffice II*, Fund. Math. 61 (1967), pp. 52–56.
[5] J. L. Kelley, *General Topology*, Princeton 1955.
[6] G. Köthe, *Topological Vector Spaces I*, New York 1969.
[7] J. H. Webb, *Sequential convergence in locally convex spaces*, Proc. Camb. Phil. Soc. 64 (1968), pp. 341–364.
[8] A. Wilansky, *Topology for Analysis*, Ginn, Waltham, Mass. 1970.

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An atriodic tree-like continuum with positive span

by

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1. Introduction. In 1964 A. Lelek defined the span of a metric space, and he proved that every chainable continuum has span zero [5], section 5. In this paper we construct an example of an atriodic tree-like continuum with positive span. The continuum is obtained as an inverse limit on simple triods using only one bonding map. The question of the existence of an atriodic tree-like continuum which is not chainable was mentioned by Bing [2], p. 45, and Anderson [1] claimed in an abstract that such an example indeed exists.

Throughout this paper the term space refers to metric space and the term mapping to continuous function. The projection of a product space onto its ith coordinate space will be denoted by π_i.

Suppose X and Y are spaces, d is a metric for Y, and f is a mapping of X into Y. The *span of* f, denoted by σf, is the least upper bound of the set of numbers ε for which there is a connected subset Z of $X \times X$ such that $\pi_1(Z) = \pi_2(Z)$ and $d(f(x), f(y)) \geqslant \varepsilon$ for each (x, y) in Z. (Of course σf may be infinite). The span of X, denoted by σX, as defined by Lelek, [5], is the span of the identity mapping on X.

Suppose $X_1, X_2, \ldots$ is a sequence of compact spaces and $f_1, f_2, \ldots$ is a sequence of mappings such that $f_i: X_{i+1} \to X_i$. The inverse limit of the inverse limit sequence $\{X_i, f_i\}$ is the subset X of $\prod_{i>0} X_i$ such that $(x_1, x_2, \ldots)$ is in X if and only if $f_i(x_{i+1}) = x_i$ for each i. We consider $\prod_{i>0} (X_i, d_i)$ metrized by

$$d(x, y) = \sum_{i>0} 2^{-i} d_i(x_i, y_i) .$$

2. The mapping f and the continuum M. Let T denote the simple triod $\{(\varrho, \theta) |\ 0 \leqslant \varrho \leqslant 1 \text{ and } \theta = 0,\ \theta = \frac{1}{2}\pi \text{ or } \theta = \pi\}$ (in polar coordinates in the plane). Define $f: T \to T$ as follows:

$$f(x, \tfrac{1}{2}\pi) = \begin{cases} (1-4x, \pi) & \text{if } 0 \leqslant x \leqslant \frac{1}{4}, \\ (4x-1, \frac{1}{2}\pi) & \text{if } \frac{1}{4} \leqslant x \leqslant \frac{1}{2}, \\ (3-4x, \frac{1}{2}\pi) & \text{if } \frac{1}{2} \leqslant x \leqslant \frac{3}{4}, \\ (4x-3, 0) & \text{if } \frac{3}{4} \leqslant x \leqslant 1 . \end{cases}$$