T-sequential topological spaces

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Let $(X, 3)$ be a topological space. If $A \subseteq X$, the sequential adherence of $A$, written $Ad_s(A)$, is the union of $A$ and the set of all points in $X$ which are limits of sequences in $A$. If we define a set function $Ad_f: P(X) \rightarrow P(X)$ such that $Ad_f(A)$ is the sequential adherence of $A$ for each $A$ in $P(X)$, then $(X, Ad_f)$ is a closure space ([1], p. 237). We shall call $(X, Ad_f)$ the sequential closure space generated by the topological space $(X, 3)$. In general, the closure space $(X, Ad_f)$ is not a topological closure space ([1], p. 239), i.e., it is not the case that $Ad_f(Ad_f(A)) = Ad_f(A)$ for each subset $A$ of $X$. A topological space $(X, 3)$ will be called topological-sequential or T-sequential if the sequential closure space $(X, Ad_f)$ generated by it is a topological closure space.

Before giving a number of equivalent characterizations of T-sequential topological spaces, we need the following definitions which are special cases of those which occur in the study of closure spaces. Let $A \subseteq X$. The sequential interior of $A$, written $Int_s(A)$, is the set $Int_s(A) = A \setminus Ad_f(X \setminus A)$. Thus, if $x \in X$, then $x \in Int_s(A)$ if and only if $x \in A$ and there is no sequence $(a_n)$ in $X \setminus A$ such that $(a_n)$ is convergent to $x$. The set $A$ is sequentially closed if $Ad_f(A) = A$. Thus $A$ is sequentially closed if and only if $A$ contains all the points of $X$ which are limits of sequences in $A$. The set $A$ is sequentially open if its complement is sequentially closed. Thus $A$ is sequentially open if and only if every sequence in $X$ which converges to a point in $A$ is ultimately in $A$. The set $A$ is a sequential neighborhood of a point $a$ in $X$ if $a \in Int_s(A)$. Thus $A$ is a sequential neighborhood of $a$ if and only if every sequence in $X$ which converges to $a$ is ultimately in $A$. Theorem 1 is an easy consequence of these definitions or of a listing of some necessary and sufficient conditions for a closure space to be a topological closure space.

**Theorem 1.** Let $(X, 3)$ be a topological space. Then the following statements are equivalent:

1. $(X, 3)$ is T-sequential.
2. The sequential adherence of every subset of $X$ is sequentially closed.
(3) The sequential interior of every subset of \(X\) is sequentially open.
(4) The sequential adherence of each subset \(A\) of \(X\) is the intersection of all the sequentially closed subsets of \(X\) containing \(A\).
(5) The sequential interior of each subset \(A\) of \(X\) is the union of all the sequentially open subsets of \(X\) contained in \(A\).
(6) At each point \(x\) of \(X\), the collection of all sequentially open sequential neighborhoods of \(x\) is a local base, i.e., given any point \(x\) of \(X\) and given any sequential neighborhood \(U\) of \(x\), there exists a sequentially open subset \(G\) of \(X\) such that \(x \in G \subseteq U\).
(7) Every sequential neighborhood of a point \(x\) in \(X\) is a sequential neighborhood of a sequential neighborhood of \(x\), i.e., given any point \(x\) in \(X\) and given any sequential neighborhood \(U\) of \(x\), there exists a sequential neighborhood \(V\) of \(x\) such that \(U\) is a sequential neighborhood of every point of \(V\).

A topological space \((X, J)\) is said to be sequential if every sequentially open subset of \(X\) is open. It is said to be neighborhood-sequential or \(N\)-sequential if every sequential neighborhood of a point is a neighborhood of that point. Sequential and \(N\)-sequential topological spaces were first introduced and studied by S. P. Franklin, [3] and [4]. Incidently, Franklin calls \(N\)-sequential topological spaces Fréchet spaces. A. Wilansky calls them closure-sequential spaces ([8], p. 30). Franklin ([4], p. 54) has asked the question: when is a sequential space \(N\)-sequential? An answer is given in the following theorem.

**Theorem 2.** Let \((X, J)\) be a topological space. Then \((X, J)\) is \(N\)-sequential if and only if \((X, J)\) is both sequential and \(T\)-sequential.

**Proof.** Assume \((X, J)\) is \(N\)-sequential. Since every sequentially open set is a sequential neighborhood of each of its points, \((X, J)\) is sequential. From Theorem 1, Part 6, we see that \((X, J)\) is \(T\)-sequential. Conversely, if \((X, J)\) is both sequential and \(T\)-sequential, every sequential neighborhood of a point is, by Theorem 1, Part 6, and the definition of a sequential space, a neighborhood of that point. Thus \((X, J)\) is \(N\)-sequential.

Every first countable topological space is \(N\)-sequential and hence both sequential and \(T\)-sequential. S. P. Franklin ([3], p. 113) and J. H. Webb ([7], p. 362) have given examples of topological spaces (and topological vector spaces) which are sequential but not \(N\)-sequential. Clearly, these examples are examples of spaces which are sequential but not \(T\)-sequential. We now give two examples of topological spaces (one a topological vector space) which are \(T\)-sequential but not sequential.

**Example 1.** Consider the real line \(R\) (or any uncountable set) with the countable topology \(J_\alpha\). Then \(J\) consists of \(R, \varnothing, \alpha\), and the complements of countable sets. A sequence \((a_n)_{n=1}^\infty\) in \(R\) is \(1\)-convergent to a point \(a\) in \(R\) if and only if all \(a_n\) in \(\text{closure of } \{a\}\). Since every subset of \(R\) is sequentially closed, \((R, J)\) is \(T\)-sequential by Theorem 1, Part 2. Given any point \(x\) in \(R\), the singleton set \(\{x\}\) is sequentially open but not closed. Thus \((R, J)\) is not sequential.

**Example 2.** Consider the sequence space \(D = \{(x_n)_{n=1}^\infty : \sum_{n=1}^\infty |x_n| < \infty\}\) with the weak topology \(\sigma(D, \ell^\infty)\). Of course, \((D, \ell^\infty)\) is a Banach space with norm \(\|\cdot\|_1 : D \rightarrow \mathbb{R}\) defined by \(\|x\|_1 = \sum_{n=1}^\infty |x_n|\) for all \(x = (x_n)\) in \(D\). The topological dual of \(D\) with the norm topology \(J\) is the space of all bounded sequences \(f : D \rightarrow \mathbb{R}\) with \(\sup_{n \geq 1} |f(x_n)| < \infty\), i.e., every \(J\)-continuous linear functional on \(D\) can be represented by a bounded sequence \(y = (y_n)\) in \(D\) and in fact \(y(x) = \sum_{n=1}^\infty x_n y_n\) for all \(x = (x_n)\) in \(D\). The space \([D, \sigma(D, \ell^\infty)]\) is a locally convex topological vector space with \(\sigma(D, \ell^\infty)\) being the vector topology on \(D\) generated by the family of semi-norms \(P_{x_1} : y \mapsto \sum_{n=1}^\infty |x_n y_n|\) for all \(x = (x_n)\) in \(D\) with \(y = (y_n)\) in \(\ell^\infty\). \(\sigma(D, \ell^\infty)\) is the weakest topology on \(D\) for which \(D\) is its topological dual. Our example depends upon the following properties of these two spaces:

1. Since \([D, \sigma(D, \ell^\infty)]\) is an infinite dimensional normed linear space, the weak topology \(\sigma(D, \ell^\infty)\) is strictly weaker than the norm topology \(\sigma(D, \ell^\infty)\) (cf. [8], p. 235; or [2]). Thus the norm \(\|\cdot\|_1 : D \rightarrow \mathbb{R}\) is \(J\)-continuous but not \(\sigma(D, \ell^\infty)\)-continuous.
2. Weak convergence and norm convergence of sequences in \(D\) are the same ([6], p. 281; or [2]), i.e., if \(x \in D\) and if \((a_n)_{n=1}^\infty\) is a sequence in \(D\), then \((a_n)_{n=1}^\infty\) is \(J\)-convergent to \(a\) if and only if \((a_n)_{n=1}^\infty\) is \(\sigma(D, \ell^\infty)\)-convergent to \(x\).

Using these facts, we can now show that \([D, \sigma(D, \ell^\infty)]\) is \(T\)-sequential but not sequential. In order to prove that \([D, \sigma(D, \ell^\infty)]\) is \(T\)-sequential, we need only to show (see Theorem 1, Part 6) that every \(\sigma(D, \ell^\infty)\)-sequential neighborhood of the zero vector \(0\) in \(D\) contains a \(\sigma(D, \ell^\infty)\)-sequentially open \(\sigma(D, \ell^\infty)\)-sequential neighborhood of \(0\). Let \(U\) be a \(\sigma(D, \ell^\infty)\)-neighborhood of \(0\). Then \(U\) is a \(\sigma(D, \ell^\infty)\)-neighborhood of \(0\). Since \((D, \sigma(D, \ell^\infty))\) is a normable locally convex topological vector space, \(U\) is a \(J\)-neighborhood of \(0\). There exists a \(J\)-open ball \(B_r(0) = \{x \in D : \|x\| < r\}\) such that \(0 \in B_r(0) \subseteq U\). Since \(B_r(0)\) is \(J\)-open, it is \(\sigma(D, \ell^\infty)\)-sequentially open and hence \(\sigma(D, \ell^\infty)\)-sequentially open. Of course, \(B_r(0)\) is \(\sigma(D, \ell^\infty)\)-sequentially open. This proves that \((D, \sigma(D, \ell^\infty))\) is \(T\)-sequential.

In order to show that \((D, \sigma(D, \ell^\infty))\) is not sequential, we must find a \(\sigma(D, \ell^\infty)\)-sequentially open subset of \(D\) which is not \(\sigma(D, \ell^\infty)\)-open. Consider the
unit ball $B_n(0) = \{ x \in P : \| x \| < 1 \}$. Since the norm $\| \cdot \| : P \to R$ is 5-continuous, $B_n(0)$ is 3-open and hence $\sigma(P, F^n)$-sequentially open. However, the norm $\| \cdot \| : P \to R$ is not $\sigma(P, F^n)$-continuous. Consequently, $B_n(0)$ is not $\sigma(P, F^n)$-open. Thus $(P, \sigma(P, F^n))$ is not sequential.

References


An atriodic tree-like continuum with positive span

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1. Introduction. In 1964 A. Lelek defined the span of a metric space, and he proved that every chainable continuum has span zero [5], section 5. In this paper we construct an example of an atriodic tree-like continuum with positive span. The continuum is obtained as an inverse limit on simple triods using only one bonding map. The question of the existence of an atriodic tree-like continuum which is not chainable was mentioned by Bing [2], p. 45, and Anderson [1] claimed in an abstract that such an example indeed exists.

Throughout this paper the term space refers to metric space and the term mapping to continuous function. The projection of a product space onto its i-th coordinate space will be denoted by $x_i$.

Suppose $X$ and $Y$ are spaces, $d$ is a metric for $Y$, and $f$ is a mapping of $X$ into $Y$. The span of $f$, denoted by $\sigma f$, is the least upper bound of the set of numbers $\varepsilon$ for which there is a connected subset $Z$ of $X \times X$ such that $x_i(Z) = x_i[Z]$ and $d(f(x), f(y)) > \varepsilon$ for each $(x, y)$ in $Z$. (Of course $\sigma f$ may be infinite). The span of $X$, denoted by $\sigma X$, is defined by $\sigma X$, as defined by Lelek, [5], is the span of the identity mapping on $X$.

Suppose $X_0, X_1, \ldots$ is a sequence of compact spaces and $f_0, f_1, \ldots$ is a sequence of mappings such that $f_i : X_{i+1} \to X_i$. The inverse limit of the inverse limit sequence $(X_i, f_i)$ is the subset $X$ of $\prod_{n=0} X_i$ such that $(x_0, x_1, \ldots )$ is in $X$ if and only if $f_i(x_{i+1}) = x_i$ for each $i$. We consider $\prod_{n=0} X_i$ metricized by

$$d(x, y) = \sum_{i=0}^{n} 2^{-i}d(x_i, y_i).$$

2. The mapping $f$ and the continuum $M$. Let $T$ denote the simple triod $((0, \theta), 0 \leq \theta < 1)$, and $T = [0, \theta = \frac{1}{2}, \theta = \frac{1}{2} = \pi]$ (in polar coordinates in the plane). Define $f: T \to T$ as follows:

$$f(x, \frac{1}{2} \pi) = \begin{cases} (1 - 4x, \pi) & \text{if } 0 \leq x \leq \frac{1}{4}, \\ (4x - 1, \frac{1}{2}) & \text{if } \frac{1}{4} \leq x \leq \frac{1}{2}, \\ (3 - 4x, \frac{1}{2}) & \text{if } \frac{1}{2} \leq x \leq \frac{3}{4}, \\ (4x - 3, 0) & \text{if } \frac{3}{4} \leq x \leq 1. \end{cases}$$