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Examples relating to mesocompact and sequentially mesocompact spaces (*)

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Introduction. It is the purpose of this paper to present examples, which relate to the structural properties and mapping properties of the mesocompact and sequentially mesocompact spaces [1]. In particular, a Tychonoff sequentially mesocompact space which is not mesocompact, is presented in Example 2.1. Example 2.2 establishes that sequential mesocompactness is not invariant under perfect mappings.

To put these examples in proper perspective, § 1 contains the definitions and statements of the main theorems which are contained in [3]. The examples are presented in § 2. All spaces are assumed to be Hausdorff and all functions are continuous surjections in this paper.

1. Characterizations and mapping theorems. A topological space is said to have *property* (k) (*property* (ω) [2]), if for each discrete collection of closed sets $\mathcal{F} = \{F_\alpha: \alpha \in A\}$, there exists a compact-finite (cs-finite) [1], collection of open sets $\mathcal{U} = \{U_\alpha: \alpha \in A\}$ such that $F_\alpha \subset U_\alpha$, for each $\alpha \in A$ and $U_\alpha \cap F_\beta = \emptyset$, if $\alpha \neq \beta$.

THEOREM 1.1. *A normal space is mesocompact (sequentially mesocompact) if and only if it is a metacompact space with property* (k) (*property* (ω)).

THEOREM 1.2. *The perfect image of a normal mesocompact space is a normal mesocompact space.*

A mapping $f: X \to Y$ is said to be *presequential*, if for each convergent sequence $\{p_i\}$ in Y, $p_i \to p$, which is not eventually equal to p, $\bigcup \{f^{-1}(p_i): i \in N, p_i \neq p\}$ is not sequentially closed.

THEOREM 1.3. *The closed presequential image of a normal sequentially mesocompact space is a normal sequentially mesocompact space.*

Theorems 1.2 and 1.3 depend on the facts that property (k) is invariant under perfect mappings and that property (ω) is invariant under

(*) This work is dedicated to the memory of Professor Hisahiro Tamano.

closed presequential mappings. Some characterizations of collectionwise normality and paracompactness, in spaces with property (k), are included in [3].

2. Examples. In this section, the examples announced in the introduction are presented and their meaning relative to the theorems in § 1 is discussed.

EXAMPLE 2.1. *A Tychonoff sequentially mesocompact space which is not mesocompact.*

Let $[0, \Omega]$ be the space consisting of the set of ordinal numbers, which are less than or equal to the first uncountable ordinal Ω, with the topology generated by the neighborhood bases $\{\{a\}\}$, for each $0 \leqslant a < \Omega$, and $\{[\beta, \Omega]: 0 \leqslant \beta < \Omega\}$, for Ω. Consider the Tychonoff product $[0, \Omega] \times \beta N$, where βN, is the Stone–Čech compactification of the natural numbers, N. Let X be the subspace of $[0, \Omega] \times \beta N$, defined by $X = [0, \Omega] \times \beta N \backslash \{(\Omega, p): p \in \beta N - N\}$.

It is clear that X is a Tychonoff space. We will establish the sequential mesocompactness of X. Let $\mathfrak{U}$ be any open covering of X. For each $a \in [0, \Omega)$, choose a finite number of basic open sets, which cover $\{(a, t): t \in \beta N\}$, and are contained in some set in $\mathfrak{U}$. For each $n \in N$, choose a basic open set which contains (Ω, n), and is contained in some set in $\mathfrak{U}$. This collection of basic open sets is a point-finite open refinement of $\mathfrak{U}$. Thus, X is metacompact. The only sequences in X that converge are those which are eventually constant, because of the topology defined on $[0, \Omega]$, and the fact that every infinite closed subset of βN has cardinality 2^c (Novák [5], Theorem 2). Accordingly, X is sequentially mesocompact. To verify that X is not mesocompact, we need only to consider any open covering of X, by basic open sets. Each of the countably infinite number of sets, which contain the points (Ω, n), $n \in N$, in any open refinement of such a covering would intersect some compact set $\{(a_0, t): a_0 < \Omega, t \in \beta N\}$. Hence X is not mesocompact.

Since X is sequentially mesocompact, it certainly has property (ω). However, X does not have property (k), because $\{\{(\Omega, n)\}: n \in N\}$ is a discrete collection of closed subsets of X, for which there does not exist the required compact-finite collection of open sets.

X is not normal, because the closed sets $[0, \Omega) \times (\beta N - N)$ and $\{\Omega\} \times N$ cannot be separated by disjoint open sets. Also, since $X - \{(\Omega, 1)\}$ is not closed and there is no compact set K whose intersection with this set is not closed in K, X is not a k-space.

EXAMPLE 2.2. *The perfect image of a sequentially mesocompact space need not be sequentially mesocompact.*

Let $[0, \Omega]$ be the same space as in Example 2.1, and let $[0, \omega]$ be the space of ordinals with the interval topology, where ω is the first infinite ordinal. Let Y be the subspace of $[0, \Omega] \times [0, \omega]$, defined by $Y = [0, \Omega] \times [0, \omega] \backslash \{(\Omega, \omega)\}$. This space was used by Dieudonné [4]. Consider the mapping, φ from the space X, from Example 2.1 onto Y, defined by $\varphi(a, n) = (a, n)$, for each $(a, n) \in [0, \Omega] \times N$, and $\varphi(a, p) = (a, \omega)$, for each $(a, p) \in [0, \Omega) \times (\beta N - N)$. The mapping φ is a perfect continuous surjection. Y is a Tychonoff metacompact space, which is not sequentially mesocompact. In fact, Y does not even have property (ω), because $\{\{(\Omega, n)\}: n \in N\}$ is a discrete collection of closed subsets of Y, for which there does not exist the required cs-finite collection of open sets. Hence, although φ is perfect, it is not presequential.

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