Analysis on topological manifolds

by

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1. Introduction. In recent years, there has been a considerable amount of research in analysis on differentiable manifolds (see, for example [4]). The concepts of stability and topological conjugacy in the spaces of diffeomorphisms and C^∞-endomorphisms of differentiable manifolds have been explored extensively [29], [28]. Let f be a diffeomorphism of a compact differentiable manifold M onto itself such that the tangent bundle of M splits continuously, $T(M) = E_s \oplus E_u$, so that the derivative of f, df, is contracting on E_s and expanding on E_u; i.e. there is a Riemannian metric $\|\cdot\|$ on $T(M)$ and there exist constants c, c' and $0 < \lambda < 1$ such that

$$\|df^n(v)\| \leqslant c\lambda^n \|v\|, \quad v \in E_s \quad \text{and} \quad \|df^n(v)\| \geqslant c'\lambda^{-n}\|v\|, \quad v \in E_u.$$

Anosov showed [1] that such a diffeomorphism is structurally stable; i.e., there exists a neighborhood $\mathfrak{U}$ of f in the space of diffeomorphisms of M such that each g in $\mathfrak{U}$ is topologically equivalent to f.

In this paper, we explore the concepts of contracting, expanding, and regular maps of topological manifolds. We are mainly interested in two questions: (1) what manifolds can support such maps and (2) determine the topological equivalence classes of such maps.

An *n-dimensional manifold* is a separable metric space each of whose points has a neighborhood homeomorphic to Euclidean n-space. We shall use int and cl to mean interior and closure respectively. $N(x, \varepsilon)$ $(N(A, \varepsilon))$ is the ε-neighborhood of the point x (the set A); $N_0(x, \varepsilon)$ will designate the path component of $N(x, \varepsilon)$ which contains x.

2. Covering spaces. We shall assume familiarity with covering space theory as given, for example, in [30].

THEOREM 1. *Let π: $\widetilde{M} \to M$ be a covering projection where M is a locally connected compact metric space with metric ϱ. Then there exists a metric $\tilde{\varrho}$ for $\widetilde{M}$ and $\eta > 0$ such that*

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(i) *if* $x, y \in \widetilde{M}$, $\widetilde{\varrho}(x, y) < \eta$, *then* $\varrho(\pi(x), \pi(y)) = \widetilde{\varrho}(x, y)$;

(ii) *the covering transformations are isometries*;

(iii) $\widetilde{\varrho}$ *is complete*;

(iv) *if* $A \subseteq \widetilde{M}$ *and the diameter of* A *is less than* η, *then the diameter of* $\pi(A)$ *is less than* η.

Proof. Let $\mathcal{U} = \{\mathcal{U}_i\}$ be a finite open cover of M such that each $\mathcal{U}_i$ is evenly covered by π. Let α be the Lebesgue number [9] of the cover $\mathcal{U}$. Hence if $A \subseteq M$ such that the diameter of A is less than α, then there exists a lift of A; — i.e. there exists a continuous map $\varphi\colon A \to \widetilde{M}$ such that $\pi\varphi(x) = x$ for each $x \in A$. Note that if A is connected and if φ_1 and φ_2 are two lifts of A such that $\varphi_1(x) = \varphi_2(x)$ for some x, then $\varphi_1 = \varphi_2$.

Let $\mathcal{V} = \{V_i\}$ be a finite open cover of M by connected sets such that the diameter of each V_i is less than $\frac{1}{2}\alpha$. Let β be the Lebesgue number of the cover $\mathcal{V}$. Define, for $x, y \in \widetilde{M}$,

$$\widetilde{\varrho}(x, y) = \begin{cases} \operatorname{minimum}\{\frac{1}{2}\beta, \varrho(\pi(x), \pi(y))\} & \text{if } x, y \text{ lie in the same component of } \pi^{-1}(V_i) \text{ for some } i, \\ \frac{1}{2}\beta & \text{otherwise.} \end{cases}$$

The only difficulty in showing that $\widetilde{\varrho}$ is a metric is the triangle inequality. Let $x, y, z \in \widetilde{M}$.

Case 1. If either $\widetilde{\varrho}(x, y)$ or $\widetilde{\varrho}(y, z)$ is $\frac{1}{2}\beta$, then clearly $\widetilde{\varrho}(x, y) + \widetilde{\varrho}(y, z) \geqslant \widetilde{\varrho}(x, z)$.

Case 2. Suppose $\widetilde{\varrho}(x, y)$ and $\widetilde{\varrho}(y, z)$ are both less than $\frac{1}{2}\beta$. Hence $x, y \in V_i^\sigma$ and $y, z \in V_j^\tau$ for some components V_i^σ and V_j^τ of $\pi^{-1}(V_j), \pi^{-1}(v_i)$ respectively. Note that the diameter of $\{\pi(x), \pi(y), \pi(z)\}$ is less than β; hence $\pi(x)$, $\pi(y)$, $\pi(z) \in V_k$ for some k.

Let V_k' be the component of $\pi^{-1}(V_k)$ which contains x. Hence $V_k' \cap V_i^\sigma \neq \emptyset$ and therefore $V_k \cup V_i$ has diameter less than α. There is a unique lift $\varphi\colon V_k \cup V_i \to \widetilde{M}$ such that $\varphi(V_k \cup V_i) = V_k' \cup V_i^\sigma$. It follows that $y \in V_k'$; similarly, $z \in V_k'$. Since x, y, z lie in the same component of $\pi^{-1}(V_k)$ and $\varrho'(a, b) = \operatorname{minimum}\{\frac{1}{2}\beta, \varrho(a, b)\}$ is a metric, $\widetilde{\varrho}(x, y) + \widetilde{\varrho}(y, z) \geqslant \widetilde{\varrho}(x, z)$. Let $\eta = \frac{1}{2}\beta$.

3. Covered and almost covered maps. Let $I = [0, 1]$ and (M, ϱ) be a metric space. Let M^I be the space of continuous maps from I to M with metric $\bar{\varrho}$ defined by $\bar{\varrho}(f, g) = \sup_{x \in I}\{\varrho(f(x), f(y))\}$. The (*Nash*) *tangent space* of M [23], $T(M)$, is the set $\{f \in M^I$ either $f(x) = f(0)$ for all $x \in I$ or $f(x) \neq f(0)$ for $x \neq 0\}$ with the topology induced from M^I. Define $p\colon T(M) \to M$ by $p(f) = f(0)$ and $i\colon M \to T(M)$ by $i(x) = c_x$ where $c_x\colon I \to M$ is the map defined by $c_x(t) = x$ for all $t \in I$. If M is a manifold, then p is a fiber map and i is a homeomorphism into.

If $f\colon M \to N$ is continuous, then f induces a continuous map $df\colon T(M) \to N^I$, defined by $df(g) = f \circ g$, the composition of f and g. If $df(T(M)) \subseteq T(N)$, f is said to be *covered*. If there exists a neighborhood $\mathcal{U}$ of $i(M)$ in $T(M)$ such that $df(\mathcal{U}) \subseteq T(N)$, f is said to be *almost covered*.

Proposition 2. *If either* (i) M *is an arcwise connected compact metric space and* N *is a non-degenerate metric space or* (ii) M *and* N *are connected* n-*dimensional manifolds, then an onto map* $f\colon M \to N$ *is covered if and only if* f *is a homeomorphism.*

Proof. Clearly if f is a homeomorphism, then f is covered. Suppose f is covered and let $x \neq y$ be points in M; there exists an embedding $\varphi\colon I \to M$ that $\varphi(0) = x$ and $\varphi(1) = y$. By hypothesis, $df(\varphi) \in T(N)$; hence $f(x) \neq f(y)$ and f is one-to-one. Compactness in (i) and invariance of domain [13] in (ii) imply that f is a homeomorphism.

Proposition 3. *If either* (i) M *is a locally compact, connected, and locally connected metric space or* (ii) M *and* N *are connected* n-*dimensional manifolds, then a map* $f\colon M \to N$ *is almost covered if and only if* f *is a local homeomorphism.*

Proof. Suppose f is a local homeomorphism; — i.e. for each $x \in M$, there exists a neighborhood V of x in M such that $f|V$ is a homeomorphism into. If $x \in M$, let $\varepsilon_x > 0$ be chosen so that $N(x, 2\varepsilon_x) \subseteq V$. Consider $N(i(x), \varepsilon_x) \subseteq T(M)$ and let $\varphi \in N(i(x), \varepsilon_x)$; note that the image of φ lies in V and hence, $df(\varphi) \in T(N)$. $U = \bigcup_{x \in M} N(i(x), \varepsilon_x)$ is the desired neighborhood.

Suppose $f\colon M \to N$ is almost covered; let $\mathcal{U}$ be a neighborhood of $i(M)$ in $T(M)$ such that $df(\mathcal{U}) \subseteq T(N)$. Let $x \in M$ and let $\varepsilon > 0$ be given so that $N(i(x), \varepsilon) \subseteq \mathcal{U}$. Let V be a connected neighborhood of x such that $\mathrm{cl}(V)$ is compact and $V \subseteq N(x, \varepsilon)$. If $y \neq z \in V$, then there is an embedding $\varphi\colon I \to V$ [9, p. 118] such that $\varphi(0) = y$ and $\varphi(1) = z$. $\varphi \in \mathcal{U}$ and we proceed as in Proposition 2 to show $f|V$ is a homeomorphism.

Proposition 4. *Let* M *be a Peano continuum, then an onto map* $f\colon M \to N$ *is almost covered if and only if* f *is a covering projection.*

Proof. Suppose f is almost covered; from the compactness of M and the previous proposition it follows that if $x \in N$, then $f^{-1}(x)$ is finite, say $f^{-1}(x) = \{x_1, x_2, \ldots, x_n\}$. Let V_i be a neighborhood of x_i such that $f|V_i$ is a homeomorphism and $V_i \cap V_j = \emptyset$ for $i \neq j$. Let $W = \bigcap_{i=l}^{m} f(V_i)$; then it is easily seen that W is evenly covered by f. Hence f is a covering projection.

The converse follows from Proposition 3.

Proposition 5. *Let* M *be a Peano continuum and let* $f\colon M \to M$ *be an almost covered onto map.* f *induces a map* $\widetilde{f}\colon \widetilde{M} \to \widetilde{M}$ *on the universal covering space of* M ($\widetilde{f}$ *is called a lifting of* f) *such that* $\widetilde{f}$ *is a homeomorphism.*

Proof. $\tilde{f}$ exists by [30, p. 76]. Let $\pi\colon \tilde{M}\to M$ be a covering projection. By the previous proposition f is a covering projection and hence $f\circ\pi$ is a covering projection. Since $f\circ\pi = \pi\circ\tilde{f}$, $\tilde{f}$ is a covering projection by [30, p. 64]. Since $\tilde{M}$ is simply-connected, $\tilde{f}$ is a homeomorphism.

DEFINITION. Let f and g be continuous maps of M onto itself. f is *topologically equivalent* to g if there exists a homeomorphism h of M onto itself such that $hf = gh$.

PROPOSITION 6. *If $f\colon M\to M$ is an almost covered onto map, if $g\colon M\to M$ is onto and if f is topologically equivalent to g, then g is almost covered.*

4. Contracting maps. Let M be a metric space; if $\varphi \in M^I$, define $\mathfrak{L}(\varphi) = \bar{\varrho}(\varphi, c_{\varphi(0)})$. A map $f\colon M\to M$ is *contracting* if there exists a neighborhood $\mathfrak{U}$ of $i(M)$ in $T(M)$, $0<\lambda<1$ and $c>0$ such that $\mathfrak{L}(df^n(\varphi)) \leqslant \lambda^n c\mathfrak{L}(\varphi)$ for all $n>0$ and $\varphi\in\mathfrak{U}$. (Note that f is actually a "local contracting" map.)

PROPOSITION 7. *If M is a locally compact locally connected metric space and $f\colon M\to M$ is a contracting map, then there exists $0<\lambda<1$ and $c>0$ such that if $x\in M$ and $\varepsilon>0$, then there exists $\delta>0$ such that if $y\in N_0(x,\delta)$, then $\varrho(f^n(x), f^n(y))\leqslant\lambda^n c\varepsilon$ for all $n>0$.*

Proof. Choose $\delta_1>0$ so that $N(i(x),\delta_1)\subseteq\mathfrak{U}$. Let $\delta=\text{minimum}\{\delta_1,\varepsilon\}$, let $y\in N_0(x,\delta)$ and let $\varphi\colon I\to N_0(x,\delta)$ be an embedding such that $\varphi(0)=x$ and $\varphi(1)=y$. Note that

$$\varrho(f^n(x), f^n(y)) \leqslant \mathfrak{L}(f^n\cdot\varphi) = \mathfrak{L}(df^n(\varphi)) \leqslant \lambda^n c\mathfrak{L}(\varphi) \leqslant \lambda^n c\varepsilon.$$

PROPOSITION 8. *If $f\colon M\to M$ is a contracting map, where M is a connected, locally connected, locally compact metric space and if $x, y\in M$, then for each $\varepsilon>0$, there exists N such that $n\geqslant N$ implies $\varrho(f^n(x), f^n(y))<\varepsilon$.*

Proof. The proof is essentially given by H. Maki in [21]. Since [21] has not yet been published, we include the proof. Define $x\sim y$ if x and y satisfy the conclusion of the Proposition. Note that "$\sim$" is an equivalence relation on M. Fix x and let $A=\{y\in M \mid x\sim y\}$. It follows from Proposition 7 that A is open in M. Suppose $\{y_n\}\subseteq A$ such that $\lim\limits_{n\to+\infty} y_n = y$. Again from Proposition 7, it follows that there exists K such that $k>K$ implies $y_k\sim y$. Hence $y\in A$; since M is connected, $A=M$.

COROLLARY 9 (H. Maki [21]). *If M is a complete, connected, locally connected, locally compact metric space and f is a contracting map of M into M, then there exists a unique fixed point of f.*

Proof(1). This is a consequence of Proposition 8 and Theorem 12 of [17].

(1) (Added in proof.) There is an error in [17]. To correct the proof of Corollary 9, replace "Theorem 12 of [17]" by "The main theorem of L.S. Husch, *Fixed points of k-regular mappings* (to appear)". The error is explained in H. Maki, "Generalizations of fixed point theorems II" (preprint Fukuoka Univ.).

Remark. If we had defined f to be a contracting map if f satisfied the conclusion of Proposition 7 with "$N_0(x,\delta)$" replaced by "$N(x,\delta)$", we could remove the hypotheses that M be locally connected and locally compact and still obtain the conclusion of Corollary 9. We will need this fact in section 5.

From the proof of Proposition 8, we also obtain

COROLLARY 10. *If f is contracting and $f(x)=x$, then $\lim\limits_{i\to+\infty} f^i(y)=x$ for all $y\in M$.*

COROLLARY 11. *If f is contracting, $f(x)=x$ and $\mathfrak{C}$ is a compact subset of M, then $\lim\limits_{i\to+\infty} f^i(\mathfrak{C})=x$.*

Proof. Let $\varepsilon>0$ be given. If $y\in\mathfrak{C}$, then there exists $\delta_y>0$ such that if $z\in N_0(y,\delta_y)$, then $\varrho(f^n(y), f^n(z))\leqslant\lambda^n c\left(\frac{\varepsilon}{2c}\right)$. Pick $y_1, y_2, \ldots, y_r\in\mathfrak{C}$ such that $\mathfrak{C}\subseteq\bigcup\limits_{i=1}^{n} N_0(y_i,\delta_{y_i})$. For each i, there exists N_i such that $n\geqslant N_i$ implies $\varrho(f^n(y_i), x)<\frac{1}{2}\varepsilon$. Let $N=\text{maximum}\{N_i\}_{i=1}^{r}$ and let $n>N$, then if $c\in\mathfrak{C}$, then $\varrho(f^n(c), x)<\varepsilon$ and the corollary follows.

COROLLARY 12. *If M is a connected absolute neighborhood retract with a complete metric ϱ and if f is a contracting homeomorphism of M onto itself, then M is contractible.*

Proof. Let x be the fixed point of f. By [12, p. 219], it suffices to show that the ith-homotopy group, $\pi_i(M,x)$ is zero. Let $\varphi\colon S^i\to M$ be a mapping of the i-sphere into M representing an element of $\pi_i(M,x)$. If V_1 is a neighborhood of x in M, then there exists a neighborhood V_2 of x, $V_2\subseteq V_1$, such that V_2 is contractible in V_1 [12, p. 96]. Choose $\varepsilon>0$ so that $N(x,\varepsilon)\subseteq V_2$. By Corollary 11, there exists n such that $f^n\varphi(S^i)\subseteq N(x,\varepsilon)$ and hence $f^n\varphi(S^i)$ is homotopically trivial in V_1 and hence in M. Since f^n induces an isomorphism of $\pi_i(M,x)$, $\pi_i(M,x)$ is trivial.

PROPOSITION 13. *Let M be as in Proposition 8 and let $f\colon M\to M$ be a contracting onto homeomorphism; then for all $y\in M$, $\{f^i(y) \mid i<0\}$ is closed in M.*

Proof. If y is the fixed point of f, the proposition is true. Suppose y is not the fixed point and suppose $z=\lim\limits_{n\to+\infty} f^{i_n}(y)$, $i_n<0$, $i_n\neq i_m$ for $n\neq m$. It follows from Proposition 7 that $y=\lim\limits_{n\to+\infty} f^{-i_n}(z)$; this contradicts Corollary 10.

THEOREM 14. *If M is a connected n-manifold with a complete metric ϱ and f is a contracting homeomorphism of M onto itself, then M is homeomorphic to n-dimensional Euclidean space. If $n\neq 4, 5$, then f is topologically equivalent to the dilation $z\to\frac{1}{2}z$.*

Proof. Let $\bar{M} = M \cup \{\infty\}$ be the one-point compactification of M. It follows from Proposition 13, that for each $y \in M$, $y \neq x$, where x is the fixed point of f, $\lim_{n\to-\infty} f^n(y) = \infty$. By an argument similar to that given for Corollary 11, for each compact set $C \subseteq M - \{x\}$, $\lim_{n\to-\infty} f^n(C) = \infty$. Let B be a closed n-cell in M such that x is in the interior of B. Hence, $\lim_{n\to-\infty} f^n(\mathrm{bdry}\, B) = \infty$. Thus, we can find a sequence of integers, $n_1 > n_2 > \ldots$, such that $B \subseteq \operatorname{int} f^{n_1}(B) \subseteq f^{n_1}(B) \subseteq \operatorname{int} f^{n_2}(B) \subseteq \ldots$ and $M = \bigcup_{i=1}^{\infty} f^{n_i}(B)$. By [3], M is homeomorphic to Euclidean n-space and by [16], f is topologically equivalent to the dilation.

Remark. If we assume in Theorem 14 that f is a homeomorphism into, then the conclusion does not necessarily follow. Whitehead [31] has given an example of a contractible open subset W of Euclidean 3-space which is not homeomorphic to Euclidean 3-space. It is easy to define a homeomorphism f of W into itself such that f is contracting.

$f\colon M\to M$ is *properly discontinuous* at $x \in M$ if there exists a neighborhood V of x in M such that $V \cap f^i(V) = \emptyset$ for all $i > 0$. f is *recurrent* at $x \in M$ if for each $\varepsilon > 0$, there exists $i > 0$ such that $\varrho(x, f^i(x)) < \varepsilon$.

PROPOSITION 15. *Let M be a locally compact locally connected metric space and let $f\colon M\to M$ be contracting such that f is not properly discontinuous at x, then f is recurrent at x.*

Proof. For each positive integer n, let $U_n = N_0(x, 1/n)$; there exists for each n, $z_n \in U_n \cap f^{i_n}(U_n)$, $z_n = f^{i_n}(w_n)$, $w_n \in U_n$, for some $i_n \neq 0$. Let $\varepsilon > 0$; by Proposition 7, there exists $0 < \lambda < 1$, $c > 0$, $\delta > 0$ such that if $y \in N_0(x, \delta)$, then $\varrho(f^n(x), f^n(y)) \leqslant \lambda^n c\left(\frac{\varepsilon}{2c}\right)$ for all $n > 0$. Choose N such $n > N$ implies $1/n < \operatorname{minimum}\{\delta, \frac{1}{2}\varepsilon\}$. If $n > N$, then

$$\varrho(x, f^{i_n}(x)) \leqslant \varrho(x, z_n) + \varrho(f^{i_n}(w_n), f^{i_n}(x)) < \varepsilon .$$

PROPOSITION 16. *Let M be as in Proposition 8 and let $f\colon M\to M$ be contracting such that f is recurrent at x, then $f(x) = x$.*

Proof. (We note that this is essentially the content of Propositions 6 and 7 of [14]; note, however, the proof of Proposition 6 in [14] is incorrect.) We wish to show that $\lim_{i\to\infty} f^i(x) = x$. Let $\varepsilon > 0$ be given and let $a = \varepsilon/4c$ where c is given in the definition of contracting. There exists $\delta > 0$ such that $\delta < ca$ and if $y \in N_0(x, \delta)$, then $\varrho(f^n(x), f^n(y)) \leqslant \lambda^n ca$. Choose $n > 0$ such that $f^n(x) \in N_0(x, \delta)$ and $\lambda^n < \frac{1}{2}$; then for all $i \geqslant 0$,

$$\varrho(x, f^{ni}(x)) \leqslant \varrho(x, f^n(x)) + \varrho(f^n(x), f^{2n}(x)) + \ldots + \varrho(f^{n(i-1)}(x), f^{ni}(x))$$
$$< ca + \lambda^n ca + \lambda^{2n} ca + \ldots + \lambda^{in} ca = ca\left(\sum_{j=0}^{i} \lambda^{nj}\right) < 2ca < \tfrac{1}{2}\varepsilon .$$

By Proposition 8, there exists for $i = 1, 2, \ldots, n-1$, an integer N_i such that if $r \geqslant N_i$, then $\varrho(f^r(x), f^{r+i}(x)) < \frac{1}{2}\varepsilon$. There exists N_0 such that if $r \geqslant N_0$, then $\varrho(x, f^{nr}(x)) < \frac{1}{2}\varepsilon$. Let $N = \operatorname{maximum}\{N_0, N_1, \ldots, N_{n-1}\}$ and let $r > nN$, say $r = np + q$ where $0 \leqslant q \leqslant n-1$. Hence $p \geqslant N$ and

$$\varrho(x, f^r(x)) \leqslant \varrho(x, f^{np}(x)) + \varrho(f^{np}(x), f^{np+q}(x)) < \varepsilon .$$

Hence

$$\lim_{i\to+\infty} f^i(x) = x \quad \text{and} \quad f(x) = f\big(\lim_{i\to+\infty} f^i(x)\big) = \lim_{i\to+\infty} f^{i+1}(x) = x .$$

PROPOSITION 17. *Let M be as in Proposition 8 and let $f\colon M\to M$ be a contracting map such that f has no fixed point; then for each compact set $C \subseteq M$, the set $\{i \mid C \cap f^i(C) \neq \emptyset\}$ is finite.*

Proof. Suppose there exists $n_1 < n_2 < \ldots$ such that $C \cap f^{n_i}(C) \neq \emptyset$. Let $z_i \in C \cap f^{n_i}(C)$, $z_i = f^{n_i}(w_i)$. By taking subsequences, we may assume that $\lim_{i\to+\infty} z_i = z$ and $\lim_{i\to+\infty} w_i = w$.

Let $\varepsilon > 0$ be given. By Proposition 7, there exists $\delta > 0$ such that if $y \in N_0(w, \delta)$, then $\varrho(f^r(y), f^r(w)) \leqslant c\lambda^r(\varepsilon/3c)$ for all $r \geqslant 0$. There exists N such that $i > N$ implies $\varrho(z, z_i) < \frac{1}{3}\varepsilon$, $\varrho(w, w_1) < \delta$ and $\varrho(f^{n_i}(w), f^{n_i}(z)) < \frac{1}{3}\varepsilon$. Hence if $i > N$, then

$$\varrho(z, f^{n_i}(z)) \leqslant \varrho(z, z_i) + \varrho(f^{n_i}(w_i), f^{n_i}(w)) + \varrho(f^{n_i}(w), f^{n_i}(z)) < \varepsilon .$$

Therefore f is recurrent at z; but this contradicts Proposition 16.

THEOREM 18. *Let M be a connected manifold and let f be a contracting homeomorphism of M onto itself such that f has no fixed points. If M has a finite number of ends [8], then the number of ends is one or two.*

Proof. By Corollary 9, M is not compact and hence has at least one end. Let M^* be the one point compactification of M and let f^* be the induced map on M^*. By the previous proposition and [20, p. 233], f^* has equicontinuous powers at each $x \in M$ using the metric ϱ^* induced from the one point compactification of M.

Let M^{**} be the Freudenthal end point compactification of M and let f^{**} be the induced map on M^{**}. If ϱ^{**} is the metric on M^{**}, it follows from Proposition 2 of [5], that f^{**} has equicontinuous powers at each $x \in M$. By [10], $M^{**} - M$ has at most two points and, hence, M has either one or two ends.

Remark. In [5], we investigated actions of open manifolds which have equicontinuous powers everywhere except at ∞. We refer the reader to [5] to obtain theorems about the structure of M as a consequence of the proof of Theorem 18. We list one of the corollaries.

COROLLARY 19. *Let M be an open connected n-manifold with two ends which has the homotopy type of a finite complex, $n \neq 4, 5$. If $n = 3$, suppose* *that M contains no fake 3-cells and if $n > 5$ suppose that the Whitehead group of $\pi_1(M)$ is trivial. Let f be a contracting homeomorphism of M onto itself such that f has no fixed points.*

Then there exists a closed submanifold N of M and homeomorphisms $\lambda\colon M \to N \times R$ (R = real numbers) and $\eta\colon N \to N$ such that f is topologically equivalent to the homeomorphism $\lambda^{-1}\varphi\lambda$ of M where φ is the homeomorphism $\varphi(x, t) = (\eta(x), t+1)$, $x \in N$, $t \in R$.

Remarks. Let R^2 be the plane with its usual metric and let $d\colon R^2 \to R^2$ be the dilation $d(x) = \frac{1}{2}x$. Then $d|R^2 - \{0\}$ is a contracting homeomorphism with no fixed point and $R^2 - \{0\}$ has two ends.

Let $A = \{(x, 0) | x = 0 \text{ or } x = 2^n \text{ where } n \text{ is an integer}\}$; then $d|R^2 - A$ is a contracting homeomorphism but $R^2 - A$ has an infinite number of ends.

5. Expanding maps. Let M be a metric space and let $f\colon M \to M$ be a map; f is *expanding* if there exists $\lambda > 1$, $c > 0$, $K > 0$ such that $\mathfrak{L}(df^n(\varphi)) \geqslant \text{minimum}\{\lambda^n c \mathfrak{L}(\varphi), K\}$ for all $\varphi \in T(M)$ and $n \geqslant 0$. We add K to the definition since if M is compact, then $\mathfrak{L}(df^n(\varphi))$ will be bounded by the diameter of M.

Let M be a locally connected compact metric space such that there exists a universal covering $\pi\colon \widetilde{M} \to M$ of M. Let $\Phi = \{g\colon \widetilde{M} \to \widetilde{M} | g$ is a lifting of some map from M into M and $g\varphi = \varphi g$ for all covering transformations $\varphi\}$. If ϱ is a metric for M, let $\widetilde{\varrho}$ be the metric given for $\widetilde{M}$ by Theorem 1 and define ϱ' on Φ by $\varrho'(g, h) = \sup_{x \in \widetilde{M}} \{\widetilde{\varrho}(g(x), h(x))\}$. It is easily seen that Φ is a complete metric space with respect to ϱ'. Let Φ_0 be the path component of the identity.

PROPOSITION 20 (see Theorem 2 of [28]). *Let M be a connected, locally connected compact metric space and let $m \in M$. Let $f\colon (M, m) \to (M, m)$ be an expanding almost covered onto map and suppose f is homotopic to a map g keeping m fixed. Suppose $\widetilde{f}$ and $\widetilde{g}$ are liftings of f and g respectively such that if for some m', $\pi(m') = m$, $\widetilde{f}(m') = \widetilde{g}(m')$. Then there exists a unique $h \in \mathrm{cl}\Phi_0$ such that $\widetilde{f}h = h\widetilde{g}$.*

Proof. Since f is homotopic to g keeping m fixed, it follows from standard covering space arguments that $\widetilde{f}(m'') = \widetilde{g}(m'')$ for all $m'' \in \pi^{-1}(m)$. Since a covering transformation is completely determined by its value at a single point, if φ, ξ are covering transformations such that $\widetilde{f}_\varphi = \xi\widetilde{f}$, then $\widetilde{g}\varphi = \xi\widetilde{g}$. Furthermore, since $\widetilde{f}$ is a homeomorphism (Proposition 5) $\pi\widetilde{f}\varphi\widetilde{f}^{-1} = f\pi\varphi\widetilde{f}^{-1} = f\pi\widetilde{f}^{-1}\varphi = \pi$, so that $\widetilde{f}\varphi\widetilde{f}^{-1}$ is a covering transformation. Therefore, given φ, there is always a covering transformation ξ such that $\widetilde{f}\varphi = \xi\widetilde{f}$.

For $k \in \Phi$, define $T(k) = \widetilde{f}^{-1}k\widetilde{g}$. If φ is a covering transformation, we get ξ by the remarks above and we have

$$\varphi T(k) = \varphi\widetilde{f}^{-1}k\widetilde{g} = \widetilde{f}^{-1}\xi k\widetilde{g} = \widetilde{f}^{-1}k\xi\widetilde{g} = \widetilde{f}^{-1}k\widetilde{g}\varphi = T(k)\varphi .$$

The map $a = \pi T(k)\pi^{-1}$ is then well defined and since $T(k)$ is a lift of a, $T(k) \in \Phi$.

To obtain h, we want to use the remark following Corollary 9. We want to show that T takes Φ_0 into itself. Let $f_t\colon (M, m) \to (M, m)$ be a homotopy, $t \in I$, such that $f_0 = g$ and $f_1 = f$. By [30, p. 67], there exists a unique homotopy $\widetilde{f}_t\colon (M, m) \to (M, m)$ such that $\widetilde{f}_0 = \widetilde{g}$. Since $\widetilde{f}_1(m') = m'$, it follows that $\widetilde{f}_1 = \widetilde{f}$. $\widetilde{f}^{-1}\widetilde{f}_t$ is then a homotopy such that $\widetilde{f}^{-1}\widetilde{f}_0 = \widetilde{f}^{-1}\widetilde{g}$ and $\widetilde{f}^{-1}\widetilde{f}_1 = $ identity. Note that if φ is a covering transformation, then $\widetilde{f}^{-1}\widetilde{f}_t\varphi = \varphi\widetilde{f}^{-1}\widetilde{f}_t$ for all t; hence $\widetilde{f}^{-1}\widetilde{f}_t$ defines a path in Φ whose endpoints are identity and T (identity).

Let ε be the constant given in Theorem 1 for the metric $\widetilde{\varrho}$; let λ, K and c be given from the definition that f is expanding. There is no loss of generality in assuming that $c\varepsilon < K$ by choosing, if necessary, a smaller ε. Let $\varphi \in N(i(\widetilde{M}), c')$; where

$$c' = \min(\varepsilon, c\varepsilon)\mathfrak{L}(\varphi) = \mathfrak{L}(\pi\varphi) = \mathfrak{L}(f^n\pi\widetilde{f}^{-n}\varphi) = \mathfrak{L}(df^n(\pi\widetilde{f}^{-n}\varphi))$$
$$\geqslant \lambda^n c\mathfrak{L}(\pi\widetilde{f}^{-n}\varphi) = \lambda^n c\mathfrak{L}(\widetilde{f}^{-n}\varphi) = \lambda^n c\mathfrak{L}(d\widetilde{f}^{-n}(\varphi))$$

provided $\mathfrak{L}(\pi\widetilde{f}^{-n}\varphi) = \mathfrak{L}(\widetilde{f}^{-n}\varphi)$. However, since $\mathfrak{L}(\varphi) < c\varepsilon$, $\mathfrak{L}(\pi\widetilde{f}^{-n}\varphi) < \varepsilon/\lambda^n < \varepsilon$ and we can apply Theorem 1 to get $\mathfrak{L}(\pi\widetilde{f}^{-n}\varphi) = \mathfrak{L}(\widetilde{f}^{-n}\varphi)$. Hence $\mathfrak{L}(d\widetilde{f}^{-n}(\varphi)) \leqslant (1/\lambda)^n(1/c)\mathfrak{L}(\varphi)$ and we have the following.

PROPOSITION 21. *If M is locally connected compact metric space and $f\colon M \to M$ is an almost covered expanding onto map, then $\widetilde{f}^{-1}\colon \widetilde{M} \to \widetilde{M}$ is a contracting homeomorphism.*

We now complete the proof of Proposition 20. Given $a \in \Phi$, we will show that given $\eta > 0$, there is a $\delta > 0$ such that if $b \in N(a, \delta)$ (in Φ_0) then $\varrho'(T^n(a), T^n(b)) < (1/\lambda)^n(1/c)\eta$.

Let δ be less than $c\varepsilon$ and such that if $x, y \in \widetilde{M}$ are points such that $\varrho(x, y) < \delta$, then x and y are connected by an arc of diameter less than $\min\{\eta, c\varepsilon\}$. Then if $b \in N(a, \delta)$, we have for each x,

$$\widetilde{\varrho}(\widetilde{f}^{-n}a(x), \widetilde{f}^{-n}b(x)) \leqslant \mathfrak{L}(d\widetilde{f}^{-n}(\varphi)) \leqslant (1/\lambda)^n(1/c)\mathfrak{L}(\varphi) \leqslant (1/\lambda)^n(1/c)\eta$$

where φ is an arc from $a(x)$ to $b(x)$ of diameterless than $\min\{\eta, c\varepsilon\}$. Taking the sup, we have $\varrho'(\widetilde{f}^{-n}a, \widetilde{f}^{-n}b) \leqslant (1/\lambda)^n(1/c)\eta$, and thus

$$\varrho'(T^n(a), T^n(b)) = \varrho'(\widetilde{f}^{-n}a\widetilde{g}^n, \widetilde{f}^{-n}b\widetilde{g}^n) \leqslant \varrho'(\widetilde{f}^{-n}a, \widetilde{f}^{-n}b) \leqslant (1/\lambda)^n(1/c)\eta.$$

T has a unique fixed point in $\mathrm{cl}\Phi_0$ by the remark following Corollary 9, and the proof is complete.

The following theorem was proved by M. Shub [28] with differentiability hypotheses; in fact, the proof is essentially his. Our contribution is that when Shub appeals to differential techniques we appeal to results we obtained above and hence drop the differentiability hypotheses.

THEOREM 22. *Let M be a compact connected absolute neighborhood retract and let f be an almost covered expanding map of M onto M. Then f has a fixed point and the universal covering space of M is contractible. If M is an n-dimensional manifold, then the universal covering space of M is homeomorphic to Euclidean n-space. If $f(m) = m$ and g is an almost covered expanding map of M onto M such that $g(m) = m$ and g is homotopic to f keeping m fixed, then g is topologically equivalent to f.*

Proof. Let $\tilde{f}$ be a lifting of f to the universal covering space $\tilde{M}$ of M. By Proposition 21, $\tilde{f}^{-1}$ is a contracting homeomorphism and by Corollary 9, $\tilde{f}^{-1}$ has a fixed point x and hence f has a fixed point $\pi(x)$. M is contractible by Corollary 12. If M is a manifold, the conclusion about $\tilde{M}$ follows from Theorem 14.

Let $m' \in \pi^{-1}(m)$ and let $\tilde{g}$ be a lifting of g such that $\tilde{g}(m') = \tilde{f}(m')$. By Proposition 20, there exist unique $h_1, h_2 \in \Phi_0$ such that

$$\tilde{f}h_1 = h_1\tilde{g} \quad \text{and} \quad \tilde{g}h_2 = h_2\tilde{f}.$$

Thus $h_2\tilde{f}h_1 = h_2h_1\tilde{g}$ and $h_1\tilde{g}h_2 = h_1h_2\tilde{f}$ and we get $\tilde{f}h_1h_2 = h_1h_2\tilde{f}$ and $\tilde{g}h_2h_1 = h_2h_1\tilde{g}$. Since $\mathrm{cl}\Phi_0$ is closed under composition and by the uniqueness part of Proposition 20, $h_1h_2 = h_2h_1 = $ identity, h_1 is the lifting of a continuous function k_i: $M \to M$, $i = 1, 2$, such that $fk_1 = k_1g$ and $gk_2 = k_2f$ and $k_1k_2 = k_2k_1 = $ identity.

6. Regular maps. Let M be a locally connected metric space and let f: $M \to M$. f is *regular* at x if for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $\varphi \in T(M)$, $\varphi(0) = x$ and $\mathfrak{L}(\varphi) < \delta$, then $\mathfrak{L}(df^n(\varphi)) < \varepsilon$ for all $n \geqslant 0$. B. v. Kerékjártó [18] introduced regularity to study homeomorphisms of the 2-sphere (1). He showed that if h is a homeomorphism of the 2-sphere such that h and h^{-1} are regular except possibly for a finite number n of points, then h is topologically equivalent to a linear action and the number n is 0, 1 or 2. If $n = 0, 1, 2$, then h is topologically equivalent to the extension of a rotation, translation or dilation, respectively, of the plane to its one-point compactification. Homma and Kinoshita [11], Kinoshita [19], [20] and Husch [14], [15], [16] considered the extension of these results to higher dimensions when $n = 1$ or 2. An example of Bing [2] showed that, in general, Kerékjártó's work could not be generalized to higher dimensions in the case $n = 0$. In this section we shall explore the case when $n = 0$.

We say f: $M \to M$ is *uniformly regular* if for each $\varepsilon > 0$, there exists $\delta > 0$ such that if $\varphi \in T(M)$, $\mathfrak{L}(\varphi) < \delta$, then $\mathfrak{L}(df^n(\varphi)) < \varepsilon$. Clearly if M is compact, then f is uniformly regular if and only if f is regular at each point of M.

(1) Kerékjártó's definition is slightly different from ours.

If f: $M \to M$, define $O(x) = \mathrm{cl}\{f^i(x) |\ i > 0\}$ and $K(x) = \bigcap_{i \geqslant 0} O(f^i(x))$ for $x \in M$. In Propositions 25 thru 28 below, we assume that $O(x)$ is complete. This will be the case if M is complete or $O(x)$ is compact.

PROPOSITION 24. *If f is uniformly regular (regular at x), then for each $\varepsilon > 0$, there is $\delta > 0$ such that if $y \in N_0(x, \delta)$, then $\varrho(f^n(x), f^n(y)) < \varepsilon$ for each $n \geqslant 0$.*

PROPOSITION 25. *If f is uniformly regular on $O(x)$ and $y, z \in O(x)$, $y = \lim_{i \to +\infty} f^{n_i}(x)$, $z = \lim_{i \to +\infty} f^{m_i}(x)$, then $\lim_{i \to +\infty} f^{m_i + n_i}(x)$ exists.*

Proof. Let $\varepsilon > 0$ and let $\delta > 0$ be given from the definition of uniform regularity for $\frac{1}{2}\varepsilon$. There exists $K_0(K_1)$ such that $i, j > K_0(K_1)$ implies $f^{n_i}(x) \in N_0(f^{n_j}(x), \delta)$ $(f^{m_i}(x) \in N_0(f^{m_j}(x), \delta))$. Hence

$$\varrho(f^{n_i+m_i}(x), f^{n_j+m_j}(x)) \leqslant \varrho(f^{n_i+m_i}(x), f^{n_i+m_j}(x)) + \varrho(f^{n_i+m_j}(x), f^{n_j+m_j}(x)) < \varepsilon.$$

Define $y \cdot z = \lim_{i \to +\infty} f^{m_i+n_i}(x)$. Clearly, $O(x)$ is algebraically a commutative semigroup.

PROPOSITION 26. *If $z_i, z, y \in O(x)$ such that $\lim_{i \to +\infty} z_i = z$, then $\lim_{i \to +\infty} y \cdot z_i = y \cdot z$.*

Proof. Suppose $y = \lim_{j \to +\infty} f^{n(j)}(x)$, $z = \lim_{j \to +\infty} f^{m(j)}(x)$, $z_i = \lim_{j \to +\infty} f^{m(i,j)}(x)$. Let $\varepsilon > 0$ be given and choose $\delta > 0$ so that $a \in N_0(b, \delta)$ implies $(f^r(a), f^r(b)) < \frac{1}{5}\varepsilon$ for all $r \geqslant 0$. There exists

(i) K such that $i > K$ implies $z_i \in N_0(z, \delta)$,
(ii) K_1 such that $j > K_1$ implies $\varrho(y \cdot z, f^{n(j)+m(j)}(x)) < \frac{1}{5}\varepsilon$,
(iii) K_2 such that $j > K_2$ implies $f^{m(j)}(x) \in N_0(z, \delta)$,
(iv) K_3 such that $j > K_3$ implies $f^{m(i,j)}(x) \in N_0(z_i, \delta)$,
(v) K_4 such that $j > K_4$ implies $\varrho(f^{n(j)+m(i,j)}(x), y \cdot z_i) < \frac{1}{5}\varepsilon$.

(Note K_1, K_2, K_3, K_4 depend upon i.)

Let $i > K$ and $j > K_1, K_2, K_3, K_4$. Then

$$\begin{aligned}\varrho(y \cdot z, y \cdot z_i) \leqslant \varrho(y \cdot z, f^{n(j)+m(j)}(x)) + \varrho(f^{n(j)+m(j)}(x), f^{n(j)}(z)) + \\ + \varrho(f^{n(j)}(z), f^{n(j)}(z_i)) + \varrho(f^{n(j)}(z_i), f^{n(j)+m(i,j)}(x)) + \\ + \varrho(f^{n(j)+m(i,j)}(x), y \cdot z_i) < \varepsilon.\end{aligned}$$

PROPOSITION 27. *If $z_i, y_i, z, y \in O(x)$ such that $\lim_{i \to +\infty} z_i = z$ and $\lim_{i \to +\infty} y_i = y$, then for $\varepsilon > 0$, there exists K such that $i > K$ implies $\varrho(y \cdot z_i, y_i \cdot z_i) < \varepsilon$.*

Proof. In addition to the notation in the previous proof, let $y_i = \lim_{j \to +\infty} f^{n(i,j)}(x)$. Let $\varepsilon > 0$ be given and choose $\delta > 0$ so that $a \in N_0(b, \delta)$ implies $\varrho(f^r(a), f^r(b)) < \frac{1}{5}\varepsilon$ for all $r \geqslant 0$. There exists

(i) K such that $i > K$ implies $y_i \in N_0(y, \delta)$,
(ii) K_1 such that $j > K_1$ implies $\varrho\big(y\cdot z_i, f^{n(j)+m(i,j)}(x)\big) < \frac{1}{5}\varepsilon$,
(iii) K_2 such that $j > K_2$ implies $f^{n(j)}(x) \in N_0(y, \delta)$,
(iv) K_3 such that $j > K_3$ implies $f^{n(i,j)}(x) \in N_0(y_i, \delta)$,
(v) K_4 such that $j > K_4$ implies $\varrho\big(f^{n(i,j)+m(i,j)}(x), y_i\cdot z_i\big) < \frac{1}{5}\varepsilon$.

Let $i > K$ and $j > K_1, K_2, K_3, K_4$. Then

$$\begin{aligned}\varrho(y\cdot z_i, y_i\cdot z_i) \leqslant\ & \varrho\big(y\cdot z_i, f^{n(j)+m(i,j)}(x)\big) + \varrho\big(f^{n(j)+m(i,j)}(x), f^{m(i,j)}(y)\big) + \\ & + \varrho\big(f^{m(i,j)}(y), f^{m(i,j)}(y_i)\big) + \varrho\big(f^{m(i,j)}(y_i), f^{m(i,j)+n(i,j)}(x)\big) + \\ & + \varrho\big(f^{n(i,j)+m(i,j)}(x), y_i\cdot z_i\big) < \varepsilon .\end{aligned}$$

COROLLARY 28. *If $f\colon M\to M$ is uniformly regular on $O(x)$, then $O(x)$ is a topological semigroup.*

COROLLARY 29. *If $f\colon M\to M$ is regular on $O(x)$ and $O(x)$ is compact, then $K(x)$ is an Abelian topological group and a minimal ideal in $O(x)$.*

Proof. This is a consequence of the previous corollary and [25], p. 109. (Recall an ideal A in a semigroup S is a subset of S such that $as \in A$ for $a \in A$ and $s \in S$.)

PROPOSITION 30. *Let f and x be as in Corollary 29. If $y \in K(x)$, then $O(y) = K(y) = K(x)$.*

Proof. Clearly $K(y) \subseteq O(y) \subseteq K(x)$. If $y = \lim_{i\to+\infty} f^{n_i}(x)$, then

$$y\cdot f(x) = \lim_{i\to+\infty} f^{n_i+1}(x) = f\big(\lim_{i\to+\infty} f^{n_i}(x)\big) = f(y);$$

by induction, it follows that for $r > 0$, $y\cdot f^r(x) = f^r(y)$.

Suppose $g \in O(x)$, $h \in K(y)$; $g = \lim_{i\to+\infty} f^{r_i}(x)$, $h = \lim_{i\to+\infty} f^{s_i}(y)$. Hence

$$\begin{aligned} h\cdot g &= \big(\lim_{i\to+\infty} f^{s_i}(y)\big)\cdot\big(\lim_{i\to+\infty} f^{r_i}(x)\big) \\ &= \big(\lim_{i\to+\infty} y\cdot f^{s_i}(x)\big)\big(\lim_{i\to+\infty} f^{r_i}(x)\big) \\ &= y\cdot\big(\lim_{i\to+\infty} f^{s_i}(x)\big)\cdot\big(\lim_{i\to+\infty} f^{r_i}(x)\big) \\ &= y\cdot \lim_{i\to+\infty} f^{s_i+r_i}(x) \\ &= \lim_{i\to+\infty} y\cdot f^{s_i+r_i}(x) = \lim_{i\to+\infty} f^{s_i+r_i}(y) \in K(y) .\end{aligned}$$

Therefore $K(y)$ is an ideal in $O(x)$ and $K(y) = K(x)$ by minimality of $K(x)$.

COROLLARY 31. *$K(z) \cap K(x) \neq \emptyset$ if and only if $K(z) = K(x)$.*

PROPOSITION 32. *Let f and x be as in Corollary 29. Then $f|K(x)\colon K(x)\to K(x)$ is a homeomorphism.*

Proof. Since $K(x)$ is compact, it suffices to show that $f|K(x)$ is one-to-one and onto. Let e be the identity element of $K(x)$. Suppose $g, h \in K(x)$ such that $f(g) = f(h)$.

$f(g) = g\cdot f(x) = (g\cdot e)\cdot f(x) = g\cdot f(e)$; hence $g\cdot f(e) = h\cdot f(e)$ and since $f(e) \in K(x)$, a group, $g = h$ and $f|K(x)$ is one-to-one.

Suppose $z \in K(x)$, $z = \lim_{n\to+\infty} f^{i_n}(x)$. Consider $\{f^{i_n-1}(x)\} \subseteq O(x)$ which is compact; hence, some subsequence converges, say, $\lim_{j\to+\infty} f^{i_{n_j}-1}(x) = r$. Then $f(r) = z$.

THEOREM 33. *Let M be a Peano continuum and let f be a regular map of M onto itself. Then f is a homeomorphism* ([1]).

Proof. By Proposition 32 and Corollary 31, it suffices to show that $M = \bigcup_{x\in M} K(x)$. For this we recall an idea of M. K. Fort [7]. If X, Y are spaces, $F\colon X\to 2^Y$ is a USC (*upper semicontinuous compact set-valued*) *function* if for each $x \in X$,

1) $F(x)$ is compact;

2) if $\mathfrak{U}$ is a neighborhood of $F(x)$, there is a neighborhood V of x such that $F(y) \subseteq \mathfrak{U}$ for each $y \in V$.

A slight modification of the arguments in [7] shows that if $\{F_i\}$ is a collection of USC functions from X to 2^Y with X compact Hausdorff and Y Hausdorff such that

3) for each i, $Y = \bigcup_{x\in X} F_i(x)$;

4) for each i and $x \in X$, $F_{i+1}(x) \subseteq F_i(x)$, then H, defined by $H(x) = \bigcap_i F_i(x)$ is a USC function and $Y = \bigcup_{x\in X} H(x)$.

If we define $F_i(x) = O\big(f^i(x)\big)$, then $K(x) = \bigcap_i F_i(x)$ and the F_i clearly satisfy 1), 3) and 4). 2) follows immediately from regularity.

COROLLARY 34. *For each $x \in M$, $x \in K(x)$.*

Proof. $x \in K(y)$ for some y; by Corollary 31, $K(x) = K(y)$.

PROPOSITION 35. *If M is a Peano continuum and $f\colon M\to M$ is an onto regular map, then $C(x) = K(x)$, where $C(x) = \mathrm{cl}\{f^i(x)|\ -\infty < i < +\infty\}$.*

Proof. Note $K(x) \subseteq C(x)$ and let $y \in C(x)$. If $y = f^i(x)$ for some i, then $y \in K(y) = K(x)$. Suppose $y = \lim_{n\to+\infty} f^{i_n}(x)$ where $i_n < 0$. Let $\varepsilon > 0$ be given; there exists $\delta > 0$ such that $a \in N_0(y, \delta)$ implies $\varrho\big(f^i(y), f^i(a)\big) < \varepsilon$ for all $i > 0$. Choose K such that $n > K$ implies $f^{i_n}(x) \in N_0(y, \delta)$. Hence $\varrho(f^{-i_n}(y), x) < \varepsilon$ and $\lim_{n\to+\infty} f^{-i_n}(y) = x$. Therefore $y \in K(x)$ and $C(x) = K(x)$.

([1]) (Added in proof.) Since this paper was written, the authors have learned that Theorem 33 was essentially proven by A.D. Wallace. See *Inverses in Euclidean Mobs*, Math. Jour. Okayama Univ. 3 (1953), pp. 1–3.

THEOREM 36 (see [6], p. 25). *If M is a Peano continuum and $f\colon M\to M$ is an onto regular map such that f^{-1} is also regular, then $\Lambda(f) = \mathrm{cl}\{f^i|\ -\infty < i < \infty\}$ in the space of continuous maps of M into M with the compact-open topology is a compact Abelian group.*

Proof. Suppose $k = \lim_{n\to+\infty} f^{i_n} \in \Lambda(f)$ and suppose $k(x) = k(y)$. $k(x) = \lim_{n\to+\infty} f^{i_n}(x)$ and hence, as argued in Proposition 35, $\lim_{n\to+\infty} f^{-i_n}k(x) = x$. Hence $x = y$, k is one-to-one and k is a homeomorphism of M onto M. Therefore $\Lambda(f)$ is an Abelian topological group.

Note that $\Lambda(f)$ is a regular transformation group; — i.e. if $\varepsilon > 0$, then there exists $\delta > 0$ such that if $\varrho(x, y) < \delta$, then $\varrho\big(g(x), g(y)\big) < \varepsilon$ for each $x, y \in M$. It follows from Ascoli's theorem [27, p. 155], that $\Lambda(f)$ is compact.

Remarks. In attempting to determine the topological conjugacy classes of homeomorphisms of manifolds by means of regular homeomorphisms, we run into difficulty with the Hilbert–Smith conjecture that a compact group which acts effectively on a manifold is a Lie group. However, if we are willing to make additional hypotheses we can apply the theory of compact Lie transformation groups. Define $\sigma(x) = \{g(x)|\ g \in \Lambda(f)\}$; it is easily seen that $\sigma(x) = C(x)$ and, by Proposition 30, $\sigma(x) = K(x)$. If we assume that if $K(x)$ is locally connected for each $x \in M$, then $\Lambda(f)$ is a Lie group [22, p. 244]. We give one example of this application.

Consider the 3-sphere S^3 as the boundary of the 4-cell, $I^2\times I^2$. Let r_1 and r_2 be rotations of the 1-sphere and extend by coning to I^2; they induce a regular homeomorphism $r_1 * r_2$ of S^3 whose inverse is also regular. Let α be the reflection of S^3 about S^2.

COROLLARY 37. *Let $f\colon S^3 \sim S^3$ be an onto regular map such that f^{-1} is also regular. If f is not periodic and for each $x \in S^3$, $K(x)$ is locally connected, then there exist rotations r_1 and r_2 of S^1 such that f is topologically equivalent to either $r_1 * r_2$ or $\alpha\cdot(r_1 * r_2)$.*

Since $\Lambda(f)$ is a compact Abelian Lie group, $\Lambda(f)$ is isomorphic to $S^1\times G$ or $S^1\times S^1\times G$ where G is a finite Abelian group. But the actions of such groups have been studied in [24] and [26] and were shown to be topologically equivalent to standard actions which reduce to that listed above when restricted to f.

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Examples relating to mesocompact and sequentially mesocompact spaces (*)

by

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Introduction. It is the purpose of this paper to present examples, which relate to the structural properties and mapping properties of the mesocompact and sequentially mesocompact spaces [1]. In particular, a Tychonoff sequentially mesocompact space which is not mesocompact, is presented in Example 2.1. Example 2.2 establishes that sequential mesocompactness is not invariant under perfect mappings.

To put these examples in proper perspective, § 1 contains the definitions and statements of the main theorems which are contained in [3]. The examples are presented in § 2. All spaces are assumed to be Hausdorff and all functions are continuous surjections in this paper.

1. Characterizations and mapping theorems. A topological space is said to have *property* (k) (*property* (ω) [2]), if for each discrete collection of closed sets $\mathcal{F} = \{F_\alpha\colon \alpha \in A\}$, there exists a compact-finite (cs-finite) [1], collection of open sets $\mathcal{U} = \{U_\alpha\colon \alpha \in A\}$ such that $F_\alpha \subset U_\alpha$, for each $\alpha \in A$ and $U_\alpha \cap F_\beta = \emptyset$, if $\alpha \neq \beta$.

THEOREM 1.1. *A normal space is mesocompact (sequentially mesocompact) if and only if it is a metacompact space with property* (k) (*property* (ω)).

THEOREM 1.2. *The perfect image of a normal mesocompact space is a normal mesocompact space.*

A mapping $f\colon X \to Y$ is said to be *presequential*, if for each convergent sequence $\{p_i\}$ in Y, $p_i \to p$, which is not eventually equal to p, $\bigcup \{f^{-1}(p_i)\colon i \in N,\, p_i \neq p\}$ is not sequentially closed.

THEOREM 1.3. *The closed presequential image of a normal sequentially mesocompact space is a normal sequentially mesocompact space.*

Theorems 1.2 and 1.3 depend on the facts that property (k) is invariant under perfect mappings and that property (ω) is invariant under

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