Measurable uniform spaces

by

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This paper concerns the interpreting of the operator which assigns to a topological space its σ-field of Baire sets as a functor in the category of separable uniform spaces. (We call a uniform space separable if it has a basis of countable covers. In metric spaces, this corresponds to topological separability.) Accordingly, a *measurable uniform space* is one derived from a σ-field $\mathcal{A} \subset 2^X$ by taking as a basis the countable covers with members from $\mathcal{A}$; this uniform space is denoted $\mathcal{A}X$. The terminology is justified by the Proposition: $\mathcal{A}X \xrightarrow{f} \varrho M$ (ϱM a metric space) is uniformly continuous iff $f^{-1}(B) \in \mathcal{A}$ for each Baire set B of M, that is, f is measurable in the usual sense. There is a functor, b, coreflecting the category of separable uniform spaces onto its subcategory of measurable spaces: $b\mu X$ carries the coarsest measurable uniformity on X, finer than μ.

A number of the results about b are routine analogs of well known results about Baire sets and functions, e.g.: $b\mu$ is associated with the σ-field generated by coz $C(\mu X)$, the class of sets $\{x: f(x) \neq 0\}$ for $f \in C(\mu X)$ (the uniformly continuous functions to R), and $C(b\mu X)$ is the least class of functions containing $C(\mu X)$ and closed under the taking of pointwise limits of sequences.

It is shown that a measurable space $\mathcal{A}X$ is weak generated from $C(\mathcal{A}X)$. Consequently, the result above on Baire functions completely describes $b\mu$. The structures $C(\mathcal{A}X)$ also have a simple algebraic description, permitting this somewhat algebraic construction of $C(b\mu X)$ (and hence $b\mu X$): $C(b\mu X)$ is the smallest uniformly closed *regular* ring containing the vector lattice $C(\mu X)$.

The operator b preserves subspaces, not topology, completeness, but not completion. More exactly, $b\mu X$ is complete if μX is; and, with γ denoting the completion functor, $b\gamma = \gamma b$ exactly on spaces which are G_δ-dense in the completion.

The completeness theorem necessitates consideration of another functor m, coreflecting separable uniform spaces onto the "$\mathcal{M}$-fine" spaces, treated in detail in [4(e)]. We have $\mu \subset m\mu \subset b\mu = bm\mu$. That is, a measurable space is $\mathcal{M}$-fine. The measurable spaces are characterized in several ways among $\mathcal{M}$-fine ones, and the functions in $C(m\mu X)$ are shown to be a special subclass of the first Baire class in $C(b\mu X)$.

I am indebted to Z. Frolík for several conversations on the general subject of this paper, to G. D. Reynolds and M. D. Rice — the proof of 6.9 was worked out while [6] was being written — and to Rice for various conversations in which we arrived at the view of "$\mathcal{M}$-O spaces" described in § 6.

1. Background. The uniform space μX is the set X with the collection of (uniform) covers, μ, satisfying the axioms to be found in [12(b)]. Not much background will be required: for the most part, the first twenty-four pages of [12(b)], and the following remarks, will suffice.

The $\mathcal{F}$-weak uniformity on X is obtained from the family $\mathcal{F}$ of maps to uniform spaces by taking for a subbasis the covers $f^{-1}(\mathcal{U})$, $f \in \mathcal{F}$ and $\mathcal{U}$ uniform in the range of f.

The precompact reflection of μX, $p\mu X$, has for basis the finite μ-covers. $p\mu X$ is weak generated by $C^*(\mu X)$, the bounded functions in $C(\mu X)$. The completion $\gamma p\mu X$ is the Samuel compactification, denoted $s\mu X$.

$T\mu X$ stands for the topological space underlying μX.

If X is a uniformizable topological space, eaX is the associated "Shirota" uniform space, whose uniformity has basis of all enumerable normal covers of X. ea is weak generated by all continuous functions to separable metric spaces. (ea is the "enumerable reflection" of the fine uniformity a.) The uniformly continuous functions $C(eaX)$ are just $C(X)$, the real-valued continuous functions. (Shirota proved that $T\gamma eaX$ is $T\gamma caX$, where ca is weak generated by $C(X)$; the latter space is Hewitt's realcompactification, vX.)

For a metrizable space M, with compatible metric ϱ, we let ϱM denote the associated metric uniform space. This has basis $\{\mathcal{S}_\varrho(\varepsilon)\colon \varepsilon > 0\}$, where $\mathcal{S}_\varrho(\varepsilon)$ (or just $\mathcal{S}(\varepsilon)$) is the cover of all ε-balls. In case M is separable, and D is countable dense, then $\{\mathcal{S}_\varrho^D(\varepsilon)\colon \varepsilon > 0\}$ is a basis for ϱM, where $\mathcal{S}_\varrho^D(\varepsilon)$ denotes the cover of all ε-balls with centers in D. So ϱM is "uniformly separable."

Finally, the structures $C(\mu X)$ will receive considerable attention. As is checked readily, any $C(\mu X)$ is a vector lattice in the pointwise operations, and is "uniformly closed", that is, contains limits of its sequences which converge uniformly on X.

2. Measurable uniform spaces. Let $\mathcal{A}$ be a σ-field of subsets of X which separates points. (That is, $\mathcal{A}$ is closed under complementation and countable union, and if $x \neq y$ there is $A \in \mathcal{A}$ with $x \in A$, $y \notin A$.) The pair $(X, \mathcal{A})$ is called a *measurable space*. A function $(X, \mathcal{A}) \xrightarrow{f} (Y, \mathcal{B})$ between measurable spaces is called measurable if $f^{-1}(B) \in \mathcal{A}$ when $B \in \mathcal{B}$.

From X and $\mathcal{A}$, we construct a uniform space as follows. Call a cover (respectively, partition) of X by countably many sets from $\mathcal{A}$, a countable $\mathcal{A}$-cover (resp., $\mathcal{A}$-partition). One checks that the collection of such covers, or partitions, satisfy the conditions for a basis (that is, the collection contains $\mathcal{U} \wedge \mathcal{V}$ when it contains $\mathcal{U}$, $\mathcal{V}$, and each such cover is $*$-refined by another — in the case of a partition, by itself.) The uniform space so generated we denote $\mathcal{A}X$, and call *measurable*.

If $\mathcal{A}X \xrightarrow{f} \nu Y$ is a uniform isomorphism, then (clearly) $f(\mathcal{A})$ is a σ-field and ν has basis of countable $f(\mathcal{A})$-covers. Thus νY is measurable, and measurability is a uniform property.

In § 6, we shall see two uniform theoretic characterizations of measurability of a uniform space, and in § 3, a characterization in terms of algebraic properties of $C(\mathcal{A}X)$.

2.1. Proposition. *$(X, \mathcal{A}) \xrightarrow{f} (Y, \mathcal{B})$ is measurable iff $\mathcal{A}X \xrightarrow{f} \mathcal{B}Y$ is uniformly continuous.*

Proof. If f is measurable, then evidently $f^{-1}(\mathcal{U})$ is a countable $\mathcal{A}$-partition if $\mathcal{U}$ is a countable $\mathcal{B}$-partition. So f is uniformly continuous.

Conversely, let f be uniformly continuous, and let $B \in \mathcal{B}$. Choose a countable $\mathcal{A}$-partition $\mathcal{U}$ refining $f^{-1}\{B, Y-B\}$. Then, $f^{-1}(B) = \bigcup \{U \in \mathcal{U}\colon U \subset f^{-1}(B)\}$. The latter is in $\mathcal{A}$, so f is measurable.

From 2.1, it follows that the uniform isomorphisms between measurable uniform spaces are the same as the one-one onto bi-measurable maps between the associated measurable spaces.

The (uniform) Baire spaces will occur periodically in the sequel. For X a uniformizable topological space, let $\mathcal{B}aX$ be the σ-field of Baire sets, by definition, the least σ-field containing $\operatorname{coz} C(X)$. The associated Baire space is the measurable uniform space $(\mathcal{B}aX)X$, which we abbreviate bX.

2.2. Proposition. *Let $\mathcal{A}X$ be measurable, ϱM separable metric, $X \xrightarrow{f} M$ a function. These are equivalent.*

(a) *$\mathcal{A}X \xrightarrow{f} \varrho M$ is uniformly continuous,*

(b) *$\mathcal{A}X \xrightarrow{f} bM$ is uniformly continuous,*

(c) *$(X, \mathcal{A}) \xrightarrow{f} (M, \mathcal{B}aM)$ is measurable,*

(d) *$f^{-1}(G) \in \mathcal{A}$ whenever G is open in M.*

For the proof of this, we need an induction involving the Baire classification for the sets in $\mathcal{B}aM$. We indicate the classification. (A full proof of a generalization appears in 4.5.)

For metrizable M, each open set is cozero, so we set $\sigma_0 =$ all open sets, $\sigma_1 =$ all G_δ's, $\sigma_2 =$ the $(G_\delta)_\sigma$'s. For limit ordinal β, let $\sigma_\beta = \bigcup\{\sigma_\alpha\colon \alpha < \beta\}$, and $\sigma_{\beta+1} = (\sigma_\beta)_\delta$. Then (theorem) $\mathcal{B}aM = \sigma_{\omega_1}$, where ω_1 is the first uncountable ordinal.

Proof of 2.2. (b) and (c) are equivalent by 2.1. (b) implies (a) because $b \supset \varrho$ (as uniformities on M), since M is separable.

Assume (a). For each n, there is a basic cover $\mathfrak{U}_n < f^{-1}(S_\varrho(1/n))$. Then $f^{-1}(G) = \bigcup\{U\colon U \in \mathfrak{U}_n$ for some n, and $U \subset f^{-1}(G)\}$. The latter is the union of countably many sets in $\mathcal{A}$, and is itself in $\mathcal{A}$. So (d) holds.

Assume (d). Then $f^{-1}(B) \in \mathcal{A}$ for $B \in \sigma_0$. If $B \in \sigma_1$, then $B = \bigcap B_n$ for $B_n \in \sigma_0$, and $f^{-1}(B) = \bigcap f^{-1}(B_n)$. Since each $f^{-1}(B_n) \in \mathcal{A}$, $f^{-1}(B) \in \mathcal{A}$ as well. Etc., by induction, through all classes σ_a. So (c) holds.

2.3. COROLLARY. *$C(\mathcal{A}X)$ coincides with the collection of measurable functions $(X, \mathcal{A}) \to (R, \mathcal{B}aR)$.*

2.4. PROPOSITION. *$\mathcal{A}X$ is weak generated by $C(\mathcal{A}X)$, or by $C(\mathcal{A}X, aN)$.*

Proof. Since $aN = \varrho N$ is a subspace of ϱR (or of bR), it suffices to show that $C(\mathcal{A}X, aN)$ generates $\mathcal{A}X$. So let $\{A_n\}_n$ be a countable $\mathcal{A}$-partition, and define $f(x) = n$ iff $x \in A_n$. Then $f \in C(\mathcal{A}X, aN)$, and $f^{-1}(\{n\}_n) = \{A_n\}$.

2.4 raises the possibility of describing the uniform spaces $\mathcal{A}X$ by describing the algebraic structures $C(\mathcal{A}X)$. We turn to this in succeeding sections.

For now, we consider uniform topology.

In general, if μX is weak generated by the family $\mathcal{F}$ of uniformly continuous maps to uniform spaces $X_f (f \in \mathcal{F})$, then $T\mu X$ is weak generated by the (continuous) maps $f\colon X \to TX_f$ $(f \in \mathcal{F})$; and, $T\mu X$ is always weak generated by $C(\mu X)$ [12(b)] (whether $C(\mu X)$ generates μX or not), and this implies that the family $\operatorname{coz} C(\mu X)$ is an open basis.

2.5. PROPOSITION. *$\operatorname{coz} C(\mathcal{A}X) = \mathcal{A}$, so $T\mathcal{A}X$ has $\mathcal{A}$ for open basis. Thus $T\mathcal{A}X$ is a P-space (i.e., G_δ's are open).*

Proof. If $A \in \mathcal{A}$, then the characteristic function $\chi_A\colon X \to \{0, 1\}$ is measurable, or uniformly continuous (see 2.3 and the proof of 2.1). So $A = \operatorname{coz}\chi_A \in \operatorname{coz} C(\mathcal{A}X)$.

If $f \in C(\mathcal{A}X)$, then $\operatorname{coz} f = f^{-1}(R - \{0\})$ and 2.2(d) applies.

For a converse of 2.5, it can be proved quickly that if X is a P-space, then $\operatorname{coz} C(X) = \mathcal{B}aX$, and thus $X = TbX$. "Usually" a P-space supports several different measurable uniformities (or different σ-algebras which induce the same topology). For example, if X = discrete R, then $\mathcal{A} = 2^R$, $\mathcal{B} = \mathcal{B}aR$ (the usual Baire sets). Or (more generally) let Y be any topological space for which not every set is Borel: then discrete Y has the topologically equivalent, but different, σ-algebras 2^Y and Borel Y. See § 6 for the exact criterion that a P-space admit a unique compatible measurable uniformity.

3. Real-valued functions, Samuel compactification, completion. We give equivalent ways of viewing measurable spaces by characterizing measurability, first, by algebraic properties of $C(\mathcal{A}X)$, and second, by topological properties of $s\mathcal{A}X$. $s\mathcal{A}X$ is shown to be the Stone space of the Boolean algebra $\mathcal{A}$, and $\gamma\mathcal{A}X$ is described as a subspace of $s\mathcal{A}X$.

3.1. THEOREM. (a) *$C(\mathcal{A}X)$ is a point-separating uniformly closed lattice and algebra with 1, which is (von Neumann) regular.*

(b) *If $\mathcal{A}X$ and $\mathcal{B}X$ are distinct measurable uniformities on X (i.e., $\mathcal{A} \neq \mathcal{B}$), then $C(\mathcal{A}X) \neq C(\mathcal{B}X)$.*

(c) *If $A \subset R^X$ is a family of functions with the properties of* (a), *then A weak-generates a measurable uniformity on X.*

(Recall that a commutative ring is *regular* if given a there is b with $a^2b = a$.)

The one-to-one correspondence described by 3.1 can be made into a categorical duality between complete measurable uniform spaces and the appropriate category of function algebras. We postpone any further discussion of this.

3.1 is closely related to [4(a), 2.3], and derives from results of Anderson [22] and Brainerd [23].

Proof of (a). Any $C(\mu X)$ is a point-separating uniformly closed vector lattice, with the constant function 1. Using 2.3, that $C(\mathcal{A}X)$ is an algebra is well known (e.g. [11]). For regularity, take f and define $g(x) = 1/f(x)$ if $f(x) \neq 0$, $g(x) = 3$ otherwise. Using, say, 2.2(d), $g \in C(\mathcal{A}X)$, and clearly, $f^2g = f$.

Proof of (b). By 2.4.

Proof of (c). Some simple lemmas are required.

3.2. LEMMA. *The linear combinations of functions χ_A ($A \in \mathcal{A}$) comprise the functions in $C(\mathcal{A}X)$ which have finitely many values. And $C^*(\mathcal{A}X)$ consists of uniform limits of sequences of these.*

Proof. Each $\chi_A \in C(\mathcal{A}X)$, of course, hence a linear combination $f = \sum_{i=1}^{n} r_i\chi_{A_i} \in C(\mathcal{A}X)$. And range f is a subset of the set of sums of the numbers $\{r_i\colon i = 1, \ldots, n\}$. Conversely, if $f \in C(\mathcal{A}X)$ takes the values $\{r_1, \ldots, r_n\}$, then each $A_i = f^{-1}(r_i) \in \mathcal{A}$, and $f = \sum_{i=1}^{n} r_i\chi_{A_i}$.

Each uniform limit of such functions is in $C^*(\mathcal{A}X)$, since $C(\mathcal{A}X)$ and $C^*(\mathcal{A}X)$ are closed under uniform convergence. If $f \in C^*(\mathcal{A}X)$, and $\varepsilon > 0$, cover range f by disjoint intervals (say, half-open) $I_1, \ldots, I_n$ of length $\frac{1}{2}\varepsilon$. By 2.2(c), each $A_i = f^{-1}(I_i) \in \mathcal{A}$. Pick $r_i \in I_i$, and note that $|f(x) - \sum_{i=1}^{n} r_i\chi_{A_i}(x)| < \varepsilon$ for all x.

3.3. LEMMA. *If $A: X \to R$ is a regular ring of functions, with $1 \in A$, then*

(a) *A is closed under inversion: $f \in A$ with $f(x) \neq 0$ for all x imply that $1/f \in A$.*

(b) *If $f \in A$, then the characteristic function of $\operatorname{coz} f$ is in A.*

Proofs. For (a), choose g with $f^2 g = f$. Then $f(fg-1) = 0$ at each $x \in X$. If $f(x) \neq 0$, then $g(x) = 1/f(x)$. If each $f(x) \neq 0$, then $g = 1/f$.

For 3.3(b), again take g with $f^2 g = f$, and set $e = (1-fg)$. Then $e^2 = e$, so e takes values 0 and 1, only. One checks that $1-e$ is the characteristic function of $\operatorname{coz} f$.

So now let A be an algebra with the properties listed. We set $\mathcal{A} = \operatorname{coz} A$, and shall prove that $\mathcal{A}$ is a σ-field, and that $A = C(\mathcal{A}X)$. Then 2.4 says that A generates $\mathcal{A}X$.

If $f_1, f_2, \ldots \in A$, then $\bigcup_n \operatorname{coz} f_n = \operatorname{coz} \sum_n 2^{-n}(|f_n| \wedge 1)$. And the latter is in $\operatorname{coz} A$. Next, if $f \in A$, let χ be the characteristic of $\operatorname{coz} f$. By 3.3(b), $\chi \in A$. Hence $1-\chi \in A$, and $\operatorname{coz}(1-\chi) = X - \operatorname{coz} f$. So $\mathcal{A}$ is a σ-field.

Now, if $f \in A$, then $f^{-1}(a, +\infty) = \operatorname{coz}(f \vee a - a) \in \mathcal{A}$. Likewise, $f^{-1}(-\infty, b) \in \mathcal{A}$. Hence, $f^{-1}(a, b) = f^{-1}(a, +\infty) \cap f^{-1}(-\infty, b) \in \mathcal{A}$. If G is open, in R, then $G = \bigcup_n I_n$, each I_n an open interval, and $f^{-1}(G) = \bigcup_n f^{-1}(I_n) \in \mathcal{A}$. By 2.2(d), $f \in C(\mathcal{A}X)$.

Conversely, first, take $f \in C^*(\mathcal{A}X)$. Then $f \in A$, by 3.2, 3.3(b), and the fact that A is uniformly closed. For general $f \in C(\mathcal{A}X)$, write $f = [f/(1+f^2)]/[1/(1+f^2)] = f_1/f_2$. Now, f_1 and f_2 are bounded, and are in $C(\mathcal{A}X)$ by 3.1(a) and 3.3(a). Hence, $f_1, f_2 \in A$. Since A is regular, 3.3(a) applies again, and $f_1/f_2 \in A$.

The proof of 3.1 is concluded.

The Samuel compactification $s\mathcal{A}X$ has the property that the restrictions $C(s\mathcal{A}X)|X = C^*(\mathcal{A}X)$. Since $C^*(\mathcal{A}X)$ determines $C(\mathcal{A}X)$ (by inversion, as at the end of the proof of 3.1(c), and $C(\mathcal{A}X)$ determines $\mathcal{A}X$ (in a strong sense, by 2.4, or in the weaker sense that $\operatorname{coz} C(\mathcal{A}X) = \mathcal{A}$), we have:

3.4. PROPOSITION. *The correspondence $\mathcal{A}X \leftrightarrow s\mathcal{A}X$ is one-to-one between measurable uniformities on X and their Samuel compactifications.*

Thus, describing measurable uniformities is essentially the same as describing their Samuel compactifications. We consider this.

Given a compact space K, let $\operatorname{clop} K$ stand for the Boolean algebra of clopen subsets. K is called Boolean if $\operatorname{clop} K$ separates the points. Given a Boolean algebra $\mathcal{B}$, there is an essentially unique Boolean space K (the Stone representation space) such that $\mathcal{B}$ is isomorphic to $\operatorname{clop} K$. (For background, see [7(b), § 21].)

Observe that $Y \subset K$, the map $\operatorname{clop} K \ni G \to G \cap Y \in (\operatorname{clop} K) \cap Y$ is a Boolean algebra homomorphism, which has kernel $\{\emptyset\}$ (i.e., is an isomorphism) if Y is dense.

3.5. THEOREM. *$s\mathcal{A}X$ is Boolean, and $\operatorname{clop} s\mathcal{A}X \ni G \to G \cap X$ is a Boolean isomorphism onto $\mathcal{A}$. Thus $s\mathcal{A}X$ is the Stone space of $\mathcal{A}$.*

Proof. We are to show that $\operatorname{clop} s\mathcal{A}X$ separates points of $s\mathcal{A}X$ (whence $s\mathcal{A}X$ is Boolean, and the Stone space of its clopen algebra), and that $(\operatorname{clop} s\mathcal{A}X) \cap X = \mathcal{A}$.

We do the latter first. If $A \in \operatorname{clop} s\mathcal{A}X$, then $\chi_A \in C(s\mathcal{A}X)$, and $\chi_A|X \in C^*(\mathcal{A}X)$. So $\operatorname{coz}(\chi_A|X) \in \mathcal{A}$, and clearly $A \cap X = \operatorname{coz}(\chi_A|X)$. And, if $A \in \mathcal{A}$, then $\chi_A \in C^*(\mathcal{A}X)$, so χ_A extends to $f \in C(s\mathcal{A}X)$. Evidently, range $f = \{0, 1\}$, so $\operatorname{coz} f \in \operatorname{clop} s\mathcal{A}X$; and $\operatorname{coz} f \cap X = A$.

If $p \neq q$ in $s\mathcal{A}X$, choose $f \in C(s\mathcal{A}X)$ with $f(p) = 0$ and $f(g) = 1$. Then $f|X \in C^*(\mathcal{A}X)$, so $f|X$ can be approximated within $\frac{1}{3}$ by a function $g \in C(\mathcal{A}X)$ with finitely many values, by 3.3. Extending over $s\mathcal{A}X$, to f and g^s, we preserve the approximation within $\frac{1}{3}$, and g^s has the same finitely many values. So $g^s(p) \leqslant \frac{1}{3}$ and $g^s(q) \geqslant \frac{1}{3}$, and $(g^s)^{-1}(g^s(p))$, $(g^s)^{-1}(g^s(q))$ are the desired separating clopen sets.

This theorem implies 3.6 below, and will be useful in studying the completion of an $\mathcal{A}X$.

A compact space is called *basically disconnected* if each open F_σ has open closure. (For normal X, the open F_σ make up exactly $\operatorname{coz} C(X)$ [14].) It is well known that these spaces arise exactly as the Stone spaces of σ-complete Boolean algebras [7(b), § 20]. Thus:

3.6. THEOREM. *$s\mathcal{A}X$ is a basically disconnected compactification of the P-space $T\mathcal{A}X$.*

Here is a short direct proof of 3.6 (from [4(a), 2.1]) based on regularity of the ring $C(\mathcal{A}X)$. If G is an open F_σ in $s\mathcal{A}X$, then $G = \operatorname{coz} h$ for an $h \in C(s\mathcal{A}X)$. Then $f = h|X \in C(\mathcal{A}X)$, so there is $g \in C(\mathcal{A}X)$ with $f^2 g = f$. Now $(fg)^2 = fg$, so range $fg \subset \{0, 1\}$, $fg \in C^*(\mathcal{A}X)$ and has the extension $e \in C(s\mathcal{A}X)$. Evidently, $\operatorname{range}(e) = \operatorname{range}(fg)$; and one checks that $\operatorname{coz} e$ is the closure of $\operatorname{coz} h$.

We now discuss how to produce measurable uniformities from compact basically disconnected spaces. The result (3.8) below) includes a converse of 3.6. We require a lemma, essentially a reformulation of 3.2.

3.7. LEMMA. *$p\mathcal{A}X$ has for basis the finite $\mathcal{A}$-partitions.*

Proof. Evidently, finite $\mathcal{A}$-partitions are in $p\mathcal{A}$. On the other hand, $p\mathcal{A}$ has the basis of covers $f^{-1}(\mathcal{S}(\varepsilon))\,(f \in C^*(\mathcal{A}X))$, and the proof of 3.2 shows that each of these is refined by a finite $\mathcal{A}$-partition $(\{f^{-1}(I_i)\}_i$, in the proof of 3.2).

3.8. THEOREM. *Let K be compact and basically disconnected, and let X be a subset of K which is a P-space in the relative topology. Then $\mathcal{A} = (\operatorname{clop} K) \cap X$ is a σ-field and $s\mathcal{A}X = \overline{X}^K$.*

Proof. Let $G \in \operatorname{clop} K$. Then

$$X - G \cap X = (K - G) \cap X.$$

Let $G_1, G_2, \ldots \in \operatorname{clop} K$, and let $G = \overline{\bigcup G_i}$. Since K is basically disconnected, $G \in \operatorname{clop} K$. Since X is a P-space, every F_σ is closed. Thus,

$$\bigcup (G_i \cap X) = \overline{\bigcup (G_i \cap X)}^X = \overline{(\bigcup G_i) \cap X}^X = \overline{\bigcup G_i}^K \cap X = G \cap X.$$

So $(\operatorname{clop} K) \cap X$ is a σ-field.

To show that $s\mathcal{A}X = \overline{X}^K$ is the same as showing that $p\mathcal{A}X = aK|X$ (a being the unique uniformity on K). Since K is Boolean, the finite $\operatorname{clop} K$-partitions form a basis for aK. If $\mathfrak{U}$ is one of these, then $\mathfrak{U} \cap X$ is a finite $\mathcal{A}$-partition, evidently in $p\mathcal{A}X$. On the other hand, by 3.7, the finite $\mathcal{A}$-partitions form a basis for $p\mathcal{A}X$. If $\{A_1, \ldots, A_n\}$ is one of these, then each $A_i = A_i' \cap X$ for $A_i' \in \operatorname{clop} K$. Then $\mathfrak{U} = \{A_1', \ldots, A_n', K - \bigcup_{i=1}^n A_i'\} \in aK$, and $\mathfrak{U} \cap X = \{A_1, \ldots, A_n\}$.

3.9. COROLLARY. *The spaces $s\mathcal{A}X$ are exactly the compact basically disconnected spaces which have dense P-subspaces.*

Thus, if K is the Samuel compactification of a measurable uniform space, the set of P-points of K is dense: call it X. And $K = s\mathcal{A}X$, with $\mathcal{A} = \operatorname{clop} K \cap X$. Evidently, if $K = s\mathcal{B}Y$, also, then $Y \subset X$ (because Y is a P-space) and $\mathcal{B}Y = \mathcal{A}X|Y$ (from 3.8). We shall see shortly that $\mathcal{A}X$ (as here) is complete, so that $\mathcal{A}X = \gamma\mathcal{B}Y$.

We now describe the completions $\gamma\mathcal{A}X$ by applying preceding theorems. (This is indirect, but avoids all annoying computation.) The first result shows $\gamma\mathcal{A}X$ is measurable using regularity of $C(\mathcal{A}X)$.

Recall that $\mu X \xrightarrow{f} \nu Y$ uniformly continuous implies the existence of a unique extension $\gamma\mu X \xrightarrow{f^\gamma} \gamma\nu Y$.

3.10. LEMMA. *Let $\mathcal{F}$ be a family of maps to uniform spaces which weak-generates μX. Then the family $\mathcal{F}^\gamma$ of extension weak-generates $\gamma\mu X$.*

Proof. Complete the range spaces for all $f \in \mathcal{F}$, and construct the uniform embedding h into the uniform product νP of these spaces. νP is complete, the $\mathcal{F}$-maps become the projections restricted to $h(X)$, $\gamma\mu X$ becomes $\overline{h(X)}$, and the $\mathcal{F}^\gamma$-maps become the projections on $\overline{h(X)}$. Evidently, these projections weak-generate $\nu P|\overline{h(X)} = \gamma\mu X$.

3.11. LEMMA. *$C(\mu X) \ni f \to f^\gamma \in C(\gamma\mu X)$ is onto, and an isomorphism of all existing pointwise operations.*

Proof. First, we have extensions f^γ: $\gamma\mu X \to \gamma\varrho R = \varrho R$; and if $g \in C(\gamma\mu X)$, evidently, $g = (g|X)^\gamma$. Second, that $f \to f^\gamma$ preserves pointwise operations is an easy consequence of density of X and continuity of the functions.

3.12. THEOREM. *$\gamma\mathcal{A}X$ is measurable; so $T\gamma\mathcal{A}X$ is a P-space.*

Proof. By 3.11, $C(\gamma\mathcal{A}X)$ is a regular algebra, and by 2.4 and 3.10, $C(\gamma\mathcal{A}X)$ weak-generates $\mathcal{A}X$. By 3.1, $\gamma\mathcal{A}X$ is measurable. That $T\gamma\mathcal{A}X$ is a P-space now follows from 2.5.

Recall that for any uniform space we have $s\gamma\mu X = s\mu X$, and $T\gamma\mu X \subset s\mu X$. The following is essentially a corollary of 3.5 and 3.10.

3.13. THEOREM. *$T\gamma\mathcal{A}X$ is the space P of all P-points in $s\mathcal{A}X$, and $\gamma\mathcal{A}$ is the measurable uniformity derived from the σ-field $(\operatorname{clop} s\mathcal{A}X) \cap P$.*

Proof. Let T be $T\gamma\mathcal{A}X$, $\mathcal{A}_T = (\operatorname{clop} s\mathcal{A}X) \cap T$ and $\mathcal{A}_P$ likewise. Since $\gamma\mathcal{A}X$ is measurable, with $s\gamma\mathcal{A}X = s\mathcal{A}X$, we have $\gamma\mathcal{A}X = \mathcal{A}_T T$, by 3.5. Now 3.12 says that $T \subset P$, and 3.9 says that $\mathcal{A}_P$ is a σ-field. Evidently, $\mathcal{A}_T = \mathcal{A}_P \cap T$, and hence $\mathcal{A}_T T \subset \mathcal{A}_P P$, as uniform subspaces. But a complete space is a proper dense subspace of no other uniform space; so $\mathcal{A}_T T = \mathcal{A}_P P$, as desired.

Various other descriptions of $T\gamma\mathcal{A}X$ can be derived. For example:

(1) *$T\gamma\mathcal{A}X$ is the G_δ-closure of X in $s\mathcal{A}X$* (the points p such that if G is a G_δ with $p \in G$, then G meets X). This has routine topological proof, using 3.13 and the open basis $\operatorname{clop} s\mathcal{A}X$.

(2) With 3.6 and the fact that the Stone space of the Boolean algebra $\mathcal{B}$ consists of the $\mathcal{B}$-ultrafilters with the hull-kernel topology, *$T\gamma\mathcal{A}X$ consists of the $\mathcal{B}$-ultrafilters with the countable intersection property* (for "cip" means P-point).

(3) *$T\gamma\mathcal{A}X$ is the largest subspace S of $s\mathcal{A}X$ for which $(\operatorname{clop} s\mathcal{A}X) \cap S$ is a σ-field* (countable supremum has to be union). Again, use 4.4 and this easy lemma (a sort of converse of 3.6): if K is Boolean, and $(\operatorname{clop} K) \cap S$ is a *σ-field*, then S is a P-space in the relative topology. When S is dense, this makes $S \subset P$.

We note explicity the criteria for completeness implicit in (2) above.

3.14. PROPOSITION. *These conditions on $\mathcal{A}X$ are equivalent.*

(a) *$\mathcal{A}X$ is complete.*

(b) *Each $\mathcal{A}$-ultrafilter with the countable intersection property is of the form $\{A \in \mathcal{A}: x \in A\}$ for (unique) $x \in X$.*

(c) *Each countably additive $\{0, 1\}$-valued measure on $\mathcal{A}$ which is not identically 0 is a point mass.*

Proof. The equivalence of (a) and (b) follows from (2), or it follows from the corresponding result in the more general class of "$\mathcal{M}$-fine" uniform spaces [4(e), § 8]. (See § 6, here.)

The equivalence of (b) and (c) follows from this. If m is a measure as in (c), then $\{A \in \mathcal{A}: m(A) = 1\}$ is an ultrafilter as in (b). If $\mathcal{F}$ is an ultrafilter as in (b), then $m(A) = 1$ iff $A \in \mathcal{F}$ defines a measure as in (c).

Completeness of Baire spaces is discussed in 6.13. Completeness of (analogously defined) *Borel* spaces is treated in [6].

With attention to computational detail, one can give a proof of 3.12 which is conceptually completely straightforward, like this.

For a general uniform space μX, and for $\mathcal{U} \in \mu$, set $\mathcal{U}^\gamma = \{\operatorname{int}\overline{U}: U \in \mathcal{U}\}$ (interior and closure in $\gamma\mu X$). Then $\mathcal{U}^\gamma \in \gamma\mu$, and if $\mathcal{B}$ is a basis for μ, $\mathcal{B}^\gamma \equiv \{\mathcal{U}^\gamma: \mathcal{U} \in \mathcal{B}\}$ is a basis for $\gamma\mu$. Now, given $\mathcal{A}X$, one proves that $\mathcal{A}^\gamma \equiv \{\operatorname{int}\overline{A}: A \in \mathcal{A}\}$ is a σ-field of subsets of $\gamma\mathcal{A}X$; and with $\mathcal{B}$ the defining basis for $\mathcal{A}X$, $\mathcal{B}^\gamma$ consists of the countable $\mathcal{A}^\gamma$-covers. Thus $\gamma\mathcal{A}X$ is measurable.

4. Measurable coreflection. We discuss the operator b, assigning to a separable uniform space μX a minimal measurable one, $b\mu X$.

For $\mathcal{S} \subset 2^s$, let $\sigma(\mathcal{S})$ be the least σ-field in 2^s containing $\mathcal{S}$. Given separable μX, $b\mu X$ is the measurable uniform space associated with $\sigma(\operatorname{coz} C(\mu X))$.

4.1. Proposition. *$b\mu$ is the coarsest measurable uniformity on X which is finer than μ.*

Proof. Being separable, μ has a basis of (some, probably not all) countable $\operatorname{coz} C(\mu X)$-covers. Each such cover is a basic $b\mu$-cover (since $\operatorname{coz} C(\mu X) \subset \sigma(\operatorname{coz} C(\mu X))$, so $\mu \subset b\mu$.

If the measurable uniformity of $\mathcal{A}X$ is finer than μ, then $C(\mathcal{A}X) \supset C(\mu X)$, so $\mathcal{A} = \operatorname{coz} C(\mathcal{A}X) \supset \operatorname{coz} C(\mu X)$, and hence $\mathcal{A} \supset \sigma(\operatorname{coz} C(\mu X))$. So the uniformity of $\mathcal{A}X$ is finer than $b\mu$.

Note that if μX is a Shirota space (i.e., of the form $e\alpha X$) then $C(\mu X) = C(X)$, $\sigma(\operatorname{coz} C(\mu X))$ consists of the Baire sets of $T\mu X$, and $b\mu X$ is what we called a Baire space in § 1. But $b\mu X$ can be Baire for various μ's; see § 6.

The first thing we do is consider the functorial nature of the operator b.

Now, a subcategory $\mathcal{B}$ of a category $\mathcal{U}$ is *coreflective* if to each object U of $\mathcal{U}$ is associated an object bU of $\mathcal{B}$ and a map $bU \xrightarrow{i} U$ such that any map $B \xrightarrow{f} U$ (B in $\mathcal{B}$) factors, $f = i \circ g$, for *unique* $B \xrightarrow{g} bU$. This readily implies that maps $U_1 \xrightarrow{f} U_2$ have unique "lifts" $bU_1 \xrightarrow{bf} bU_2$ (with $f \circ i_1 = i_2 \circ bf$). Thus b is functorial; we call it the *coreflection*.

We are in this situation with b the "measurable" operator.

4.2. Theorem. *b coreflects the category of separable uniform spaces onto its subcategory of measurable uniform spaces.*

We prove this shortly.

That the category of measurable uniform spaces is coreflective can be derived from general theorems of Kennison and or Isbell [15], [12(a)], [9], by showing that measurability is additive and divisible. Resulting from this, and an analysis of the proofs (as explained to me by M. D. Rice), is 4.1. While here, 4.1 is essentially all there is to say about $b\mu$ (though we shall go on at some length), in other cases the analogous description is not incisive; see § 6, here, for an example.

To return to 4.2, of course, the required map $b\mu X \to \mu X$ is the identity 1_x. Uniform continuity is asserted by 4.1. The factorization property is the following.

4.3. Proposition. *If $\mathcal{A}Y \xrightarrow{f} \mu X$ is uniformly continuous, with $\mathcal{A}Y$ measurable, then $\mathcal{A}Y \xrightarrow{f} b\mu X$ is uniformly continuous.*

To prove this, it is required (by 2.2) that $f^{-1}(E) \in \mathcal{A}$ whenever $E \in \sigma(\operatorname{coz} C(\mu X))$. We shall proceed by an easy transfinite induction, but for this it is necessary that we indicate a Baire classification for the sets in $\sigma(\operatorname{coz} C(\mu X))$. It will be useful to do this in somewhat more generality. Henceforth, A will denote an arbitrary vector lattice of functions: $X \to R$, with $1 \in A$. We shall consider $\operatorname{coz} A$ and its derived σ-field.

4.4. Lemma. *$\operatorname{coz} A$ is closed under finite intersection. If $f \in A$, then $f^{-1}(a, b) \in \operatorname{coz} A$.*

Proof. $\bigcap_{i=1}^{n} \operatorname{coz} f_i = \operatorname{coz}(|f_1| \wedge \ldots \wedge |f_n|)$.

$$f^{-1}(a, +\infty) = \operatorname{coz}((f-a) \vee 0), \quad f^{-1}(-\infty, b) = \operatorname{coz}((f-b) \wedge 0),$$

and

$$f^{-1}(a, b) = f^{-1}(a, +\infty) \cap f^{-1}(-\infty, b).$$

Let $\sigma_{-1} = \operatorname{coz} A$, $\sigma_0 = (\operatorname{coz} A)_\sigma$ (all countable unions), $\sigma_1 = (\sigma_0)_\delta$ (all countable intersections of σ_0-sets), etc.; at a limit ordinal β, set $\sigma_\beta = \bigcup_{\alpha<\beta} \sigma_\alpha$, and then $\sigma_{\beta+1} = (\sigma_\beta)_\delta$. Let ω_1 be the first uncountable ordinal.

4.5. Proposition. $\sigma(\operatorname{coz} A) = \sigma_{\omega_1}$.

(For Baire sets, this is well known.)

Proof. Clearly, $\sigma_{\omega_1} \subset \sigma(\operatorname{coz} A)$, and $\sigma_{\omega_1} = (\sigma_{\omega_1})_\sigma$. So it suffices that σ_{ω_1} be closed under complementation. We prove this by transfinite induction, by showing (*) if $E \in \sigma_\alpha$ then $X - E \in \sigma_{\alpha+2}$.

First, if $f \in A$ then $X - \operatorname{coz} f = \bigcap_n f^{-1}(-1/n, 1/n) \in (\operatorname{coz} A)_\delta \subset (\sigma_0)_\delta$. (Here we used 4.2.) Next, let $\beta \geqslant 0$, let (*) be true for all $\alpha < \beta$, and let $E \in \sigma_\beta$. If β is limit, then $E \in \sigma_\alpha$ for an $\alpha < \beta$, and $X - E \in \sigma_{\alpha+2} \subset \sigma_\beta$. If β is not limit, then σ_β is either $(\sigma_{\beta-1})_\sigma$ or $(\sigma_{\beta-1})_\delta$. If the first case,

$E = \bigcup E_n$ with $E_n \in \sigma_{\beta-1}$, so $X - E = \bigcap (X - E_n) \in (\sigma_{(\beta-1)+2})_\delta = (\sigma_{\beta+1})_\delta = \sigma_{\beta+1}$. The second case is similar.

Proof of 4.3. Taking $A = C(\mu X)$, 4.5 shows that $\sigma(\operatorname{coz} C(\mu X)) = \bigcup_{\alpha<\omega_1} \sigma_\alpha$, where $\sigma_{-1} = \sigma_0$ because $C(\mu X)$ is uniformly closed.

For $\operatorname{coz} g \in \sigma_0$, $f^{-1}(\operatorname{coz} g) = \operatorname{coz}(g \circ f) \in \operatorname{coz} C(\mathcal{A}Y) = \mathcal{A}$, because f, and $f \circ g$, are uniformly continuous. For $E_1, E_2, \ldots \in \sigma_0$, $f^{-1}(\bigcap E_i) = \bigcap f^{-1}(E_i)$; thus $f^{-1}(E) \in \mathcal{A}$ if $E \in \sigma_1$. Etc., by induction.

We consider the topology of $b\mu X$. Of course, for any measurable $\mathcal{A}X$, $T\mathcal{A}X$ is a P-space, with the σ-field $\mathcal{A}$ being an open basis. For $b\mu$, this can be sharpened.

4.6. Theorem. *$Tb\mu X$ carries the coarsest P-space topology on X which contains $\operatorname{coz} C(\mu X)$; and the complementary family $\mathfrak{Z}(C(\mu X))$ is an open basis.*

Proof. Of course, $Tb\mu$ is a P-space topology, and it contains $\operatorname{coz} C(\mu X)$ because $C(\mu X) \subset C(Tb\mu X)$.

If $\mathcal{T}$ is any P-space topology on X, then $\operatorname{coz} C(\mathcal{T}X)$ is a σ-field (because any $\operatorname{coz} f$ is closed, as well as open, hence a zero-set). Thus, if $\mathcal{T} \supset \operatorname{coz} C(\mu X)$ then $\operatorname{coz} C(\mathcal{T}X) \supset \sigma(\operatorname{coz} C(\mu X))$. So $\mathcal{T}$ contains the basis for $Tb\mu$, and $\mathcal{T} \supset Tb\mu$.

To show that $\mathfrak{Z}(C(\mu X))$ is an open basis, we show by induction, using 4.5, that whenever $p \in E \in \sigma(\operatorname{coz} C(\mu X))$ there is $Z \in \mathfrak{Z}(C(\mu X))$ with $p \in Z \subset E$. If $E \in \sigma_0 = \operatorname{coz} C(\mu X)$, say $E = \operatorname{coz} f$, then with $0 < a < |f(p)|$, $Z = |f|^{-1}[a, +\infty)$ works. ($Z \in \mathfrak{Z}(C(\mu X))$ by 4.4.) Suppose $E \in \sigma_\beta$, and the assertion holds for all sets in σ_α, $\alpha < \beta$. This is trivial, if β is limit, and if $\sigma_\beta = (\sigma_{\beta-1})_\sigma$. In case $\sigma_\beta = (\sigma_{\beta-1})_\delta$, then $E = \bigcap E_n$ with $E_n \in \sigma_{\beta-1}$. Since $p \in E_n$ for each n, there is $Z_n \in \mathfrak{Z}(C(\mu X))$ with $p \in Z_n \subset E_n$. Let $Z = \bigcap_n Z_n$, so $Z \in \mathfrak{Z}(C(\mu X))_\delta = \mathfrak{Z}(C(\mu X))$.

The topology of a Baire space $be\alpha X$ will be called the *Baire topology* associated with the topological space X. This topology has been studied by Lorch [16] (and called the *ι-topology*).

4.7. Corollary. *$Tb\mu X$ carries the coarsest P-space topology on X which is finer than $T\mu$. Hence, this topology is the Baire topology associated with $T\mu X$, and in particular, if $T\mu X = T\nu X$ then $Tb\mu X = Tb\nu X$.*

Proof. $b\mu \supset \mu$, hence $Tb\mu \supset T\mu$. Use 4.6.

We return to descriptions of $b\mu$.

By 2.4, measurable $\mathcal{A}X$ is weak generated by $C(\mathcal{A}X)$. Thus, describing $b\mu X$ is essentially the same as describing $C(b\mu X)$.

At little expense, we proceed more generally with a vector lattice $A\colon X \to R$, with $1 \in A$. The results which follow apply to describe a $C(b\mu X)$ by taking $A = C(\mu X)$, whence $\sigma(\operatorname{coz} A)$ generates $b\mu$.

4.8. Corollary. *$A \subset C(\sigma(\operatorname{coz} A)X)$.*

Proof. We are to show that $f^{-1}(E) \in \sigma(\operatorname{coz} A)$, if $f \in A$ and $E \in \mathcal{B}aR$. By 4.4, it is true if $E = (a, b)$. If G is open, $G = \bigcup I_n$, with each I_n an (a, b), and $f^{-1}(G) \in \sigma(\operatorname{coz} A)$. Now proceed by transfinite induction, using 4.5 with $A = C(R)$ (so $\mathcal{B}aR = \bigcup_{\alpha<\omega_1} \sigma_\alpha$).

We now have an analog of 4.1.

4.9. Theorem. *$\sigma(\operatorname{coz} A)X$ carries the coarsest measurably uniformity on X which is finer than the A-weak uniformity, and $C(\sigma(\operatorname{coz} A)X)$ is the smallest uniformly closed and regular algebra of functions on X which contains A.*

Proof. Let μ_A be the A-weak uniformity. By 4.8, the uniformity of $\sigma(\operatorname{coz} A)X$ is finer than μ_A. If $\mathcal{B}$ is a σ-field, and the uniformity of $\mathcal{B}X$ is finer than μ_A, then $C(\mathcal{B}X) \supset C(\mu_A X) \supset A$, hence $\operatorname{coz} C(\mathcal{B}X) \supset \operatorname{coz} C(\mu_A X) \supset \operatorname{coz} A$. But by 2.5, $\operatorname{coz} C(\mathcal{B}X) = \mathcal{B}$. Thus $\mathcal{B} \supset \sigma(\operatorname{coz} A)$, and therefore the uniformity of $\mathcal{B}X$ is finer than that of $\sigma(\operatorname{coz} A)X$.

By 3.1 and 4.8, $C(\sigma(\operatorname{coz} A)X)$ is a uniformly closed and regular algebra containing A. Let B_1 and B_2 be uniformly closed regular algebras, and let μ_{B_1} and μ_{B_2} be the measurable uniformities which correspond via 3.1. The proof of 3.1 shows that $\mu_{B_i}X = (\operatorname{coz} B_i)X$; hence $B_1 \supset B_2$ iff μ_{B_1} is finer than μ_{B_2}. The desired result now follows.

This theorem does not yield an explicit construction of $C(\sigma(\operatorname{coz} A)X)$ from A. We describe such a construction, generalizing the well known generation of Baire measurable functions using pointwise limits. Some notation is helpful.

Given $\mathcal{F}\colon X \to R$, let $p\mathcal{F}$ be those $f\colon X \to R$ for which there are $f_1, f_2, \ldots \in \mathcal{F}$ with $f_n \to f$ (pointwise). Let $\mathcal{R}_0 = A$, $\mathcal{R}_1 = p\mathcal{R}_0, \ldots, \mathcal{R}_\beta = p \bigcup_{\alpha<\beta} \mathcal{R}_\alpha, \ldots$ Set $\mathcal{R} = \bigcap \{\mathcal{F}\colon\ \mathcal{F} \supset A,\ p\mathcal{F} = \mathcal{F}\}$. Evidently, we have $\mathcal{R} = \bigcup_{\alpha<\omega_1} \mathcal{R}_\alpha$.

4.10. Theorem. *$C(\sigma(\operatorname{coz} A)X)$ is the smallest family of functions on X which contains A, and is closed under formation of pointwise limits (that is, $\mathcal{R}$).*

Before the proof, we interject a corollary of 4.10 and 4.6.

4.11 Theorem. *With $A = C(\mu X)$: $Tb\mu$ is the weak topology generated by $\mathcal{R}_1$.*

Proof. By 4.10, the $\mathcal{R}_1$-topology $\subset Tb\mu$. So, it suffices to show that the sets in $\mathfrak{Z}(C(\mu X))$ are open in the $\mathcal{R}_1$-topology, and apply 4.6.

For $f \in C(\mu X)$, $1 - n(|f| \wedge 1/n) \to \chi_{Z(f)}$. Thus $\chi_{Z(f)} \in \mathcal{R}_1$, and $Z(f)$ is $\mathcal{R}_1$-open.

4.11 is perhaps surprising, since $\mathcal{R}_1$ is really quite far from $\mathcal{R}$, e.g., in case $\mu X = \varrho R$. Then, $\mathcal{R}_\alpha \not\subseteq \mathcal{R}_\beta$ whenever $\alpha < \beta$, a result of Lebesgue [18]. Actually, 4.11 is related to another theorem of Lebesgue, that for $\mu X = \varrho R$, $f \in \mathcal{R}_1$ iff $f^{-1}(G)$ is F_σ, for open G. See 6.4.

We prove 4.10. Half is immediate from the following.

4.12. *For any σ-field* $\mathcal{A}$, $pC(\mathcal{A}X) = C(\mathcal{A}X)$. *Thus* $\mathcal{R} \subset C(\sigma(\operatorname{coz} A)X)$.

Proof. This is easy, and well known. See [11, 11.14].

The other half of 4.10 requires more work.

4.13. $\mathcal{R}$ *is a uniformly closed vector lattice.*

Proof. Since $p\mathcal{R} = \mathcal{R}$, $\mathcal{R}$ is uniformly closed. That $\mathcal{R}$ is a vector lattice follows by an easy transfinite induction using the $\mathcal{R}_\alpha$'s; one starts with A, and uses the lemma: if $\mathcal{F}$ is a vector lattice so is $p\mathcal{F}$.

4.14. $\sigma(\operatorname{coz} A) \subset \{E\colon \chi_E \in \mathcal{R}\} = \operatorname{coz}\mathcal{R}$.

Proof. First, $\{E\colon \chi_E \in \mathcal{R}\} \subset \operatorname{coz}\mathcal{R}$, obviously. For the opposite conclusion: if $f \in \mathcal{R}$, then $f_n = n(|f| \wedge 1/n) \in \mathcal{R}$ (by 4.13); and $f_n \to \chi_{\operatorname{coz} f}$.

So call the collection σ. Evidently, $\operatorname{coz} A \subset \sigma$. And σ is a σ-field: If $E \in \sigma$ then $\chi_E \in \mathcal{R}$, so $1 - \chi_E \in \mathcal{R}$ (by 4.13) but $1 - \chi_E = \chi_{X-E}$. Thus $X - E \in \sigma$. If $E_1, E_2, \ldots \in \sigma$, then all $\chi_{E_i} \in \mathcal{R}$, and hence $f = \sum 2^{-i}\chi_{E_i} \in \mathcal{R}$ (by 4.13). And $\bigcup E_i = \operatorname{coz} f \in \sigma$.

Finally, consider $C(\sigma(\operatorname{coz} A)X) = C$. Evidently, $pC^* \supset C$ (because $(f\wedge n)\vee(-n) \to f$), so it suffices that $C^* \subset \mathcal{R}$. Let $f \in C^*$. By 3.2, there is a sequence $s_n \to f$, each s_n being a linear combination of functions χ_E, $E \in \sigma(\operatorname{coz} A)$. By 4.14 and 4.13, each $s_n \in \mathcal{R}$. Hence $f \in \mathcal{R}$.

4.15. Remarks. The above proof of 4.10 is based on the sketch in [11, 11.41] for continuous, and Baire, functions.

Mauldin [18] has recently discussed the Baire system derived from a vector lattice, and his results imply 4.10. In § 6, we discuss further Mauldin's results and the class $\mathcal{R}_1$.

5. b on subspaces and completion. We shall show that b preserves subspaces. (Products are treated briefly and incompletely at the end of the section.) It will follow quickly that $\gamma b\mu X$ is a subspace of $b\gamma\mu X$, if we know the latter is complete. It is, but this is a difficult theorem whose proof we postpone to § 6. Assuming this, we describe the subspace, and show that $b\gamma = \gamma b$ exactly on spaces which are G_δ-dense in their completion.

Let $\mu X|Y$ stand for the uniform space obtained by relativizing μ to $Y (\subset X)$; the uniformity consists of the covers $\{\mathfrak{U} \cap Y\colon \mathfrak{U} \in \mu\}$ (where $\mathcal{S} \cap Y \equiv \{S \cap Y\colon S \in \mathcal{S}\}$, for any $\mathcal{S} \subset 2^X$). Comparing $b(\mu X|Y)$ with $b\mu X|Y$ is the same as comparing $\sigma(\operatorname{coz} C(\mu X|Y))$ with $\sigma(\operatorname{coz} C(\mu X)) \cap Y$.

5.1. Lemma. $\operatorname{coz} C(\mu X|Y) = (\operatorname{coz} C(\mu X)) \cap Y$.

5.2. Lemma. $\sigma(\operatorname{coz} C(\mu X|Y)) = \sigma(\operatorname{coz} C(\mu X)) \cap Y$.

5.3. Theorem. $b(\mu X|Y) = b\mu X|Y$.

Proof of 5.1. We have $C(\mu X|Y) \supset C(\mu X)|Y$ (generally without equality), so that $\operatorname{coz} C(\mu X|Y) \supset (\operatorname{coz} C(\mu X)) \cap Y$. For the opposite inclusion, if $E \in \operatorname{coz} C(\mu X|Y)$, then (by truncating) $E = \operatorname{coz} f$ for $f \in C^*(\mu X|Y)$. By Katetov's theorem [13], [12(b)], f extends to $f' \in C(\mu X)$. So $\operatorname{coz} f = (\operatorname{coz} f') \cap Y$.

Proof of 5.2. In general, $\sigma(\mathcal{S} \cap Y) = \sigma(\mathcal{S}) \cap Y$, by [7(a), p. 25]. Apply this and 5.1.

Proof of 5.3. If $\mathfrak{U}$ is $b(\mu X|Y)$-basic, then $\mathfrak{U}$ is a countable $\sigma(\operatorname{coz} C(\mu X(Y)))$-cover, say $\{E_n\}_n$. By 5.2, each $E_n = E'_n \cap Y$ for $E'_n \in \sigma(\operatorname{coz} C(\mu X))$. Let $E = X - \bigcup_n E'_n$, and $\mathfrak{U}' = \{E'_n\}_n \cup \{E\}$. So $\mathfrak{U}' \cap Y = \mathfrak{U} \in b\mu X|Y$. Thus the uniformity of $b(\mu X|Y)$ is contained in that of $b\mu X|Y$.

For the opposite inclusion: If $\mathfrak{U}$ is $b\mu X|Y$-basic, then $\mathfrak{U} = \{E_n\}_n$ with each $E_n \in \sigma(\operatorname{coz} C(\mu X))$. By 5.2, each $E_n \cap Y \in \sigma(\operatorname{coz} C(\mu X|Y))$, so $\mathfrak{U} \cap Y \in b(\mu X|Y)$.

(A topological version of the equality in 5.3 would read "$\mathcal{B}aY = (\mathcal{B}aX) \cap Y$." This generally fails, because the topological version of 5.1, "$\operatorname{coz} C(Y) = (\operatorname{coz} C(X)) \cap Y$," generally fails. An uncountable discrete Y in its one-point compactification X is an example for each. See [1(a)] and [5].)

Since μX is a subspace of $\gamma\mu X$, it is immediate that $b\mu X$ is a subspace of $b\gamma\mu X$, from 5.3.

5.4. Theorem. *If* νY *is complete, then* $b\nu Y$ *is complete.*

As mentioned, we prove this later. That 5.4 is at least somewhat subtle can be seen from the compact space $2^{\aleph_1}$. By 5.4, the Baire space $ba2^{\aleph_1}$ is complete. But, the Borel space $(\mathcal{B} \circ 2^{\aleph_1})2^{\aleph_1}$ is not [6].

5.5. Corollary. $T\gamma b\mu X$ *is the closure of* X *in* $Tb\gamma\mu X$, *and* $\gamma b\mu$ *is the relativization of* $b\gamma\mu$.

Proof. Completion is obtained by closing in any complete superspace. We use $b\gamma\mu X$, by 5.4 and the remark preceding 5.4.

5.5 is to be read with care: the closure refers to the topology $Tb\gamma\mu$. This has basis the σ-field $\sigma(\operatorname{coz} C(\gamma\mu X))$, or by 4.6, $\mathfrak{Z}(C(\gamma\mu X))$. Using this, the description in 5.5 can be simplified. When $A \subset B$ (topological spaces), the G_δ-closure of A in B is $\{p \in B\colon$ each G_δ around p hits $A\}$. (This operation seems to have been studied first by Mrówka.)

5.6. Corollary. *The set on which* $\gamma b\mu X$ *lives is the* G_δ*-closure of* X *in* $\gamma\mu X$.

Proof. From the preceding remarks, all there is to show is this: for $p \in \gamma\mu X$, each G_δ around p hits X iff each $Z \in \mathfrak{Z}(C(\gamma\mu X))$ around p hits X. Since Z's are G_δ's, the implication ⇒ is immediate. Conversely, if $G = \bigcap_n G_n$ is a G_δ around p, then $p \in$ each G_n, and there is $f_n \in C(\gamma\mu X)$

with $f_n(p) = 0$ and $f_n(X - G_n) = 1$ [12(b)]. Then $p \in Z = \bigcap Z(f_n) \subset G$, and $Z \in \mathfrak{Z}(C(\gamma\mu X))$. Since Z hits X, so does C.

A is G_δ-dense in B if the G_δ-closure of A is B.

5.7. Theorem. *$\gamma b\mu X = b\gamma\mu X$ iff X is G_δ-dense in $\gamma\mu X$.*

Proof. By 5.5, $\gamma b\mu X \subset b\gamma\mu X$ (as a uniform subspace); so equality holds iff $\gamma b\mu X$ lives on all of $\gamma\mu X$. G_δ-density is the exact criterion, by 5.6.

(A similar result holds for the $\mathcal{M}$-fine operator m [4(e), § 6]; that result is more difficult.)

We consider briefly products, and confine attention to products of just two spaces. The main questions (which we do not resolve) are these: When is the product of two measurable spaces measurable? When does $b(\mu X \times \nu Y) = b\mu X \times b\nu Y$ hold? The questions do not seem totally inaccessible, based on the related treatments of fine spaces in [12(b), Ch. VII], [4(d)] and the references given there; but we settle now for a few comments.

The uniform space μX is said to "admit $\aleph_0$" if each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \mu$ has a common refinement $\mathcal{U} \in \mu$. The condition is quite restrictive, implying that $T\mu X$ is P; indeed, from [12(b), Ch. VII], it can be shown that a fine space αX admits $\aleph_0$ iff X is P, and that a Shirota space $e\alpha X$ admits $\aleph_0$ iff X is pseudo-$\aleph_1$-compact P. (X is pseudo-$\aleph_1$-compact means αX is separable.)

Now, a measurable space is subfine (because measurable $\Rightarrow$ $\mathcal{M}$-fine (§ 6) and separable $\mathcal{M}$-fine $\Rightarrow$ subfine [4(e)]), and [12(b), Ch. VII, §§ 22 & 28] imply that a product of two separable subfine spaces is subfine iff each admits $\aleph_0$. Hence:

5.8. *If $\mathcal{A}X \times \mathcal{B}Y$ is measurable, then each of $\mathcal{A}X$ and $\mathcal{B}Y$ admits $\aleph_0$.*

I suspect the converse holds. In any event, we need to know when a measurable space $\mathcal{A}X$ admits $\aleph_0$. I suspect this occurs if $T\mathcal{A}X$ is pseudo-$\aleph_1$-compact (but the converse fails, by 5.9(c)).

5.9. Examples. (a) Let $\mathcal{A} = 2^D$, so $\mathcal{A}D = e\alpha D$ (D discrete). Then, $\mathcal{A}D \times \mathcal{A}D$ is measurable iff its uniformity is $e\alpha(D \times D)$, which occurs if $|D| \leqslant \aleph_0$ (i.e., iff discrete D is pseudo-$\aleph_1$-compact). This can be derived from [4(d)].

(b) Let X be the uncountable discrete set D, with an additional point adjoined whose neighborhoods have countable complement, and let $\mathcal{A}X = \alpha X$ (the latter being measurable because X is a Lindelöf P-space). Then $\mathcal{A}X \times \mathcal{A}X = \alpha(X \times X)$, and is measurable (because $X \times X$ is Lindelöf P). Again, $T\mathcal{A}X$ is pseudo-$\aleph_1$-compact.

(c) Let $\mathcal{A}X$ be as in (b), and consider $\mathcal{A}X|D = \mathcal{B}D$, where $\mathcal{B}$ is the σ-field of countable and cocountable subsets of D. Then $\mathcal{B}D \times \mathcal{B}D$ $= (\mathcal{A}X \times \mathcal{A}X)|D \times D$, and is measurable (as a subspace of a measurable space). Here, $T\mathcal{B}D$ is discrete and uncountable, hence not pseudo-$\aleph_1$-compact.

6. $\mathcal{M}$-fine spaces. μX is called *$\mathcal{M}$-fine* (or, *fine relative to the class $\mathcal{M}$ of metric spaces*) if $\mu X \xrightarrow{f} \varrho M$ uniformly continuous, with ϱM metric, implies uniform continuity of $\mu X \xrightarrow{f} \alpha T\varrho M$ (where α denotes the fine uniformity). Separable $\mathcal{M}$-fine spaces are studied in detail in [4(e)]. We shall indicate here the facts necessary to establish the connection with measurable spaces.

The $\mathcal{M}$-fine spaces form a coreflective subcategory of uniform spaces, and in separable spaces, the coreflecting functor m has this description: $m\mu$ has basis of all countable $\operatorname{coz} C(\mu X)$-covers. It results that $Tm\mu = T\mu$, and that $m\mu X \xrightarrow{f} \varrho M$ (metric) is uniformly continuous iff $f^{-1}(G) \in \operatorname{coz} C(\mu X)$ for all open G.

Evidently, $\mu \subset m\mu \subset b\mu = bm\mu$. Comparing the criterion in 2.2 for uniform continuity of a map $b\mu \xrightarrow{f} \varrho M$ indicates the close connection of, say, $C(m\mu X)$ with the functions of the first Baire class in $C(b\mu X)$. We discuss this more carefully below.

6.1. Theorem. [4(e), 10.1] *The measurable spaces are exactly the separable hereditarily $\mathcal{M}$-fine ones.*

Part of the proof can be indicated. From 5.3, measurability is hereditary, so we show that a measurable space is $\mathcal{M}$-fine. For $\mathcal{A}X$ to be $\mathcal{M}$-fine requires $m\mathcal{A}X = \mathcal{A}X$. Since $\operatorname{coz} C(\mathcal{A}X) = \mathcal{A}$, this is clear. (The result also follows from $m\mu \subset b\mu$.)

Another characterization of measurable spaces among $\mathcal{M}$-fine ones comes about as follows. In [4(e), § 6] is proved this analogue of 3.1: if μX is $\mathcal{M}$-fine, then $C(\mu X)$ is a uniformly closed algebra, closed under inversion (see 3.3), and $\mu X \to C(\mu X)$ is a one-to-one correspondence between all separable $\mathcal{M}$-fine uniformities on X and the point-separating algebras of functions with the properties mentioned (but here, $C(\mu X)$ does not weak-generate μX, in contrast to the situation for measurable spaces). Since a regular ring is closed under inversion (3.3), and since measurable spaces are $\mathcal{M}$-fine, we find:

6.2. Theorem. *Let μX be separable and $\mathcal{M}$-fine. Then, μX is measurable iff $C(\mu X)$ is regular.*

What's involved in this proof of 6.2 is essentially just the observation that regularity of $C(\mu X)$ forces $\operatorname{coz} C(\mu X)$ to be a σ-field (see the proof of 3.1), so $\mu = m\mu = b\mu$.

It is shown in [4(e), § 5] and [5] that uniformizable X has a unique compatible $\mathcal{M}$-fine uniformity iff X is either almost compact or Lindelöf.

(Almost compact means a unique uniformity, or, of each pair of disjoint zero-sets, one is compact.) It follows that for such spaces, $b\mu X = bX$ (the Baire space) for each compatible μ. The exact topological condition for this uniqueness comes quickly from a result of Frolík [1 (a)].

Call uniformizable X almost Lindelöf if of each pair of disjoint zero-sets, one is Lindelöf [5, 4.3].

6.3. THEOREM. *For uniformizable X, these are equivalent.*

(a) *X is almost Lindelöf.*

(b) *$b\mu X = bX$ for each compatible μ.*

(c) *If A is any vector lattice of continuous functions on X which generates X's topology, then the smallest uniformly closed and regular algebra containing A is the algebra of all Baire functions.*

Proof. (b) is clearly equivalent to this: $\sigma(\operatorname{coz} C(\mu X)) = \mathcal{B}aX$ for each compatible μ. It is not hard to prove that this is equivalent to: $\mathcal{B}aX$ is the smallest σ-field containing a base for the topology in X. Frolík [1(a)] has shown the equivalence of this with (a).

The equivalence of (b) and (c) follows from 4.9.

[1(a)] contains some other equivalences of 6.3(a), involving functions of the first Baire class.

We turn to a more careful description of $C(m\mu X)$ in $C(b\mu X)$.

There are two explicit constructions of $C(m\mu X)$ from $C(\mu X)$, in [4(c)] and [4(e), 6.3]. The former relates closely to the first Baire class, via recent results of Mauldin [18]. It is convenient to describe this in slightly more generality. Let A be a uniformly closed vector lattice of functions on X, so that $\operatorname{coz} A = (\operatorname{coz} A)_\sigma$. Let $\mathcal{R}_1(A)$ be the first Baire class derived from A (i.e., pA, in the notation of § 4), let $LS(A)$ be those f which are limits of increasing sequences from A, and when $\mathcal{F}$ is a class of functions, let $\mathrm{ul}\mathcal{F}$ be the "uniform closure" of $\mathcal{F}$. Let mAX and bAX be respectively the $\mathcal{M}$-fine and measurable coreflections of the A-weak uniformity on X. So, from § 4, $C(bAX)$ is the Baire system of functions derived from A; and $A \subset C(mAX) \subset \mathcal{R}_1(A) \subset C(bAX)$. Let $\mathfrak{Z}(A)_\sigma$ stand for the collection of countable unions of the zero-sets from A. Since

$$\operatorname{coz} f = \bigcup_n \{x\colon |f(x)| \geqslant 1/n\}, \qquad \operatorname{coz} A \subset \mathfrak{Z}(A)_\sigma \ \big(\subset \sigma(\operatorname{coz} A)\big).$$

6.4. THEOREM. (a) *$f \in C(mAX)$ (respectively, $\mathcal{R}_1(A)$, $C(bAX)$) iff for each open $G \subset R$, $f^{-1}(G) \in \operatorname{coz} A$ (respectively, $\mathfrak{Z}(A)_\sigma$, $\sigma(\operatorname{coz} A)$).*

(b) *$f \in \big(LS(A)\big)^*$ iff f is bounded and $\{x\colon f(x) < a\} \in (\operatorname{coz} A)_\delta$ for each $a \in R$.*

(c) *If $f \in C(mAX)$, and f is bounded below, then $f \in LS(A)$. Since $f = (f \vee 0) - \big((-f) \vee 0\big)$, $C(mAX) \subset LS(A) - LS(A)$.*

(d) $\mathcal{R}_1(A) = \mathrm{ul}\big(LS(A) - LS(A)\big)$.

(b), (d), and the assertion about $\mathcal{R}_1(A)$ in (a), are due to Mauldin [18]. Note that the latter generalizes the Lebesgue theorem for $A = C(R)$, whence $\mathfrak{Z}(A)_\sigma$ is the collection of F_σ's. ([18] contains the more general result describing $\mathcal{R}_a(A)$.) The rest of (a) is from [4(e)], and § 2, here (c) is from [4(d)].

In this connection, it would seem at least of passing interest to consider the function algebra $\mathcal{R}_1 \cap C(T\mu X)$ derived from $C(\mu X)$. This is inverse-closed, since both $\mathcal{R}_1$ and $C(T\mu X)$ are, and so there is an associated $\mathcal{M}$-fine space, say $m_0\mu X$, with $Tm_0\mu = T\mu$. The operator m_0 is probably a coreflection. Whether this process is of serious interest, I don't know. (Also, it can be shown that for each countable ordinal $a > 0$, $\mathcal{R}_a$ is inverse-closed, so there is related $\mathcal{M}$-fine space $m_a\mu X$ — with $Tm_a\mu \neq T\mu$, usually. Evidently, $b\mu = \bigcup_a m_a\mu$.)

We consider for a moment the form of the definition of $\mathcal{M}$-fine spaces. What is given is a class $\mathcal{M}$ of spaces (metric) and an operator O (here, aT) defined initially just on $\mathcal{M}$, and behaving like a coreflection in that there is a map $OM \xrightarrow{i_M} M$, say such that each map $M_1 \xrightarrow{f} M_2$ has unique lift, $OM_1 \xrightarrow{Of} OM_2$ (such that $i_{M_2} \circ Of = f \circ i_{M_1}$). In this rough generality, call a space μX (separable, if you will) an $\mathcal{M}$-O space if $\mu X \xrightarrow{f} M \in \mathcal{M}$ implies unique factorization $f = g \circ i_M$ ($\mu X \xrightarrow{g} OM$). Then (with a little attention to detail), the $\mathcal{M}$-O spaces form a coreflective subcategory, whose functor, say c, extends O over the category $\mathcal{U}$ of uniform spaces, or separable uniform spaces.

Trivially, if c is any coreflection on $\mathcal{U}$, then c (or its range, $c\mathcal{U}$, the associated coreflective subcategory) can be given this form: $c\mathcal{U}$ consists exactly of the $\mathcal{U}$-c spaces.

Now, this trivial representation is uninteresting, but in most of the few special cases which have been studied, non-trivial and useful representations can be found. For example, for Isbell's subfine coreflection l, $l\mathcal{U} = \gamma\mathcal{M} - aT$ (complete-$\mathcal{M}$-fine), when $\mathcal{U}$ is separable spaces, and where $\gamma\mathcal{M}$ denotes complete metric spaces. This result is essentially due to Isbell, Ginsberg, and Corson, is explicitly noted in [4(e), § 9], and is used to derive much of what (little) is known about l [12(b), Ch. VII].

We shall represent b in this way.

6.5. THEOREM. *For separable μX, these are equivalent.*

(a) *μX is measurable.*

(b) *$\mu X \xrightarrow{f} \varrho M$ uniformly continuous, with ϱM (separable) metric, implies that $\mu X \xrightarrow{f} b\varrho M$ is uniformly continuous.*

(c) *$\mu X \xrightarrow{f} \varrho R$ uniformly continuous implies that $\mu X \xrightarrow{f} b\varrho R$ is uniformly continuous (R being the reals).*

Note that (b) implies that μX is $\mathcal{M}$-fine: $b\varrho M$ is finer than $aT\varrho M$, so uniform continuity of $\mu X \xrightarrow{f} aT\varrho M$ follows.

The reduction in (c), to measurable $= \mathcal{M}$-O, with $\mathcal{M}$ the singleton $\{\varrho R\}$, seems unusually drastic.

Proof of 6.5. That (a) $\Rightarrow$ (b) is part of 2.1, and (b) $\Rightarrow$ (c) is obvious. Assume (c). We show that $b\mu = \mu$, i.e., that each countable $\sigma(\operatorname{coz} C(\mu X))$-partition is actually in μ.

First, if $f \in C(\mu X)$, then by (c), $\mu X \xrightarrow{f} b\varrho R$ is uniformly continuous; with χ the characteristic function of the Baire set $R - \{O\}$, $\chi \circ f \in C(\mu X)$ and $\operatorname{coz} \chi \circ f = Z(f)$. Thus, $\operatorname{coz} C(\mu X)$ is closed under complementation, and $\sigma(\operatorname{coz} C(\mu X)) = \operatorname{coz} C(\mu X)$ follows.

Next, if $\{A_n\}_n$ is a basic $b\mu$-cover, i.e., a countable $\operatorname{coz} C(\mu X)$-partition, then each χ_n, the characteristic function of A_n, is in $C(\mu X)$, and so is $f \equiv \sum 2^{-n}\chi_n$, by uniform convergence. Thus, $f \in C(\mu X, b\varrho R)$, so that $f^{-1}\{B_n\}_n \in \mu$ whenever $\{B_n\}_n$ is a basic $b\varrho$-cover. Take $B_0 = R - \{2^{-n}\colon n = 1, 2, \ldots\}$, and $B_n = \{2^{-n}\}$; then $\{A_n\}_n = f^{-1}\{B_n\}_n \in \mu$.

In application of 6.5, we reprove 3.12. We shall need to know that $b\varrho R$ is complete, a special case of 5.4, or from [10]; further comments appear below.

6.6. Corollary. *The completion of a measurable space is measurable.*

Proof. Let $\gamma \mathcal{A} X \xrightarrow{f} \varrho R$ be uniformly continuous. Then $\mathcal{A} X \xrightarrow{f|X} \varrho R$ is, and by 6.5 (c), so is $\mathcal{A} X \xrightarrow{f|X} b\varrho R$. Since $b\varrho R$ is complete, $f|X$ has the uniformly continuous extension over $\gamma \mathcal{A} X$ into $b\varrho R$; this must be f. By 6.5 (c), $\gamma \mathcal{A} X$ is measurable.

Finally, we prove the completeness theorem

6.7. Theorem. *$bm\mu X$ is complete iff $m\mu X$ is complete.*

Since $m\mu X \xrightarrow{1_X} \mu X$ is a uniformly continuous *homeomorphism* ($Tm\mu = T\mu$), $m\mu X$ is complete if μX is [4 (e), 4.10]. Thus 5.4 follows from 6.7.

To prove 6.7, we require the following.

6.8. Lemma. [4 (e), 8.4] *$m\mu X$ is complete iff each $\mathfrak{Z}(C(\mu X))$-ultrafilter with the countable intersection property is fixed.*

6.8 applies to a measurable space $\mathcal{A} X$, because $m\mathcal{A} X = \mathcal{A} X$, and reads: $\mathcal{A} X$ is complete iff each $\mathcal{A}$-ultrafilter with cip is fixed (because $\mathfrak{Z}(C(\mathcal{A} X)) = \mathcal{A}$). This is 3.14 (whose explicit proof we omitted).

From now on, all hypothesized families $\mathcal{F}, \mathcal{S}, \ldots \subset 2^X$ are to be closed under countable intersection.

To prove 6.7, we take $\mathfrak{Z} = \mathfrak{Z}(C(\mu X))$ in the following.

6.9. Theorem. *Let $\mathfrak{Z} \subset 2^X$ have the property: if $Z \in \mathfrak{Z}$, there are $Z_1, Z_2, \ldots \in \mathfrak{Z}$ with $X - Z = \bigcup Z_n$. Then, $\mathcal{F} \to \mathcal{F} \cap \mathfrak{Z}$ is a one-to-one correspondence of the $\sigma(\mathfrak{Z})$-ultrafilters with cip with the $\mathfrak{Z}$-ultrafilters with cip, and $\mathcal{F}$ is fixed iff $\mathcal{F} \cap \mathfrak{Z}$ is.*

6.9 can be derived from either of the more general results of [1 (b)] or [8]. We sketch a short and direct proof, somewhat resembling [8].

6.10. Proposition. *Let $\mathfrak{Z}$ be as in 6.9, and let $2^X \supset \mathcal{S} \supset \mathfrak{Z}$. If $\mathcal{F}$ is an $\mathcal{S}$-ultrafilter with cip, then $\mathcal{F} \cap \mathfrak{Z}$ is a $\mathfrak{Z}$-ultrafilter with cip.*

Proof. $\mathcal{F} \cap \mathfrak{Z}$ is always a $\mathfrak{Z}$-filter with cip. For maximality, let $Z \in \mathfrak{Z}$ with $Z \cap F \neq \emptyset$ for each $F \in \mathcal{F} \cap \mathfrak{Z}$. Write $X - Z = \bigcup Z_n$; since each $Z \cap Z_n = \emptyset$, $Z_n \notin \mathcal{F}$ and there is $F_n \in \mathcal{F}$ with $Z_n \cap F_n = \emptyset$. Then $Z \supset \bigcap F_n$, so $Z \in \mathcal{F} \cap \mathfrak{Z}$.

6.10 shows that the map $\mathcal{F} \to \mathcal{F} \cap \mathfrak{Z}$ of 6.9 goes into the $\mathfrak{Z}$-ultrafilters with cip. For the rest, we need the following.

6.11. Proposition. *Let $\mathfrak{Z} \subset 2^X$, and let $\mathcal{F}$ be a $\mathfrak{Z}$-ultrafilter with cip. Let $\mathcal{S}(\mathcal{F}) \equiv \{S \in 2^X\colon S$ either contains, or misses, some $F \in \mathcal{F}\}$.*

(a) $\mathcal{S}(\mathcal{F})$ is a σ-field containing $\mathfrak{Z}$.

(b) $\mathcal{F}^* \equiv \{S \in \mathcal{S}(\mathcal{F})\colon S$ contains an $F \in \mathcal{F}\}$ is an $\mathcal{S}(\mathcal{F})$-ultrafilter with cip, with $\mathcal{F}^* \cap \mathfrak{Z} = \mathcal{F}$, and $\mathcal{F}^*$ is fixed iff $\mathcal{F}$ is.

6.11 is an exact analogue of the development in [20, Ch. 12] on extending pre-measures to measures, obtained by specializing to $\{0, 1\}$-valued set functions (which stand in natural correspondence with the ultrafilters in question, per 3.14). The proof of 6.11 is short and routine, and we can omit it.

The next result is immediate from 6.10 and 6.11.

6.12. Proposition. *Let $\mathfrak{Z} \subset \mathfrak{W} \subset \mathcal{S}(\mathcal{F})$, with $\mathfrak{W}$ having the property of 6.9 (e.g., $\mathfrak{W}$ a σ-field). Then, $\mathcal{F}^* \cap \mathfrak{W}$ is a $\mathfrak{W}$-ultrafilter with cip, is the only $\mathfrak{W}$-filter containing $\mathcal{F}$, and is fixed iff $\mathcal{F}$ is.*

6.9 now follows from 6.12, using $\mathfrak{Z} \subset \sigma(\mathfrak{Z}) \subset \bigcap_{\mathcal{F}} \mathcal{S}(\mathcal{F})$.

6.13. Remarks. Gordon has recently defined and discussed what he calls "zero-set spaces" [3]. It can be shown that these are in one-one correspondence with separable $\mathcal{M}$-fine uniform spaces. In this context, he discusses the Baire system derived from a zero-set space, and shows how it can be viewed as a zero-set space; this is a version of the theorem "measurable $\Rightarrow$ $\mathcal{M}$-fine". He defines a realcompact zero-set space: Translated to $\mathcal{M}$-fine spaces, his definition is the condition in 6.8. He then proves that a zero-set space is realcompact iff the derived Baire system is. This is, then, a version of 6.7.

Something resembling a first version of 6.7, is due to Marczewski and Sikorski [17]: for metric M, bM is complete iff $|M|$ is nonmeasurable. The latter was somewhat later shown to be equivalent to completeness of eaM (whence 6.7, for metric spaces), with results of Hewitt, Katetov, and Shirota.

A little later, Hewitt showed that for uniformizable X, bX is complete iff X is realcompact [10]. With Shirota's theorem [2, 12(b)], the latter is equivalent to completeness of eaX. In [16], Lorch reproved half this theorem for compact X.

More recently, Hayes [8] and Frolík [1(b)] have given new set-theoretic proofs of results a bit more general than 6.9.

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Reçu par la Rédaction le 24. 7. 1971