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Spaces of ANR's

by

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1. Introduction. For a finite dimensional compactum X, let 2_h^X denote the hyperspace of ANR's lying in X, with the metric ϱ_h introduced and studied by K. Borsuk [3]. Among many results established by Borsuk, we mention here that 2_h^X is complete and separable, and the topology of 2_h^X is characterized by homotopic convergence: a sequence $\{A_i\}$ converges to A in 2_h^X if and only if (1) $\{A_i\}$ converges to A in the Hausdorff sense and (2) for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for each i, every subset of A_i of diameter less than δ is contractible to a point in a subset of A_i of diameter less than ε. Thus two ANR's in X which are "close" relative to the metric ϱ_h have similar homotopy properties. In particular, as was shown in [3], for each $A \in 2_h^X$, all ANR's in X which are sufficiently close to A in 2_h^X are homotopically equivalent to A.

The aim of the present paper is to investigate topological properties of the space 2_h^X, primarily for $X = S^2$.

It is evident that the subspace $\mathfrak{C}_X$ of 2_h^X consisting of all connected ANR's in X is open and closed in 2_h^X. Our attention will frequently be directed to this (complete) subspace of 2_h^X rather than to the whole space. For notational convenience, $\mathfrak{C}_{S^2}$ will be denoted simply by $\mathfrak{C}$.

We show that each pair of homotopically equivalent elements of $\mathfrak{C}$ can be joined by an arc in $2_h^{S^2}$, thus characterizing the components of $\mathfrak{C}$ as precisely the sets $[C] = \{A \in 2_h^{S^2} \mid A \underset{h}{\cong} C\}$, for $C \in \mathfrak{C}$. It is clear that S^2 is an isolated point of $2_h^{S^2}$, since no ANR properly contained in S^2 is homotopically equivalent to S^2, but there are no other isolated points in $2_h^{S^2}$. In fact, $2_h^{S^2}$ is infinite dimensional at every point of $2_h^{S^2} - \{S^2\}$, and is not locally compact at any point except S^2.

As partial answers to questions posed by Borsuk ([3], p. 201, [4], p. 221), we show that the set of polyhedra properly contained in S^2 is dense in $2_h^{S^2}$ and is of the first (Baire) category. On the other hand, the set of topological polyhedra in S^2 is of the second category (in fact, residual) in $2_h^{S^2}$.

While a number of our results are given for spaces more general than S^2, most have severely limited applicability. It would appear that the space 2^X_h warrants much further study, and several specific questions are posed in the final section of this paper.

2. Definitions and notation. Given a compactum X with metric ϱ, we will, following [3], denote the Hausdorff metric for the set of closed subsets of X by ϱ_s, and will use ϱ_h for the "metric of homotopy" on the set of ANR's in X, as defined in [3]. For convenience, we will use $s(A, \delta, \varepsilon)$ to denote the statement "every subset of A of diameter less than δ is contractible to a point in a subset of A of diameter less than ε." We remark that $\varrho_s(A, B) \leqslant \varrho_h(A, B)$ whenever these distances are defined.

Convergence relative to one of the metrics under discussion will be indicated in some obvious manner; e.g., $\{A_i\} \underset{\varrho_h}{\to} A$ will mean that $\lim_{i\to\infty} \varrho_h(A_i, B) = 0$. As remarked above, $\{A_i\} \underset{\varrho_h}{\to} A$ if and only if the sequence $\{A_i\}$ converges homotopically to A, in the sense that

(1) $\{A_i\} \underset{\varrho_s}{\to} A$ and

(2) for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $s(A_i, \delta, \varepsilon)$ holds for every i.

(It is shown in [3] that condition (2) implies that A is an ANR.)

We use $\bar{S}$ to denote the closure of the set S, and $\mathrm{Bd}\,S$ and $\mathrm{Int}\,S$ to denote, respectively, the boundary and interior of S in the point set sense; i.e., $\mathrm{Bd}\,S$ is the intersection of the closure of S and the closure of the complement of S, and $\mathrm{Int}\,S = S - \mathrm{Bd}\,S$. The ε-neighborhood of a point p will be denoted by $N_\varepsilon(p)$, and $N_\varepsilon(S)$ will denote the union of the ε-neighborhoods of the points of S.

A subset X of a Euclidean space E^n is called a *polyhedron* if X is the union of a finite number of closed geometric simplexes of E^n; any homeomorphic image of a polyhedron will be called a *topological polyhedron*. We always consider S^2 (or S^n) to be a polyhedron in E^3 (E^{n+1}).

3. Density and category. The set of all polyhedra properly contained in S^2 will be denoted by $\mathfrak{T}$, the set of all topological polyhedra by $\hat{\mathfrak{T}}$. By an *annulus* in S^2 we mean a continuum $A \subset S^2$ such that $\mathrm{Bd}\,A$ is the union of a finite number of disjoint simple closed curves; in particular, we consider S^2 itself to be an annulus. The set of all polyhedral annuli in S^2 will be denoted by $\mathcal{A}$, and the set of all annuli in S^2 by $\hat{\mathcal{A}}$.

The principal results of this section are that $\mathcal{A}$ is dense, $\mathfrak{T}$ is of the first category, and $\hat{\mathfrak{T}}$ is residual in $2^{S^2}_h$. Several of the lemmas, particularly 3.3, will have later applicability.

3.1. Lemma. *If C is a locally connected continuum in E^2 such that $E^2 - C$ is connected, then there exists a sequence $\{Q_i\}_{i=1}^\infty$ of polyhedral disks in E^2 such that*

(1) *for each i, $Q_{i+1} \subset Q_i$ and $Q_{i+2} \subset \mathrm{Int}\,Q_i$,*

(2) $C = \bigcap Q_i$,

(3) *for each i, $\mathrm{Bd}\,Q_i \cap \mathrm{Bd}\,Q_{i+1}$ is a finite set,*

(4) *there exists a sequence $\{\varepsilon_i\}_{i=1}^\infty$ of positive numbers such that $\sum_{i=1}^\infty \varepsilon_i < \infty$ and, for each i, every component of $Q_i - Q_{i+1}$ has diameter less than ε_i.*

Proof. It follows from a construction given by Borsuk ([4], pp. 132–137) that there exist polyhedral disks $P_1, P_2, P_3, \ldots$ in E^2 bounded by simple closed (polygonal) curves $J_1, J_2, J_3, \ldots$, respectively, and a sequence $\{\varepsilon_i\}_{i=1}^\infty$ of positive numbers, with $\varepsilon_i \to 0$, such that

(1) for each i, $P_{i+1} \subset P_i$,

(2) $C = \bigcap P_i$,

(3) for each i, $J_i \cap C$ is a finite set,

(4) for each i, $J_i \cap J_{i+1}$ is the union of a finite number of disjoint arcs,

(5) if $\overline{ab}$ is a component of $J_i \cap J_{i+1}$, then $\overline{ab} \cap C$ is a single point, p, different from a and b, and $\overline{ab} = \overline{ap} \cup \overline{pb}$, where $\overline{ap}$ and $\overline{pb}$ are intervals having only p in common; moreover, $\overline{ab} \cap J_{i+2}$ is a component of $J_i \cap J_{i+2}$, and is the union of two intervals $\overline{a'p}$ and $\overline{pb'}$, with $a' \in \overline{ap} - \{a\}$ and $b' \in \overline{pb} - \{b\}$,

(6) for each i, every component of $P_i - C$ has diameter less than $\frac{1}{2}\varepsilon_i$.

Since these properties hold for any subsequence of $\{P_i\}$, it may be assumed that $\sum_{i=1}^\infty \varepsilon_i < \infty$.

Let $J_1 \cap C = \{p_1, p_2, \ldots, p_k\}$ and let $\overline{a_1b_1}, \ldots, \overline{a_kb_k}$ be the components of $J_1 \cap J_2$, with $\overline{a_jb_j} = \overline{a_jp_j} \cup \overline{p_jb_j}$ for $1 \leqslant j \leqslant k$. Since each of the intervals $\overline{a_jp_j}$ and $\overline{p_jb_j}$ lies on the boundary of some component of $P_1 - C$ and therefore has diameter less than $\frac{1}{2}\varepsilon_1$, $\mathrm{dia}\,\overline{a_jb_j} < \varepsilon_1$ for $j = 1, \ldots, k$. Hence there exist disjoint polygonal arcs $\alpha_1, \ldots, \alpha_k$ such that for each j, $1 \leqslant j \leqslant k$, α_j is an arc from a_j to b_j, $\alpha_j - \{a_j, b_j\} \subset E^2 - P_1$, and $\mathrm{dia}(\alpha_j \cup \overline{a_jb_j}) < \varepsilon_1$. For $j = 1, \ldots, k$, let D_j denote the disk bounded by $\alpha_j \cup \overline{a_jb_j}$ and note that $\mathrm{dia}\,D_j < \varepsilon_1$. Let $K_1 = (J_1 - \bigcup_{j=1}^k \overline{a_jb_j}) \cup (\bigcup_{j=1}^k \alpha_j)$. Then K_1 is a simple closed curve and bounds the disk $Q_1 = P_1 \cup \bigcup_{j=1}^k D_j$. Since $J_1 \cap C \subset \bigcup_{j=1}^k \overline{a_jb_j}$ and $\bigcup_{j=1}^k \alpha_j$ does not intersect C, $K_1 \cap C = \emptyset$. Since each component of $P_1 - C$ has diameter less than ε_1, $P_1 \subset N_{\varepsilon_1}(C)$, and since $Q_1 = P_1 \cup \bigcup_{j=1}^k D_j$ and each D_j has diameter less than ε_1, $Q_1 \subset N_{\varepsilon_1}(P_1)$, and it follows that $Q_1 \subset N_{2\varepsilon_1}(C)$. It is clear that $K_1 \cap J_2 = \{a_1, b_1, a_2, b_2, \ldots, a_k, b_k\}$, so $K_1 \cap J_3 = \emptyset$ and hence $P_3 \subset \mathrm{Int}\,Q_1$. If D is a component of $Q_1 - P_2$, then either $\bar{D}$ is one of the disks $D_1, \ldots, D_k$, or else D is a subset of a component of $P_1 - C$. Hence every component of $Q_1 - P_2$ has diameter less than ε_1.

Suppose disks $Q_1, \ldots, Q_l$, with boundaries $K_1, \ldots, K_l$, respectively, have been defined so that

(a) $Q_{i+1} \subset Q_i$, for $i = 1, \ldots, l-1$,

(b) $K_i \cap K_{i+1}$ is finite, for $i = 1, \ldots, l-1$,

(c) $K_i \cap K_{i+2} = \emptyset$, for $i = 1, \ldots, l-2$,

(d) every component of $Q_i - Q_{i+1}$ has diameter less than ε_i, for $i = 1, \ldots, l-1$,

(e) $K_i \cap C = \emptyset$, for $i = 1, \ldots, l$,

(f) $Q_i \subset N_{2\varepsilon_i}(C)$, for $i = 1, \ldots, l$,

(g) $P_l \subset Q_l$ and $P_{l+2} \subset \operatorname{Int} Q_l$,

(h) $K_l \cap J_{l+1}$ is finite,

(i) every component of $Q_l - P_{l+1}$ has diameter less than ε_l.

Let $p_1, \ldots, p_k$ be the points of $J_{l+1} \cap C$ and $\overline{a_1 b_1}, \ldots, \overline{a_k b_k}$ the corresponding components of $J_{l+1} \cap J_{l+2}$. A simple closed curve K_{l+1} may be constructed from J_{l+1} in virtually the same way K_1 was obtained from J_1, the only essential difference being that the arcs a_j should be chosen so that for each j, $a_j - \{a_j, b_j\}$ is a subset of a component of $Q_l - P_{l+1}$; this is possible since $\overline{a_j b_j} \subset P_{l+2} \subset \operatorname{Int} Q_l$ and hence $\overline{a_j b_j}$ is on the boundary of some component of $Q_l - P_{l+1}$. If K_{l+1} is constructed in this way and Q_{l+1} denotes the disk bounded by K_{l+1}, it is easy to verify that conditions (a)–(i) are satisfied with l replaced by $l+1$ throughout. Thus by induction we obtain sequences $\{Q_i\}_{i=1}^{\infty}$ and $\{K_i\}_{i=1}^{\infty}$ satisfying (a)–(f) for every integer i. Condition (c) implies that $Q_{i+2} \subset \operatorname{Int} Q_i$ for every i, and (f) shows that $C = \bigcap Q_i$. Thus the sequence $\{Q_i\}_{i=1}^{\infty}$ satisfies all the required conditions.

3.2. Corollary. *If C is a connected* ANR *properly contained in S^2, then there exists a sequence $\{A_i\}$ of polyhedral annuli in S^2 such that*

(1) *for each i, $A_{i+1} \subset A_i$ and $A_{i+2} \subset \operatorname{Int} A_i$,*

(2) $C = \bigcap A_i$,

(3) *for each i, $\operatorname{Bd} A_i \cap \operatorname{Bd} A_{i+1}$ is a finite set,*

(4) *there exists a sequence $\{\varepsilon_i\}_{i=1}^{\infty}$ of positive numbers such that $\sum_{i=1}^{\infty} \varepsilon_i < \infty$ and, for each i, every component of $A_i - A_{i+1}$ has diameter less than ε_i.*

Proof. Since C is an ANR, $S^2 - C$ has only a finite number of components. Let $D_1, \ldots, D_n$ be the components of $S^2 - C$ and, for $j = 1, \ldots, n$, let $C_j = S^2 - D_j$. Then C_j is a locally connected continuum, $S^2 - C_j \neq \emptyset$, and C_j does not separate S^2. It therefore follows from Lemma 3.1 that for each j, $1 \leqslant j \leqslant n$, there exist a sequence $\{Q_i^j\}_{i=1}^{\infty}$ of polyhedral disks in S^2 and a sequence $\{\varepsilon_i^j\}_{i=1}^{\infty}$ of positive numbers satisfying, with respect to C_j, all the conditions of that lemma. If for each i, $A_i = \bigcap_{j=1}^{n} Q_i^j$, it is evident that A_i is an annulus and that conditions (1)–(3) are satisfied by the sequence $\{A_i\}_{i=1}^{\infty}$.

For each i, let $\varepsilon_i = \sum_{j=1}^{n} \varepsilon_i^j$. Then $\sum_{i=1}^{\infty} \varepsilon_i < \infty$ and, moreover, if D is a component of $A_i - A_{i+1}$, then $D \subset S^2 - C$, hence D is contained in some D_j and therefore in some component of $Q_i^j - Q_{i+1}^j$, so $\operatorname{dia} D < \varepsilon_i^j < \varepsilon_i$. Hence condition (4) is satisfied.

Our usage of the terms *homotopy*, *deformation*, *isotopy*, etc. is standard except that we find it convenient not to insist that the interval used be always $[0, 1]$; for example, by a *deformation retraction* of A onto B we mean a mapping $h\colon A \times [a, b] \to A$, for some interval $[a, b]$ of real numbers, such that h_a is the identity map on A and h_b is a retraction of A onto B. (We adhere, of course, to the standard notation h_t for $h|A \times \{t\}$, and alternate as convenient between the notations $h(x, t)$ and $h_t(x)$.)

A mapping $h\colon A \times [a, b] \to X$ is called a *pseudo-isotopy* if h_t is a homeomorphism for $t \in [a, b)$. We will say that a mapping $h\colon A \times [a, b] \to X$ is *strongly contracting* if $a \leqslant u \leqslant v \leqslant b$ implies $h_u h_v(A) \subset h_v(A) \subset h_u(A)$.

3.3. Lemma. *Suppose C, $\{A_i\}_{i=1}^{\infty}$ and $\{\varepsilon_i\}_{i=1}^{\infty}$ satisfy the conditions of Corollary 3.2, and let $\{t_i\}_{i=1}^{\infty}$ be an increasing sequence of real numbers converging to 1, with $t_1 = 0$. Then there exists a map $h\colon A_1 \times [0, 1] \to A_1$ such that*

(1) *h is a strong deformation retraction of A_1 onto C,*

(2) *h is strongly contracting,*

(3) *for each i, $h|A_1 \times [0, t_{i+1}]$ is a strong deformation retraction of A_1 onto A_{i+1},*

(4) *for each i, $h|A_i \times [t_i, t_{i+1}]$ is a strongly contracting pseudo-isotopy of A_i onto A_{i+1}.*

Proof. If i is a positive integer and $D_1, \ldots, D_m$ are the components of $A_i - A_{i+1}$, then for each j, $1 \leqslant j \leqslant m$, $\bar{D}_j$ is a disk, of diameter less than ε_i, bounded by the union of an arc $\alpha_j \subset \operatorname{Bd} A_i$ and an arc $\beta_j \subset \operatorname{Bd} A_{i+1}$, with α_j and β_j intersecting only in their endpoints. It is easily seen that for each j, there is a strongly contracting pseudo-isotopy of D_j onto β_j. Hence there is a strongly contracting pseudo-isotopy $\varphi^i\colon A_i \times [t_i, t_{i+1}] \to A_i$ of A_i onto A_{i+1} such that for $j = 1, \ldots, m$, $\varphi_t^i(D_j) \subset D_j$ for every $t \in [t_i, t_{i+1}]$, and such that $\varphi^i(x, t) = x$ for every $x \in A_{i+1}$, $t \in [t_i, t_{i+1}]$. Since $\operatorname{dia} D_j < \varepsilon_i$ for every j, $1 \leqslant j \leqslant m$, it follows that $\varrho(x, \varphi^i(x, t)) < \varepsilon_i$ for every $x \in A_i$, $t \in [t_i, t_{i+1}]$.

For $t \in [t_1, t_2)$, let $h_t = \varphi_t^1$ and for $i > 1$ and $t \in [t_i, t_{i+1})$, let $h_t = \varphi_t^i \circ \varphi_{t_i}^{i-1} \circ \ldots \circ \varphi_{t_3}^2 \circ \varphi_{t_2}^1$. If $h(x, t) = h_t(x)$ for $x \in A_1$ and $t \in [0, 1)$, then h is a continuous function from $A_1 \times [0, 1)$ into A_1. It is not difficult to see that for each positive integer k, if $t \geqslant t' > t_k$, then $\varrho\big(h(x,t), h(x,t')\big) < \sum_{i=k}^{\infty}\varepsilon_i$ for every $x \in A_1$. Since $\sum_{i=k}^{\infty}\varepsilon_i \to 0$ as $k\to\infty$, it follows that h is uniformly continuous on $A_1\times[0,1)$ and hence (e.g., [12], p. 28) h has a continuous extension $\tilde{h}\colon A_1\times[0,1]\to A_1$. It can be shown that $\tilde{h}$ has all the desired properties.

3.4. Lemma. *Suppose X is a finite dimensional compactum, A and B are ANR's in X and $h\colon A\times I\to A$ is a strongly contracting, strong deformation retraction of A onto B. If $\{t_i\}$ is an increasing sequence of numbers in I converging to 1 and for each i, $A_i = h_{t_i}(A)$ is an ANR, then $\{A_i\}\underset{\varrho_h}{\to} B$.*

Proof. Since it is evident that $\{A_i\}\underset{\varrho_s}{\to} B$, it will be sufficient to show that for every positive number ε, there is a positive number δ such that $s(A_i,\delta,\varepsilon)$ holds for each i. (Recall that $s(A,\delta,\varepsilon)$ denotes the statement "every subset of A of diameter less than δ is contractible to a point in a subset of A of diameter less than ε.") Since each A_i is an ANR, it is clearly enough to prove that $s(A_i,\delta,\varepsilon)$ holds for all sufficiently large i.

It will first be shown that for every $\delta>0$, there is an integer n such that $\varrho\big(x, h_t(x)\big)<\delta$ for every $x\in A_n$, $t\in I$. The supposition that for some $\delta>0$ there is no such n implies the existence of a sequence $\{x_i\}$ of points of A and a sequence $\{s_i\}$ of numbers in I such that $\{x_i\}\to x\in A$, $\{s_i\}\to s\in I$, and for each i, $x_i\in A_i$ and $\varrho\big(x_i, h_{s_i}(x_i)\big)\geqslant\delta$. Since h is continuous, $\{h_{s_i}(x_i)\}\to h_s(x)$. Since $x_i\in A_i$ for every i, $x\in B$ and therefore, since h is a strong deformation retraction, $h_s(x)=x$. Thus both $\{h_{s_i}(x_i)\}$ and $\{x_i\}$ converge to x, contrary to the supposition that $\varrho\big(x_i, h_{s_i}(x_i)\big)\geqslant\delta$ for all i.

Next we observe that for each i, $h(A_i\times I)\subset A_i$. This follows from the fact that h is strongly contracting, since if $0\leqslant t\leqslant t_i$, $h_t(A_i)=h_t h_{t_i}(A)\subset h_{t_i}(A)=A_i$ and if $t_i\leqslant t\leqslant 1$, $h_t(A_i)\subset h_t(A)\subset h_{t_i}(A)=A_i$.

Now suppose $\varepsilon>0$. Since B is an ANR, there exists a positive number $\eta<\frac{1}{2}\varepsilon$ such that $s(B,\eta,\frac{1}{2}\varepsilon)$ is true. Let $\delta=\frac{1}{3}\eta$ and choose an integer n such that $\varrho\big(x,h_t(x)\big)<\delta$ for every $x\in A_n$, $t\in I$. Suppose $i\geqslant n$ and let M be a subset of A_i with diameter less than δ. Let $f=h|M\times I$; then f is a homotopy of M onto $M'=f_1(M)=h_1(M)\subset B$, and since $h(A_i\times I)\subset A_i$, $f(M\times I)\subset A_i$. Since $\varrho\big(x,h_t(x)\big)<\delta$ for every $x\in A_i$, $t\in I$, it follows that $\operatorname{dia} f(M\times I)<\operatorname{dia}M+2\delta<3\delta=\eta<\frac{1}{2}\varepsilon$. Hence in particular, $\operatorname{dia}M'<\eta$ and since $M'\subset B$ and $s(B,\eta,\frac{1}{2}\varepsilon)$ is true, there is a homotopy $g\colon M'\times[1,2]\to B$ such that $\operatorname{dia} g(M'\times[1,2])<\frac{1}{2}\varepsilon$ and $g_1(M')$ is a point. Clearly f followed by g is a homotopy taking M to a point in a subset of A_i of diameter less than ε, so $s(A_i,\delta,\varepsilon)$ is true. It follows that $\{A_i\}\underset{\varrho_h}{\to} B$, as required.

3.5. Theorem. *Every connected ANR properly contained in S^2 is the homotopic limit of a sequence $\{B_i\}$ of polyhedral annuli, with $B_{i+1}\subset\operatorname{Int}B_i$ for every i.*

Proof. Suppose C is a connected ANR properly contained in S^2 and let $\{A_i\}$, $\{t_i\}$ and $h\colon A_1\times I\to A_1$ be as in Lemma 3.3. It follows from Lemma 3.4 that $\{A_i\}\underset{\varrho_h}{\to} B$. Since $A_{i+2}\subset\operatorname{Int}A_i$ for every i, if for each i, $B_i=A_{2i}$, then $\{B_i\}\underset{\varrho_h}{\to} C$ and for each i, $B_{i+1}\subset\operatorname{Int}B_i$.

3.6. Corollary. *The set of polyhedra in S^2 is dense in $2^{S^2}_h$.*

Proof. Suppose $C\in 2^{S^2}_h-\{S^2\}$ and let $C_1,\ldots,C_n$ be the components of C. For $j=1,\ldots,n$, let $\{A_i^j\}_{i=1}^{\infty}$ be a sequence of polyhedral annuli converging homotopically to C_j. Since $C_j=\bigcap_{i=1}^{\infty}A_i^j$, it follows that $A_i^1, A_i^2,\ldots,A_i^n$ are disjoint for i sufficiently large, and hence it may be assumed that $A_1^1, A_1^2,\ldots,A_1^n$ are disjoint.

Suppose $\varepsilon>0$ and for $j=1,\ldots,n$, let δ_j be a positive number such that $s(A_i^j,\delta_j,\varepsilon)$ holds for all i. Let $\eta=\min\{\varrho(A_1^j,A_1^k)|\ 1\leqslant j<k\leqslant n\}$ and let $\delta=\min(\eta,\delta_1,\ldots,\delta_n)$. If $M\subset A_i$ and $\operatorname{dia}M<\delta$, then $M\subset A_i^j$ for some j, $1\leqslant j\leqslant n$, and hence M is contractible to a point in a subset of A_i^j of diameter less than ε. It follows that $s(A_i,\delta,\varepsilon)$ is true for all i, and hence that $\{A_i\}\underset{\varrho_h}{\to} C$.

We next consider the Baire category in 2^X_h of the set $\mathcal{P}_X$ of subpolyhedra of the polyhedron X, and show, in effect, that $\mathcal{P}_X$ is a first category set in all instances in which it is not trivially of the second category. In particular, the set $\mathcal{P}$ of polyhedra properly contained in S^2 is a first category subset of $2^{S^2}_h$. Perhaps surprisingly, the corresponding set $\hat{\mathcal{P}}$ of topological polyhedra is a second category subset of $2^{S^2}_h$.

3.7. Lemma. *Suppose X is a finite dimensional compactum, $A\in 2^X_h$, and $\{A_i\}_{i=1}^{\infty}$ is a sequence of elements of 2^X_h such that $\{A_i\}\underset{\varrho_h}{\to} A$. If $\{B_i\}_{i=1}^{\infty}$ is a null sequence (i.e., $\operatorname{dia}B_i\to 0$) of absolute retracts in X such that for each i, $A_i\cap B_i$ is a non-empty absolute retract, then $\{A_i\cup B_i\}\underset{\varrho_h}{\to} A$.*

Proof. Suppose $\varepsilon>0$. Since $\{A_i\}\underset{\varrho_h}{\to} A$, there exists a positive number $\delta<\frac{1}{4}\varepsilon$ such that $s(A_i,2\delta,\frac{1}{2}\varepsilon)$ is true for every i.

Since $\operatorname{dia}B_i\to 0$, there exists an integer i_0 such that $\operatorname{dia}B_i<\delta$ for every $i>i_0$. Suppose i is an integer greater than i_0 and M is a subset of $A_i\cup B_i$ with $\operatorname{dia}M<\delta$. If $M\cap B_i=\emptyset$, then M is contractible to a point in a subset of A_i of diameter less than ε, so suppose $M\cap B_i\neq\emptyset$.

Since $A_i\cap B_i\subset B_i$ and $A_i\cap B_i$ and B_i are absolute retracts, $A_i\cap B_i$ is a strong deformation retract of B_i (e.g., [6], p. 33). Hence A_i is a strong deformation retract of $A_i\cup B_i$, and it follows that M is contractible in $M\cup B_i$ to $M'=(A_i\cap B_i)\cup(M\cap A_i)$. Since $\operatorname{dia}M<\delta$ and $\operatorname{dia}B_i<\delta$

and $M \cap B_i \neq \emptyset$, $\operatorname{dia}(M \cup B_i) < 2\delta$. Hence, in particular, $\operatorname{dia} M' < 2\delta$ and therefore, since $s(A_i, 2\delta, \frac{1}{2}\varepsilon)$ is true, M' is contractible to a point in a subset of A_i of diameter less than $\frac{1}{2}\varepsilon$. Since M is contractible to M' in $M \cup B_i$ and $\operatorname{dia}(M \cup B_i) < 2\delta < \frac{1}{2}\varepsilon$, it follows that M is contractible to a point in a subset of $A_i \cup B_i$ of diameter less than ε. Hence $s(A_i \cup B_i, \delta, \varepsilon)$ holds for all $i > i_0$, from which it easily follows that $\{A_i \cup B_i\} \underset{\varrho_h}{\to} A$.

3.8. THEOREM. *If X is a connected polyhedron with no 1-dimensional open subset, then the set $\mathcal{T}_X$ of all polyhedra properly contained in X is a first category subset of 2^X_h.*

Proof. For each positive integer m, let $\mathcal{T}_m$ denote the set of all elements of $\mathcal{T}_X$ which can be expressed as the union of m or fewer geometric simplexes.

Suppose $A \in 2^X_h$ and $\{A_i\}$ is a sequence of elements of $\mathcal{T}_X$ such that $\{A_i\} \underset{\varrho_\varrho}{\to} A$. Since each A_i is a proper subset of X and X has no 1-dimensional open subset, it follows that for each i, there is a 2-simplex $\sigma_i \subset X$ such that $\sigma_i \cap A_i$ is a vertex, p_i, of σ_i. For each i, let B_i be an arc in σ_i such that p_i is an endpoint of B_i, $\operatorname{dia} B_i < 1/i$, and B_i contains an arc of a circle. Since $A_i \cap B_i = \{p_i\}$ and $\operatorname{dia} B_i \to 0$ it follows from Lemma 3.7 that $\{A_i \cup B_i\} \underset{\varrho_h}{\to} A$. Since B_i contains a circular arc, for no integer m is there a sequence of elements of $\mathcal{T}_m$ converging to B_i.

It follows that every open subset of 2^X_h which intersects $\overline{\mathcal{T}}_X$ contains an element of 2^X_h which is not in $\overline{\mathcal{T}}_m$ for any m. Hence each $\mathcal{T}_m$ is nowhere dense in 2^X_h, and since $\mathcal{T}_X = \bigcup_{m=1}^{\infty} \mathcal{T}_m$, $\mathcal{T}_X$ is a first category subset of 2^X_h.

Remarks. (1) If X is not connected, then 2^X_h may have isolated points different from X. In this case, $\mathcal{T}_X$ would be of the second category in 2^X_h, but for a trivial reason. Clearly (in view of Lemma 3.7), every isolated point of 2^X_h must be a component of X, and hence the requirement in Theorem 3.8 that X be connected could be deleted if $\mathcal{T}_X$ were replaced by the set $\mathcal{T}'_X$ of all polyhedra in X which contain no component of X.

(2) If X has a 1-dimensional open subset, then there is an interval $\overline{ab} \subset X$ such that $U = \overline{ab} - \{a, b\}$ is open in X. If $\mathcal{U}$ is the set of all closed intervals lying in U, then $\mathcal{U}$ is open in 2^X_h and hence $\mathcal{U}$ is topologically complete. Since $\mathcal{U} \subset \mathcal{T}_X$, it follows that $\mathcal{T}_X$ is of the second category in 2^X_h. Hence the requirement that X have no 1-dimensional open subset is essential in Theorem 3.8.

3.9. LEMMA. *Suppose X is a finite dimensional compactum and $\{A_i\} \underset{\varrho_h}{\to} A$, where each A_i is a connected ANR in X. If a is an arc in A with endpoints p and q, and for each i, p_i and q_i are points of A_i such that $\{p_i\} \to p$ and $\{q_i\} \to q$, then there exists, for each i, an arc a_i from p_i to q_i in A_i such that $\{a_i\} \underset{\varrho_s}{\to} a$.*

Proof. Suppose $\varepsilon > 0$ and let δ be a positive number less than $\frac{1}{2}\varepsilon$ such that $s(A_i, 3\delta, \frac{1}{2}\varepsilon)$ is true for all i. Let $\overline{a_0 a_1}, \overline{a_1 a_2}, \ldots, \overline{a_n a_{n+1}}$ be a finite sequence of subarcs of a, with $a_0 = p$ and $a_{n+1} = q$, such that $\operatorname{dia} \overline{a_j a_{j+1}} < \delta$ for $j = 0, 1, \ldots, n$. There exists an integer i_0 such that for $i > i_0$, $\varrho(p_i, p) < \delta$, $\varrho(q_i, q) < \delta$, and for $j = 1, \ldots, n$, $A_i \cap N_\delta(a_j) \neq \emptyset$. For $j = 1, \ldots, n$, let r_j be a point of $A_i \cap N_\delta(a_j)$ and let $r_0 = p_i$, $r_{n+1} = q_i$. Then for $0 \leqslant j \leqslant n$, $\varrho(r_j, r_{j+1}) \leqslant \varrho(r_j, a_j) + \varrho(a_j, a_{j+1}) + \varrho(a_{j+1}, r_{j+1}) < 3\delta$ and hence, since $s(A_i, 3\delta, \frac{1}{2}\varepsilon)$ is true, there is an arc $\overline{r_j r_{j+1}} \subset A_i$ with $\operatorname{dia} \overline{r_j r_{j+1}} < \frac{1}{2}\varepsilon$. Since for each j, $0 \leqslant j \leqslant n$, $\varrho(a_j, r_j) < \delta < \frac{1}{2}\varepsilon$, it follows that $\overline{r_j r_{j+1}} \subset N_\varepsilon(a)$. Since clearly $\bigcup_{j=0}^{n} \overline{r_j r_{j+1}}$ contains an arc from p_i to q_i, it follows that for every open set U containing a, there is an integer i_0 such that for $i > i_0$, $U \cap A_i$ contains an arc from p_i to q_i.

Let $U_0 = X$ and for $i > 0$, let $U_i = N_{1/i}(a)$. There exists an increasing sequence $i_1, i_2, \ldots$ of positive integers such that, for each j, if $i > i_j$, then $U_i \cap A_i$ contains an arc from p_i to q_i. Let $i_0 = 0$ and for each positive integer i, let a_i be an arc from p_i to q_i in $A_i \cap U_{i_j}$, where $i_j < i \leqslant i_{j+1}$. If $K = \lim_{\varrho_s} a_i$, it is clear that $K \subset a$, p and q belong to K, and K is connected. Hence $K = a$, so $\{a_i\} \underset{\varrho_s}{\to} a$.

Remark. It is not difficult to modify the above argument to show that if $\{A_i\} \underset{\varrho_h}{\to} A$ and a is an arc in A, then there exists a sequence $\{a_i\}$ of arcs with $a_i \subset A_i$ and $\{a_i\} \underset{\varrho_h}{\to} a$. The analogous proposition with a and the a_i's replaced by arbitrary ANR's is false, however, as may be seen by considering in E^3 a sequence $\{A_i\}$ of irreducible 2-dimensional AR's (see [2]) converging homotopically to a planar disk A and choosing a to be any 2-dimensional AR properly contained in A.

3.10. LEMMA. *If X is a finite dimensional compactum and $\mathcal{K}$ is the set of all connected ANR's in X which have a local cut point, then $\mathcal{K}$ is an F_σ subset of 2^X_h.*

Proof. It is easily seen that if p is a local cut point of a locally connected continuum K, then there exists a positive number ε such that if U is an open subset of X containing p and having diameter $\leqslant \varepsilon$ and C is the component of $K \cap U$ containing p, then there are at least two components of $\overline{C} - \{p\}$ which intersect $\operatorname{Bd} U$. A local cut point of K for which this condition is satisfied for a given $\varepsilon > 0$ will be said to have *magnitude* ε.

Suppose $\varepsilon > 0$ and let $\mathcal{K}_\varepsilon$ denote the set of all elements of $\mathcal{K}$ which have a local cut point of magnitude ε. We wish to show that $\overline{\mathcal{K}}_\varepsilon \subset \mathcal{K}$, so suppose $\{K_i\} \underset{\varrho_h}{\to} K$ with each $K_i \in \mathcal{K}_\varepsilon$, and for each i, let p_i be a local cut point of K_i of magnitude ε. It may be assumed that $p_i \to p \in K$.

Since K is locally connected, there exists an open subset U of X such that $p \in U \subset N_{\varepsilon/2}(p)$ and such that $K' = K \cap \overline{U}$ is a locally connected continuum. Assume that $p_i \in U$ for each i, and let C_i denote the component of $K_i \cap N_\varepsilon(p)$ which contains p_i. Since p_i has magnitude ε, there are at least two components of $\overline{C}_i - \{p_i\}$ which intersect $\operatorname{Bd} N_\varepsilon(p)$, and hence there exist points $a_i, b_i \in K_i \cap \operatorname{Bd} U$ such that p_i separates a_i from b_i in C_i. It may be assumed that $a_i \to a$, $b_i \to b$; then $a, b \in K' - \{p\}$. Suppose α is an arc (possibly degenerate) from a to b in K'. By Lemma 3.9, for each i, there is an arc α_i from a_i to b_i in K_i such that $\alpha_i \underset{\varrho_s}{\to} \alpha$. Since $\alpha \subset N_\varepsilon(p)$, there is an integer i_0 such that $\alpha_i \subset N_\varepsilon(p)$ for all $i > i_0$. It follows, since p_i separates a_i from b_i in C_i, that $p_i \in \alpha_i$ for all $i > i_0$ and hence $p \in \alpha$. Thus every arc from a to b in K' contains p, and since K' is a locally connected continuum, this implies that p separates a from b in K'. Therefore p is a local cut point of K, so $K \in \mathcal{K}$.

Hence for every $\varepsilon > 0$, $\overline{\mathcal{K}}_\varepsilon \subset \mathcal{K}$. Since $\mathcal{K} = \bigcup_{n=1}^{\infty} \mathcal{K}_{1/n}$, it follows that $\mathcal{K}$ is an F_σ in 2^X_h.

3.11. Lemma. *Every connected* ANR *in* S^2 *which has no local cut point is an annulus.*

Proof. If A is a connected ANR in S^2 which has no local cut point, then A is a locally connected continuum with at most a finite number of complementary domains; since A has no cut point, every complementary domain of A is bounded by a simple closed curve ([9], p. 199, Th. 46) and since A has no local cut point, no two boundaries of complementary domains of A can intersect ([11], p. 308, Th. 6). It follows that A is an annulus.

3.12. Theorem. *The set* $\hat{\mathcal{T}}$ *of all topological polyhedra in* S^2 *is a residual set (i.e., a dense* G_δ*) in* $2^{S^2}_h$.

Proof. Let $\mathcal{C}$ denote the set of connected ANR's in S^2 and let $\mathcal{K} = \{K \in \mathcal{C} | K \text{ has a local cut point}\}$. It is clear that $\hat{\mathcal{A}}$, the set of all annuli in S^2, is a subset of $\mathcal{C} - \mathcal{K}$ and by Lemma 3.11, $\mathcal{C} - \mathcal{K} \subset \hat{\mathcal{A}}$, so $\mathcal{C} - \mathcal{K} = \hat{\mathcal{A}}$. By Theorem 3.5, $\hat{\mathcal{A}}$ is dense in $\mathcal{C}$ and therefore, since $\mathcal{K}$ is an F_σ by Lemma 3.10, it follows that $\mathcal{K} = \mathcal{C} - \hat{\mathcal{A}}$ is a first category subset of $\mathcal{C}$, so $\hat{\mathcal{A}}$ is a residual set in $\mathcal{C}$.

For $k > 1$, let $\mathcal{C}_k$ denote the set of all ANR's in S^2 which have exactly k components and let $\hat{\mathcal{A}}_k$ denote the set of all elements A of $\mathcal{C}_k$ such that every component of A is an annulus. It follows from the proof of Corollary 3.6 that $\hat{\mathcal{A}}_k$ is dense in $\mathcal{C}_k$, and obvious modifications of Lemmas 3.10 and 3.11 then suffice to show, as above, that $\hat{\mathcal{A}}_k$ is a residual subset of $\mathcal{C}_k$. Since each $\mathcal{C}_k$ is open and closed in $2^{S^2}_h$, it follows that $\hat{\mathcal{T}}$ is a residual subset of $2^{S^2}_h$.

4. Arcs in 2^X_h. The existence or non-existence of an arc joining two elements A, B of 2^X_h would seem to be of some topological significance. In order that such an arc should exist, it is obviously necessary that A and B be homotopically equivalent, but this is not sufficient. On the other hand, it is sufficient that A and B be isotopic in X, but of course this is not necessary. And while it is easy to find examples of ANR's which are homotopic in X but cannot be joined by an arc in 2^X_h, the converse question is more elusive.

It is shown below that two connected ANR's in S^2 can be joined by an arc in $2^{S^2}_h$ if and only if they are homotopically equivalent. This result fails for non-connected ANR's in S^2 and also for connected AR's in an arbitrary AR. We do not know whether it holds for connected ANR's in S^3.

4.1. Lemma. *If X is a finite dimensional compactum, $\{A_i\} \to A$ in 2^X_h and for each i, g_i is an ε_i-homeomorphism of A_i onto a subset B_i of X, where $\varepsilon_i \to 0$ as $i \to \infty$, then $\{B_i\} \underset{\varrho_h}{\to} A$.*

Proof. Since it is clear that $\{B_i\} \underset{\varrho_s}{\to} A$, it is sufficient to show that for each $\varepsilon > 0$, there is a $\delta > 0$ such that $s(B_i, \delta, \varepsilon)$ holds for almost all i.

Since $\{A_i\} \underset{\varrho_h}{\to} A$, for every $\varepsilon > 0$ there is a $\delta > 0$ such that $s(A_i, 2\delta, \frac{1}{2}\varepsilon)$ is true for all i. Let i_0 be a positive integer such that $\varepsilon_i < \min(\frac{1}{2}\delta, \frac{1}{4}\varepsilon)$ for all $i > i_0$. Suppose $i > i_0$ and $M \subset B_i$, with $\operatorname{dia} M < \delta$. Since $\varrho(x, g_i(x)) < \varepsilon_i$ for every $x \in A_i$, it follows that $\operatorname{dia} g^{-1}(M) < \delta + 2\varepsilon_i < 2\delta$, and hence $g^{-1}(M)$ is contractible to a point in a subset K of A_i, with $\operatorname{dia} K < \frac{1}{2}\varepsilon$. It readily follows that M is contractible to a point in the subset $g_i(K)$ of B_i. Since $\operatorname{dia} K < \frac{1}{2}\varepsilon$, $\operatorname{dia} g_i(K) < \frac{1}{2}\varepsilon + 2\varepsilon_i < \varepsilon$, and hence $s(B_i, \delta, \varepsilon)$ holds for every $i > i_0$. It follows that $\{B_i\} \underset{\varrho_h}{\to} A$.

4.2. Lemma. *Suppose X is a finite dimensional compactum, $A \in 2^X_h$ and $f\colon A \times I \to X$ is an isotopy. If for each $t \in I$, $\varphi(t) = f_t(A)$, then φ is a continuous mapping of I into 2^X_h.*

Proof. For each $u, v \in I$, let $g_{uv} = f_u \circ f_v^{-1}$. Then g_{uv} is a homeomorphism of $f_v(A)$ onto $f_u(A)$, and it follows from the continuity of f that for each $\varepsilon > 0$, there is a $\delta > 0$ such that g_{uv} is an ε-homeomorphism whenever $|u - v| < \delta$. Hence if $\{s_i\} \to s_0 \in I$ and $\varepsilon_i = \max\{\varrho(x, g_{s_i s_0}(x)) | x \in f_{s_0}(A)\}$, then $g_{s_i s_0}$ is an ε_i-homeomorphism of $f_{s_0}(A)$ onto $f_{s_i}(A)$, and $\varepsilon_i \to 0$ as $i \to \infty$. Hence by Lemma 4.1, $\{f_{s_i}(A)\} \underset{\varrho_h}{\to} f_{s_0}(A)$. Thus $\{\varphi(s_i)\} \underset{\varrho_h}{\to} \varphi(s_0)$ whenever $\{s_i\} \to s_0$, so φ is continuous at each $s_0 \in I$.

4.3. Corollary. *If X is a finite dimensional compactum and A and B are elements of 2^X_h which are isotopic in X, then there is an arc from A to B in 2^X_h.*

Proof. By Lemma 4.2, there is a mapping $\varphi\colon I\to 2^X_h$ with $\varphi(0)=A$ and $\varphi(1)=B$; since $\varphi(I)$ is a locally connected continuum, it follows that there is an arc from A to B in 2^X_h.

4.4. Theorem. *Every two homotopically equivalent connected ANR's in S^2 are joined by an arc in $2^{S^2}_h$.*

Proof. Suppose C is a connected ANR properly contained in S^2. Let $\{A_i\}$, $\{t_i\}$ and $h\colon A_1\times I\to A_1$ satisfy the conditions of Lemma 3.3 with respect to C, and for each $t\in I$, let $\varphi(t)=h_t(A_1)$.

For each n, $h|A_n\times[t_n, t_{n+1}]$ is a strongly contracting pseudo-isotopy and a strong deformation retraction of A_n onto A_{n+1}. Since for each $t\in[t_n, t_{n+1}]$, $\varphi(t)=h_t(A_1)=h_t(A_n)$, it follows from Lemma 4.2 that φ is continuous at each $t\in[t_n, t_{n+1})$ and from Lemma 3.4 that φ is continuous at t_{n+1}. Since h is itself a strongly contracting, strong deformation retraction, it follows from Lemma 3.4 that φ is continuous at $t=1$. Hence φ is continuous on I, and therefore there is an arc from A_1 to C in $2^{S^2}_h$.

Now suppose C and C' are any two homotopically equivalent connected ANR's in S^2. By the argument above, there exist annuli A and A' in S^2 and arcs $\mathcal{A}$ and $\mathcal{A}'$ in $2^{S^2}_h$ joining A to C and A' to C', respectively. Since A and A' are homotopically equivalent and each is an annulus, A and A' are isotopic in S^2. Thus by Corollary 4.3, there is an arc $\mathcal{B}$ from A to A' in $2^{S^2}_h$; clearly $\mathcal{A}\cup\mathcal{B}\cup\mathcal{A}'$ contains an arc from C to C', as desired.

4.5. Theorem. *If n is a positive integer, $A\in 2^{S^n}_h$ and P and Q are continua lying in S^n-A, then there is a neighborhood $\mathcal{U}$ of A in $2^{S^n}_h$ such that either every element of $\mathcal{U}$ separates P from Q in S^n or no element of $\mathcal{U}$ does so.*

Proof. If A does not separate P from Q in S^n, there is a continuum K containing P and Q and lying in S^n-A. If $U=S^n-K$ and $\mathcal{U}=\{B\in 2^{S^n}_h|\ B\subset U\}$, then $\mathcal{U}$ is open in $2^{S^n}_h$ and no element of $\mathcal{U}$ separates P from Q in S^n.

Conversely, suppose A separates P from Q in S^n. It is easy to show that there is a positive number ε such that every subset of S^n which is the image of A under an ε-map is homotopic to A in $S^n-(P\cup Q)$, and hence (e.g. [7], p. 473, Th. 2) every such set separates P from Q in S^n. There is a neighborhood $\mathcal{U}$ of A in $2^{S^n}_h$ such that for each $B\in\mathcal{U}$, $\varrho_h(A, B)<\varepsilon$ and $B\cap(P\cup Q)=\emptyset$. If $B\in\mathcal{U}$, it follows from the definition of ϱ_h (given in [3]) that there is an ε-map $f\colon A\to B$; then $f(A)$ separates P from Q in S^n and therefore, since $f(A)\subset B\subset S^n-(P\cup Q)$, so does B.

4.6. Corollary. *Suppose $A\in 2^{S^2}_h$ and p and q are points of S^2-A. If $\{A_i\}\to A$ in $2^{S^2}_h$ and $\{p_i\}\to p$, $\{q_i\}\to q$ in S^2, then A separates p from q in S^2 if and only if for almost all i, A_i separates p_i from q_i in S^2.*

Proof. Let P and Q be topological disks such that $p\in\operatorname{Int}P$, $q\in\operatorname{Int}Q$ and $P\cup Q\subset S^2-A$. Then A separates p from q if and only if it separates P from Q and since $p_i\in P$ and $q_i\in Q$ for almost all i, the conclusion follows immediately from Theorem 4.5.

4.7. Example. *There exist two homotopically equivalent ANR's in S^2 which cannot be joined by an arc in $2^{S^2}_h$.*

Proof. Let C denote the union of a circle J in S^2 and two points p, q in different components of S^2-J, and let C' be the union of J and two points p', q' in the same component of S^2-J. Suppose φ is a homeomorphism of $[0,1]$ into $2^{S^2}_h$ with $\varphi(0)=C$ and $\varphi(1)=C'$, and for each $t\in I$, let $C_t=\varphi(t)$. Then for each t, since $C_t\underset{h}{\cong}C$, C_t is the union of three disjoint continua A_t, P_t, Q_t such that A_t has the homotopy type of a circle and P_t and Q_t are AR's. If U denotes the set of all $t\in I$ such that A_t separates P_t from Q_t, and $V=I-U$, it follows easily from Corollary 4.6 that U and V are open in I, which is a contradiction since U and V are non-empty. Hence there is no arc from C to C' in $2^{S^2}_h$ even though $C\underset{h}{\cong}C'$ and, indeed, C is homotopically deformable onto C' in S^2. Thus the requirement that C and C' be connected is essential in Theorem 4.4.

Remark. It was shown in [3] that for any finite dimensional compactum X and any $C\in 2^X_h$, the set $[C]_X=\{A\in 2^X_h|\ A\underset{h}{\cong}C\}$ is open and closed in 2^X_h. Theorem 4.4 implies that if $C\in 2^{S^2}_h$ and C is connected, so is $[C]$; minor modifications of the argument for Example 4.7 show that $[C]$ need not be connected if C is not. For the set of that example, $[C]$ has precisely two components, but in general the component structure of $[C]$ is quite complicated.

4.8. Example. *There exists in E^3 a 2-dimensional absolute retract X such that 2^X_h is not locally connected.*

Proof. It was shown by Borsuk [2] that there is a 2-dimensional AR in E^3 which is irreducible in the sense that no proper 2-dimensional subset of it is an AR.

Let $X_1, X_2, \ldots$ be a sequence of irreducible 2-dimensional AR's in E^3 such that $\{X_i\}\underset{\varrho_s}{\to}\{p\}$, for some $p\in E^3$, and such that for each i, $X_i\cap X_{i+1}$ is a single point and $X_i\cap X_j=\emptyset$ if $|i-j|>1$. It is easily seen that the set $X=\{p\}\cup\bigcup_{i=1}^{\infty}X_i$ is an AR.

For each n, let $\mathcal{U}_n$ denote the set of all AR's in X which contain X_n. Suppose $A\in\mathcal{U}_n$ and $\{A_i\}\to A$ in 2^X_h. It is not difficult to show that $\{A_i\cap X_n\}\underset{\varrho_h}{\to}X_n$ and hence for almost all i, $A_i\cap X_n$ is a 2-dimensional AR. Since X_n is irreducible, this implies that $A_i\cap X_n=X_n$ for almost all i. It follows that $\mathcal{U}_n$ is open in 2^X_h, and it is evident that $\mathcal{U}_n$ is closed. Hence since $\{X_i\} \underset{\varrho_h}{\rightarrow} \{p\}$, no neighborhood of $\{p\}$ in 2_h^X is connected. Thus 2_h^X is not locally connected at $\{p\}$.

Remarks. (1) For the space X of Example 4.8 (or, indeed, for X a single irreducible 2-dimensional AR and $p \in X$), there is no arc in 2_h^X from X to $\{p\}$ even though X and $\{p\}$ are homotopically equivalent and connected and, in fact, $\{p\}$ is a strong deformation retract of X. Hence Theorem 4.4 cannot be generalized by replacing S^2 by an arbitrary 2-dimensional ANR, or even AR.

(2) It seems intuitively clear that if X is a 2-dimensional torus and C and C' are simple closed curves in X such that C is nullhomotopic and C' is not, then there is no arc from C to C' in 2_h^X. Hence it appears that Theorem 4.4 fails also for connected ANR's in a 2-manifold.

5. Dimension and compactness. We show in this section that, for X a finite dimensional ANR, 2_h^X is usually infinite dimensional and not locally compact; more exactly, 2_h^X is locally compact if and only if $\dim X \leqslant 1$ and is infinite dimensional whenever $\dim X > 1$. In particular, if X is an n-manifold ($n > 1$), then 2_h^X is infinite dimensional at each non-isolated point and fails to be locally compact at each such point.

5.1. THEOREM. *If X is a finite dimensional* ANR, *then 2_h^X is locally compact if and only if $\dim X \leqslant 1$.*

Proof. Suppose X is an ANR and $\dim X \leqslant 1$. Since no simple closed curve is contractible to a point in a 1-dimensional set, there is a positive number d such that X contains no simple closed curve of diameter less than d.

Let A be an element of 2_h^X. It is easily shown that there exist a neighborhood $\mathfrak{U}$ of A in 2_h^X and a positive number η such that $s(B, \eta, \frac{1}{2}d)$ is true for every B in $\overline{\mathfrak{U}}$. Suppose ε is a positive number less than d; it will be shown that there is a positive number δ such that $s(B, \delta, \varepsilon)$ is true for every B in $\overline{\mathfrak{U}}$, and it will follow ([3], p. 198, Corollary 6) that $\overline{\mathfrak{U}}$ is compact.

Since X is compact and locally connected, there is a positive number γ such that every two points of X at a distance apart less than γ can be joined by a $\frac{1}{2}\varepsilon$-arc in X. Let $\delta = \min(\gamma, \eta)$ and suppose $B \in \overline{\mathfrak{U}}$ and M is a subset of B of diameter less than δ. If p and q are points of B, then since $\varrho(p, q) < \eta$ and $s(B, \eta, \frac{1}{2}d)$ is true, there is a $\frac{1}{2}d$-arc α from p to q in B, and since $\varrho(p, q) < \gamma$, there is a $\frac{1}{2}\varepsilon$-arc β from p to q in X. Since $\frac{1}{2}\varepsilon < \frac{1}{2}d$ and no two points of X can belong to more than one arc of diameter less than $\frac{1}{2}d$, it follows that $\alpha = \beta$. Hence every two points of M can be joined by a $\frac{1}{2}\varepsilon$-arc in B, and hence M is contained in a subcontinuum K of B with $\operatorname{dia} K < \varepsilon$. Since X contains no simple closed curve of diameter less than d, X does not contain infinitely many simple

closed curves [13] and hence every component of X is a regular curve [10] and therefore is hereditarily locally connected ([12], p. 99). Since $\operatorname{dia} K < d$, K is an acyclic locally connected continuum and hence is contractible. Thus M can be contracted to a point in the subset K of B of diameter less than ε, and it follows that $s(B, \delta, \varepsilon)$ is true.

Conversely, suppose X is a finite dimensional ANR with $\dim X \geqslant 2$, and let X' be a component of X with $\dim X' \geqslant 2$. Since X' is not a regular curve, it contains infinitely many simple closed curves and hence contains arbitrarily small ones, so there is a null sequence $\{C_i\}$ of simple closed curves in X and a point p of X such that $\{C_i\} \underset{\varrho_s}{\rightarrow} \{p\}$. It follows from Lemma 3.7 that if $\mathfrak{U}$ is a neighborhood of $\{p\}$ in 2_h^X, there is a positive number ε such that every ε-arc in X with one endpoint p is an element of $\mathfrak{U}$. Since X is locally connected at p, there exist a positive integer j and an arc α (possibly degenerate) irreducible from p to C_j such that $\operatorname{dia}(\alpha \cup C_j) < \varepsilon$. There is a sequence $\{A_i\}$ of arcs in $\alpha \cup C_j$ such that each A_i has one endpoint p, $A_i \subset A_{i+1}$ for every i, and $\bigcup_{i=1}^{\infty} A_i = \alpha \cup C_j$. Since each $A_i \in \mathfrak{U}$ and no subsequence of $\{A_i\}$ converges in 2_h^X, $\overline{\mathfrak{U}}$ is not compact.

5.2. COROLLARY. *If X is an n-manifold, $n \geqslant 2$, then 2_h^X is not locally compact at any non-isolated point.*

Proof. If A is a non-isolated point of 2_h^X, then A is not a component of X and hence some point of A is the (Hausdorff) limit of a null sequence of simple closed curves in $X - A$. It follows as above that no neighborhood of A in 2_h^X has a compact closure.

Remark. The arguments given above can easily be modified to show that, for any finite dimensional locally connected compactum X, the space 2_h^X is compact if and only if X contains no simple closed curve and is locally compact if and only if X contains at most a finite number of simple closed curves.

5.3. THEOREM. *If X is a finite dimensional* ANR, *D is a disk lying in X and A is an element of 2_h^X which intersects but does not contain D, then 2_h^X is infinite dimensional at A.*

Proof. Let $\mathfrak{U}$ be any neighborhood of A in 2_h^X, and let p be a point of A that is accessible from $D - A$. There is, for each positive integer n, an n-od P_n in $(D - A) \cup \{p\}$ emanating from p and having diameter less than $1/n$. It follows from Lemma 3.7 that there is a positive integer k such that if $n \geqslant k$ and Q is an n-od contained in P_n, then $A \cup Q \in \mathfrak{U}$. It will be shown that for each n, the set $K_n = \{A \cup Q \mid Q \text{ is an } n\text{-od in } P_n\}$ is an n-cube in 2_h^X, and since $K_n \subset \mathfrak{U}$ for $n \geqslant k$, it will follow that $\mathfrak{U}$ is infinite dimensional.

Denote by $a_1, a_2, \ldots, a_n$ the arcs in P_n such that $P_n = \bigcup_{i=1}^{n} a_i$, p is an endpoint of each a_i, and $a_i \cap a_j = \{p\}$ for $1 \leqslant i < j \leqslant n$. For each i, there is a homeomorphism φ_i from a_i onto $[0,1]$ such that $\varphi_i(p) = 0$, $i = 1, 2, \ldots, n$. For each n-od Q lying in P_n, denote by $Q(i)$ the endpoint of Q, other than p, that lies in a_i. The function f from K_n into the unit n-cube that takes $A \cup Q$ to the n-tuple $[\varphi_1(Q(1)), \ldots, \varphi_n(Q(n))]$ is one-to-one and onto, and f is continuous since the homotopy metric and the Hausdorff metric are equivalent on $2_h^{P_n}$.

5.4. COROLLARY. *If X is an n-manifold, $n \geqslant 2$, then 2_h^X is infinite dimensional at every non-isolated point.*

Proof. If A is a non-isolated point of 2_h^X, then A is not a component of X and it follows that there is a disk D in X such that $A \cap D \neq \emptyset$ and $D \not\subset A$. Hence by Theorem 5.3, 2_h^X is infinite dimensional at A.

Remark. It follows from the argument for Theorem 5.3 that if X is any compactum which contains an n-od, then 2_h^X is at least n-dimensional. If X is a locally connected continuum which has Menger order at least n at some point, then X contains an n-od [8] and therefore $\dim 2_h^X \geqslant n$; in particular, if $\dim X \geqslant 2$, then 2_h^X is infinite dimensional.

6. Questions. The result that the set of polyhedra in S^2 is dense in $2_h^{S^2}$ answers a minor case of an important problem posed by Borsuk: If X is a polyhedron, is the set of subpolyhedra of X dense in 2_h^X? An affirmative answer for $X = S^n$ (all n) would imply that every finite dimensional ANR has the homotopy type of a polyhedron. This latter result would also follow from an affirmative answer to the following weaker form of Borsuk's question.

6.1. *Is every ANR in E^n the homotopic limit of a sequence of polyhedra in E^{n+k}, for some k?*

It was shown by R. H. Bing ([1], Th. 10) that every topological polyhedron in E^3 is the homeomorphic limit of a sequence of polyhedra in E^3, and it follows from a result recently announced by J. L. Bryant [5] that this is true in E^n for topological polyhedra of dimension $\leqslant n-3$. Homeomorphic approximation, of course, is much stronger than approximation relative to the metric ϱ_h, which suggests the next question.

6.2. *Is every topological polyhedron in E^n the homotopic limit of a sequence of polyhedra in E^n?*

For $X = S^n$, $n \geqslant 2$, the set of proper subpolyhedra of X is of the first category in 2_h^X, but for $n = 2$, this is not true of the set of topological polyhedra properly contained in X. Our proof of this latter result depends on the fact that every connected ANR in S^2 which has no local cut point is a topological polyhedron, and thus there is little chance of modifying this argument to apply to higher dimensional spaces. Indeed, it seems likely that no such extension is possible.

6.3. *For $n \geqslant 3$, is the set of topological polyhedra properly contained in S^n a first category subset of $2_h^{S^n}$?*

There are many natural questions concerning the existence of arcs in 2_h^X, of which we mention but two.

6.4. *If X is a finite dimensional compactum and A and B are ANR's in X which can be joined by an arc in 2_h^X, must A be homotopically deformable onto B in X?*

6.5. *For which compacta X is 2_h^X locally arcwise connected? In particular, is this true for $X = S^n$, $n \geqslant 2$?*

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