predicate symbols be interpreted on \(|A'_i|\) so as to make \(A_i\) admissible and \(f_i\) an isomorphism of \(A_i\). Let \(E_i\) be the unique equivalence relation on \(|A'_i|\) such that \(R_i\) is isomorphic to \(E_i\) via \(f_i\). Then \(f_i\) induces an \(L(A)\)-isomorphism between \(A_i\) and \(E_i\). By the induction hypothesis \(A_i, E_i) \rightarrow A_i, E_i)

Thus some extension of \((A', E')\) forces \(A_i, E_i)

This completes the proof of the lemma.

Using the lemma we can now prove that \(B \leq 3^i\). We first observe that if \(A \in S(L)\) then \(3^i(A) = T\) if and only if \((A, R) \nrightarrow A\) for some \(n \in \omega\). Let \(A \in S(L)\) and suppose that \((A, R) \nrightarrow A\). We define an extension \((A', E')\) of \((A, R)\) as follows. Let \(|A'| = |A| \cup \{a\}\) where \(a = (n, m)\) is chosen in \(\omega \times \omega - |A|\) so that \(a \in \mathcal{L}(A)\). Let the predicate symbols be interpreted on \(|A'|\) so that \(A'\) is admissible and so that \(f_i: (A', E') \rightarrow (A', E')\) is an isomorphism of \(A'\). This follows from the lemma that \((A', E') \nrightarrow A\) since \(A' = A\). Thus if \(A(a)\) is true in \(3^i\) so is \(A[i]\) for some \(a \neq i\). This completes the proof that \(B \leq 3^i\).

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An isomorphism theorem of the Hurewicz-type in Borsuk’s theory of shape

by

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Introduction. In Hurewicz’s well-known paper [6] is a homomorphism \(\varphi\) defined from the \(n\text{-th}\) homotopy group \(\pi_n(X)\) into the \(n\text{-th}\) singular homology group \(H_n(X)\) with integral coefficients, for any compact, pathwise-connected space \(X\), and it is proved there (for \(n \geq 2\)) that if the space \(X\) is \((n-1)\text{-connected}\) (that is, if \(\pi_n(X) \approx \pi_{n-1}(X) \approx \cdots \approx \pi_0(X) \approx 0\)), then the homomorphism \(\varphi\) is an isomorphism.

In this note an analogous homomorphism with similar properties will be constructed on the ground of Borsuk’s theory of shape (introduced in [1]).

The singular homology groups of a pointed compactum \((X, x_0)\) will be replaced by the Vietoris–Čech homology groups of \((X, x_0)\), denoted by \(\check{H}_n(X, x_0)\), and the homotopy groups \(\pi_n(X, x_0)\) will be replaced by the so called fundamental groups \(\pi_n(X, x_0)\), defined by K. Borsuk (see also [1]).

In the general case, Hurewicz’s assumption of the \((n-1)\text{-connectedness}\) of \(X\) will be replaced by approximative \(q\text{-connectedness}\) (for \(q = 0, 1, \ldots, n-1\)) of \((X, x_0)\) (see for instance [2], p. 266, or Definition 3.1 in this paper). But if the pointed compactum \((X, x_0)\) is connected and movable (see [2], § 4) then the assumption of the approximative \(q\text{-connectedness}\) (for \(q = 0, 1, \ldots, n-1\)) is equivalent to \(\pi_n(X, x_0) \approx \pi_q(X, x_0) \approx \cdots \approx \pi_0(X, x_0)\).

\(\S\) 1 of this paper contains a modified definition of the homology groups \(\check{H}_n(X, x_0)\) and a proof of the equivalence of this definition to the original Vietoris definition. \(\S\) 2 contains a construction of a homomorphism \(\varphi: \pi_n(X, x_0) \rightarrow \check{H}_n(X, x_0)\), called the limit Hurewicz homomorphism. In \(\S\) 3 the following theorem is proved:

If the pointed compactum \((X, x_0)\) is approximatively \(q\text{-connected}\) for \(q = 0, 1, \ldots, n-1 (n \geq 2)\), then the limit Hurewicz homomorphism \(\varphi\) is an isomorphism.

\(\S\) 1. The groups \(\Gamma_q(X, x_0)\). Let \(Q\) denote the Hilbert-cube, \(X\) — a non-empty closed subset of \(Q\) and \(x_0\) — a point lying in \(X\). For any positive
real number \( \varepsilon \), the open neighbourhood of \( X \) consisting of all points \( x \in Q \) with \( d(x, X) < \varepsilon \) will be denoted by \( U^c \). The term "mapping" will always denote continuous mapping.

In this paper we will base ourselves on the definition of the singular homology groups and all concepts related to this definition, described in [5], Chapter VII.

1.1. Definition. A singular \( q \)-simplex \( T: A_q \to Q \) such that the set
\[ T(A_q) \]
contains a set \( A \subset \mathbb{Q} \) is said to be lying in \( A \). If \( A = U^c \), then \( T \) is called an \((\varepsilon, q)\)-simplex. If \( A = (x_q) \), then \( T \) is said to be lying at \( x_q \). Now, let \( a = \sum_{i=1}^{n} a_i T_i \) be a singular \( q \)-chain of \( Q \). Let \( T_i \) be singular \( q \)-simplexes. The chain \( a \) is said to be lying in \( A \) (resp. at \( x_q \)) if each \( T_i \) is lying in \( A \) (resp. at \( x_q \)); it is said to be an \((\varepsilon, q)\)-chain if each \( T_i \) is an \((\varepsilon, q)\)-simplex, and it is said to be smaller than \( 0 \geq 0 \) whenever for each \( T_i \), the image \( T_i(\partial^0) \) is of diameter less than \( \delta \).

1.2. Definition. A sequence \( \lambda = (\lambda_k) \) be a sequence of singular chains. The sequence \( \lambda \) is called an infinite singular \( q \)-chain of \( X \) if there exists a sequence of positive real numbers converging to zero such that \( \lambda_k \) is a \((\varepsilon_k, q)\)-chain. The infinite singular \( q \)-chain \( \lambda = (\lambda_k) \) is said to be lying at \( x_q \) if each \( \lambda_k \) is lying at \( x_q \).

Let \( \lambda = (\lambda_k) \) be an infinite singular \( q \)-chain of \( X \), then the sequence \( (\partial \lambda_k) \) is a singular \((q-1)\)-chain of \( X \); this chain will be denoted by \( \partial \lambda \). If \( \partial \lambda \) is lying at \( x_q \), then \( \lambda \) will be called a singular \((q-1)\)-cycle of the pair \( (X, x_q) \).

An infinite singular \( q \)-cycle \( a \) of \((X, x_q)\) is said to be homologous to zero in \((X, x_q)\) (written \( a \sim 0 \)) if there exists an infinite singular \((q-1)\)-chain \( \lambda \) such that \( a = \partial \lambda \) is lying at \( x_q \).

An infinite singular \( q \)-cycle \( a = (a_k) \) of \((X, x_q)\) is called a fundamental \( q \)-cycle of \((X, x_q)\) whenever the infinite singular cycle \( \beta = (a_k - x_k) \) is homologous to zero in \((X, x_q)\). Fundamental cycles of \((X, x_q)\) will be denoted by underlined Greek letters, e.g., \( \alpha = (\alpha_k) \). It is easy to see that the set of all fundamental \( q \)-cycles of \((X, x_q)\) is a subgroup of the group of all infinite singular \( q \)-chains; this subgroup will be denoted by \( Z_q(X, x_q) \). The subgroup of \( Z_q(X, x_q) \) consisting of all infinite singular \( q \)-cycles of \((X, x_q)\) which are homologous to zero in \((X, x_q)\) will be denoted by \( B_q(X, x_q) \). Let \( I_q(X, x_q) \) denote the factor group \( Z_q(X, x_q)/B_q(X, x_q) \).

Let \( \varepsilon \) be a positive number. A \( q \)-dimensional \( \varepsilon \)-simplex lying in \( X \) is an ordered set of \( q+1 \) points of \( X \) (called the vertices of the \( \varepsilon \)-simplex) with diameter less than \( \varepsilon \). An \( \varepsilon \)-simplex is said to be lying at \( x_q \), if any vertex of it is equal to \( x_q \). A formal finite linear combination \( a = \sum_{i=1}^{n} b_i \sigma_i \) of \( q \)-dimensional \( \varepsilon \)-simplexes \( \sigma_i \) with integral coefficients \( b_i \) is called a \( q \)-dimensional \( \varepsilon \)-chain in \( X \). If any \( \varepsilon \)-simplex of this combination is lying at \( x_q \), then the \( \varepsilon \)-chain \( a \) is said to be lying at \( x_q \).

For any two \( q \)-dimensional \( \varepsilon \)-chains \( a = b_1 \sigma_1 + \ldots + b_n \sigma_n \) and \( a' = b'_1 \sigma_1' + \ldots + b'_n \sigma_n' \) define the sum \( a + a' \) as the \( q \)-dimensional \( \varepsilon \)-chain \( b_1 \sigma_1 + \ldots + b_n \sigma_n + b'_1 \sigma_1' + \ldots + b'_n \sigma_n' \). This addition is a group operation on the set of all \( q \)-dimensional \( \varepsilon \)-chains. Equivalently, the group of all \( q \)-dimensional \( \varepsilon \)-chains can be defined as the free Abelian group generated by the set of all \( q \)-dimensional \( \varepsilon \)-simplexes.

Let \( \sigma = (\tau_0, \tau_1, \ldots, \tau_q) \) be a \( q \)-dimensional \( \varepsilon \)-simplex. The \((q-1)\)-dimensional \( \varepsilon \)-chain \( \partial \sigma = \sum_{i=0}^{q} (-1)^i \tau_i (\tau_0, \tau_1, \ldots, \tau_{i-1}, \tau_{i+1}, \ldots, \tau_q) \) is called the boundary of \( \sigma \). Then one extends linearly the boundary-operation \( \partial \) on the whole group of \( q \)-dimensional \( \varepsilon \)-chains, i.e., \( \partial \sum_{i=0}^{q} b_i \sigma_i = \sum_{i=0}^{q} b_i \partial \sigma_i \).

A sequence \( \gamma = (\gamma_k) \) of \( q \)-dimensional \( \varepsilon \)-chains, where \( \gamma_k \) converges to \( a \), is called an infinite \( q \)-dimensional chain (in the Vietoris sense). If each \( \gamma_k \) is lying at \( x_q \), then the infinite \( q \)-dimensional chain \( \gamma \) is said to be lying at \( x_q \). For any two infinite \( q \)-dimensional chains \( \gamma = (\gamma_k) \) and \( \gamma' = (\gamma'_k) \) define the sum \( \gamma + \gamma' \) as the infinite \( q \)-dimensional chain \( (\gamma_k + \gamma'_k) \). The set of all infinite \( q \)-dimensional chains together with this addition is an Abelian group.

For an infinite \( q \)-dimensional chain \( \gamma = (\gamma_k) \), the infinite \((q-1)\)-dimensional chain \( \partial \gamma = (\partial \gamma_k) \) is called the boundary of \( \gamma \) and denoted by \( \partial \gamma \). An infinite \( q \)-dimensional chain \( \gamma \) is said to be an infinite \( q \)-dimensional cycle in \((X, x_q)\), if \( \partial \gamma \) is lying at \( x_q \). An infinite \( q \)-dimensional cycle \( \gamma \) is said to be homologous to zero in \((X, x_q)\), if there is an infinite \((q-1)\)-dimensional chain \( \kappa \) such that the infinite \( q \)-dimensional chain \( \partial \kappa = -\gamma \) is lying at \( x_q \). An infinite \( q \)-dimensional cycle \( \gamma \) is said to be a \( q \)-dimensional true cycle in \((X, x_q)\), if the infinite cycle \( \gamma_k \) is homologous to zero in \((X, x_q)\).

Let \( Z_q(X, x_q) \) denotes the group of all \( q \)-dimensional true cycles in \((X, x_q)\) and \( B_q(X, x_q) \) the group of all \( q \)-dimensional infinite cycles in \((X, x_q)\), which are homologous to zero in \((X, x_q)\). Certainly, \( B_q(X, x_q) \) is a subgroup of \( Z_q(X, x_q) \). The quotient group \( Z_q(X, x_q)/B_q(X, x_q) \) is denoted by \( H_q(X, x_q) \) and called the \( q \)th homology group (in the Vietoris sense) of the pointed compactum \((X, x_q)\), with coefficients, as the coefficients.
1.3. Theorem. The groups $T_q(X, a_0)$ and $\tilde{H}_q(X, a_0)$ are isomorphic.

Proof. Let $p : Q \to X$ be a (not necessary continuous) function such that $q(x, p(x)) = q(x, X)$ for each $x \in Q$.

Let $T : A_\gamma \to Q$ be an $(\varepsilon, r)$-simplex of diameter less than $\varepsilon$ and let $d^0, d^1, \ldots, d^n$ denote the vertices of $A_\gamma$. Then the system

$$\{p(T(d^0)), p(T(d^1)), \ldots, p(T(d^n))\}$$

is an $r$-dimensional $\varepsilon$-simplex (in the Vietoris sense) lying in $X$, let it be denoted by $P(T)$. Now, if $\lambda = \sum_{c_i} \lambda_i T_i$ is an $(\varepsilon, r)$-chain smaller than $\lambda$, then $\sum_{c_i} a_i P(T_i)$ is an $r$-dimensional $\varepsilon$-chain (in the Vietoris sense) lying in $X$, let it be denoted by $P(\lambda)$. Observe, that $P(\lambda_i + \lambda_j) = P(\lambda_i) + P(\lambda_j)$ and $P(\lambda_i - \lambda_j) = \partial P(\lambda_i)$. Observe also, that for any $(\varepsilon, r)$-cycle $\lambda$ there exists an $(\varepsilon, r)$-cycle $\lambda'$ which is smaller than $\lambda$ such that $\lambda'$ and $\lambda'$ are homologous in $(X, a_0)$ (for instance, $\lambda'$ can be obtained as the result of an iterated barycentric subdivision of $\lambda$; compare Theorem 8.2 in [3], chap. VII, p. 197).

Let $\alpha$ be a fundamental $g$-cycle in $(X, a_0)$. There exists an element $\alpha' = (a_0)\alpha$ of the homology class $[\alpha]$, such that $a_0 \alpha$ is an $(\varepsilon, q)$-cycle smaller than $\varepsilon a_0$ (where $0 < \varepsilon a_0 < 0$ for $k = 1, 2, \ldots$). For this element $\alpha'$, define $P(\alpha') = \{P(a_0\alpha)\}$. The sequence $[P(a_0\alpha)]$ is an infinite cycle (in the Vietoris sense) in $(X, a_0)$. In order to prove, that it is a true cycle, observe, that there exists an infinite singular $(q+1)$-chain $\mu = (\mu_k)$ such that $(\mu_k - a_0\alpha_k)$ is $\mu_k$ lying at $a_0$, since $a_0 = (a_0)$ is a fundamental $g$-cycle in $(X, a_0)$. Moreover, the sequence $\mu$ can be assumed to be such that $\mu_k$ is a $(\varepsilon, q+1)$-chain smaller than $\varepsilon a_0 k$, where $0 < \varepsilon k a_0$.

Therefore, $P(\alpha') = \alpha' - \partial P(\alpha)$ is an infinite chain in $(X, a_0)$, and the sequence $P(\alpha)$ is a Vietoris true cycle. An analogous argumentation shows, that the homology class $[P(\alpha')]$ does not depend of the choice $\alpha'$ in the homology class $[\alpha]$. Thus, the formula $\omega([\alpha]) = [P(\alpha')]$ defines a homomorphism $\omega : T_q(X, a_0) \to \tilde{H}_q(X, a_0)$ ($q = 0, 1, \ldots$).

Now, let $\sigma = (\varepsilon_0 a_1, \ldots, \varepsilon_n a_n)$ be an $r$-dimensional (Vietoris) $\varepsilon$-simplex in $X$. The linear mapping $T : A_\sigma \to Q$ with $T(d^0) = 0$ for all $0 < \varepsilon_i < q$ is a singular $(\varepsilon, r)$-simplex in $T_q(X, a_0)$. Denote this singular simplex by $P(\sigma)$. If $\gamma = \sum_{i=0}^{n} b_i P(\sigma_i)$ is a singular $(\varepsilon, r)$-chain such that $P(\gamma) = \gamma$, then $[P(\gamma)]$ is an infinite $(\varepsilon, r)$-dimensional chain, and the formula $\omega([\sigma]) = [P(\gamma)]$ is a singular $(\varepsilon, r)$-chain such that $\omega([\sigma]) = \omega([\gamma])$. Observe, that $\omega([\sigma]) = P(\gamma)$, and $P(\gamma') + P(\gamma'') = P(\gamma'') + P(\gamma'')$. This shows that

if $\gamma$ is a true (Vietoris) $g$-dimensional cycle, then $P(\gamma')$ is a fundamental $g$-cycle such that $P(\gamma') = \gamma$. This proves that $\omega$ is an epimorphism.

To prove that $\omega$ is a monomorphism, suppose that $a = \sum_{i=1}^{n} a_i T_i$ is a singular $(\varepsilon, g)$-cycle smaller than $\varepsilon$ and $T_i' : A_\gamma \to Q$ is the linear map with $T_i(d^i) = P_i a_i T_i$ for $0 < \varepsilon_i < g$ and each $1 < j < m$, and the cycle $\sum_{i=1}^{n} a_i (T_i - T_i')$ is equal to the boundary of the $(q+1)$-dimensional prism-chain (see [5], chap. VII, § 6.7) $D = \sum_{i=1}^{n} a_i DT_i$ where the mapping $DT_i : I \times A_\gamma \to Q$

for each $i$ is defined by the formula

$$DT_i(t, v) = t T_i(v) + (1-t) T_i'(v)$$

for $t \in I$, $v \in A_\gamma$ (the sign $+$ denotes here addition in the linear Hilbert space $H \supset Q$).

Therefore, if $\alpha$ is a fundamental $g$-cycle, then $a = \omega(\alpha)$ and $P(\omega(\alpha))$ are homologous in $(X, a_0)$ and, in particular, if $P(\alpha)$ is homologous to zero in $(X, a_0)$, then $\omega(a)$ is also homologous to zero in $(X, a_0)$, i.e. $\omega$ is a monomorphism, which completes the proof.

§ 2. The limit Hurewicz homomorphism. The fundamental groups $\pi_1(X, a_0)$ are defined by K. Borsuk ([1], pp. 246–252) as follows:

Let $X$ denote, as before, a closed subset of the Hilbert-cube $Q$ and let $a_0 \in X$. The $g$-sphere will be denoted by $S^g$; Let $s \in S^g$ be a base-point of $S^g$. A sequence of pointed mappings $T_k : (S^g, s_0) \to (Q, q_k)$ ($k = 1, 2, \ldots$) will be called an approximative map of $(S^g, s_0)$ towards $(X, a_0)$ whenever for any neighbourhood $U$ of $X$ the pointed homotopy $\xi_k : S^g \to U$, $\xi_k(z) = s_k, z \in S^g$, holds for almost all $k$. This approximative map is denoted by $(\varepsilon_k, S^g, s_0) \to (X, a_0)$ or, more briefly, by $\varepsilon$.

Two approximative maps

$$\varepsilon = (\varepsilon_k, S^g, s_0) \to (X, a_0) \quad \text{and} \quad \varepsilon' = (\varepsilon_k', S^g, s_0) \to (X, a_0)$$

are said to be homotopic (written $\varepsilon \simeq \varepsilon'$) whenever for any neighbourhood $U$ of $X$ the pointed homotopy $\xi_k \simeq \xi_k'$ in $(U, x_0)$ holds for almost all $k$, i.e., whenever the "mixed" sequence $\xi_1, \xi_2, \xi_3, \ldots$ is an approximative map of $(S^g, s_0)$ towards $(X, a_0)$. The equivalence class of the approximative map $\varepsilon$ under the relation $\simeq$ is called the homotopy class of $\varepsilon$ or the approximative class from $(S^g, s_0)$ towards $(X, a_0)$, represented by the approximative map $\varepsilon$.

Now, let $([\varepsilon])$ and $([\varepsilon'])$ be approximative classes, where $\varepsilon = (\varepsilon_k, S^g, s_0) \to (X, a_0)$ and $\varepsilon' = (\varepsilon_k', S^g, s_0) \to (X, a_0)$). Define $[\varepsilon] + [\varepsilon'] = ([\varepsilon_k] + [\varepsilon_k'])$. If $\gamma = ([\varepsilon] + [\varepsilon'])$ is a fundamental $g$-dimensional cycle, then $P(\gamma)$ is a fundamental $g$-cycle such that $P(\gamma) = \gamma$. This proves that $\omega$ is an epimorphism.
An isomorphism theorem of the Hurewicz-type

Let us remember that the $i$th $(q-1)$-dimensional face $T^0$ of the singular q-dimensional simplex $T: A_0 \to T (0 \leq i \leq q)$ is in [5] (chap. VII) defined as the composition $T^0 = T \circ q_i$, where $q_i$ is the suitable simplicial inclusion of $A_{i-1}$ into $A_i$. Let us write $A_0^0 = q_0(A_{-1})$ and $A_q = \bigcup_{i=0}^{q} A_i^0$.

For any subset $U$ of $Q$, the group of all $q$-dimensional singular chains lying in $U$ will be denoted by $C_q(U)$. Observe that for any homotopy $h: I \times A \to B$ (where $I$ is the unit interval $[0, 1]$ and $A$, $B$ are topological spaces) there is a unique corresponding family of mappings $h_t: A \to B (0 \leq t \leq 1)$ defined by the formula $h_t(x) = h(t, x), x \in A, t \in I$. Thus the homotopy will be sometimes considered as the family $h_t$.

3.3. LEMMA. If the pair $(X, x_0)$ is approximately $s$-connected for all $s = 0, 1, ..., n-1 \geq 2$, then for any neighbourhood $U \supset X$ there exists a neighbourhood $U \supset X$ such that to each singular q-simplex $T: A_0 \to U (q = 0, 1, 2, ...) can be assigned a homotopy $h^T: I \times A_0 \to U$ such that the following conditions are satisfied:

(i) $h^T_0 = T$

(ii) If $T^0$ is the $i$th face of $T (0 \leq i \leq q)$, then $h^T_0$ is the $i$th face of the simplex $h^T_i$ for each $t \in I$, i.e. $h^T_0 = h^T_0 = q_i^T$

(iii) $h^T_2$ is a reduced singular simplex;

(iv) If $T$ is a reduced singular simplex, then $h^T = T$ for each $t \in I$.

Proof. Observe first that approximately $s$- connectivity for all $s = 0, 1, ..., n-1$ implies that

(*) there is a system of neighbourhoods $U_0, U_1, ..., U_n$ with $X \subseteq U_0 \subseteq U_1 \subseteq ... \subseteq U_n = U$ such that each mapping $f: (S^q, x_0) \to (V, x_0)$ is essential in $(U_{k+1}, x_0)$ for $q = 0, 1, ..., n-1$.

The homotopy $h^T$ will be defined inductively with respect to the dimension of the singular simplex $T$.

1. Let $T$ be a $0$-dimensional singular simplex. If $T$ maps $A_0$ into $x_0$, then $h^T$ is the constant map into $x_0$. Otherwise, let $h^T$ be a homotopy in $U_k (U_k$ as defined in (*)) such that $h^T_0 = T$ and $h^T_1$ is the constant map into $x_0$ for any $t \in [1/n, 1]$.

2. Suppose $1 \leq q \leq n-1$ and the homotopies $h^T$ are defined for all singular simplices $T$ of dimension $q-1$, so that the conditions (i)-(iv) are satisfied and, moreover, it $T$ is of dimension $q-1$, then for any $t \in [q/n, 1]$ $h^T_{q/n}$ is the constant map into $x_0$.

Let $T: A_0 \to U_0$ be a $q$-dimensional singular simplex. If $T$ is reduced, then $h^T_{q/n} = T$, for each $t \in I$. Suppose now that $T$ is not reduced. Let $T^{00}: A_{i-1} \to U_0$ and $T^{01}: A_{i-1} \to U_1$ denote respectively the $i$th and the $j$th
face of $T$. Observe that if $T': A_q \to U_q$ is a common face of $T^{(0)}$ and $T^{(0)}$, i.e. if there are simplicial inclusions $e': A_q \to A_{q-1}$ and $f': A_q \to A_q$ such that $T = T^{(0)} \circ e' = T^{(0)} \circ f'$, then by (ii) the equality
\[ h^{(0)} \mid_{X \times I \times T^{(0)}} = h^{(0)} \mid_{X \times I \times T^{(0)}} \] holds.

Thus, a continuous function
\[ g: \left[ 0, \frac{q}{n} \right] \times A_q \cup \{0\} \times A_q \to U_q \]
is well defined by the following formula:
\[ g(t, v) = \begin{cases} h^{(0)}(u) & \text{if} \ v = e_i'(u) \text{ for some} \ i \text{ and} \ u \in A_{q-1}, \\ T(v) & \text{if} \ t = 0. \end{cases} \]

It is easy to see that there is a retraction
\[ r: \left[ 0, \frac{q}{n} \right] \times A_q \to \left[ 0, \frac{q}{n} \right] \times A_q \cup \{0\} \times A_q; \]
let us write
\[ h' = r \circ g: \left[ 0, \frac{q}{n} \right] \times A_q \to U_q. \]

It follows by the inductive assumption that
\[ h' \left( \left[ 0, \frac{q}{n} \right] \times A_q \right) = \{x_0\}, \]
and by (s) there exists a homotopy
\[ h': \left[ 0, \frac{q + 1}{n} \right] \times A_q \to U_{q+1} \]
such that
\[ h' \mid_{[0, \frac{q}{n}] \times A_q} = h' \mid_{[0, \frac{q}{n}] \times A_q} \quad \text{and} \quad h' \left( \left[ 0, \frac{q + 1}{n} \right] \times A_q \right) = \{x_0\}. \]

The homotopy $h^F: I \times A_q \to U_{q+1}$, defined by the following formula
\[ h^F(t, v) = \begin{cases} h'(t, v) & \text{for} \ t \in [0, \frac{q}{n}], \\ h''(t, v) & \text{for} \ t \in \left[ \frac{q}{n}, \frac{q + 1}{n} \right], \\ x_0 & \text{for} \ t \in \left[ \frac{q + 1}{n}, 1 \right], \end{cases} \]
satisfies the conditions (i)-(iv) and, moreover, $h^F(I \times A_q) \subset U_{q+1}$ and $h^F(v)$
\[ = x_0 \text{ if} \ \frac{q + 1}{n} \leq t \leq 1. \]

3° Suppose now that $q \geq n$ and assume that the homotopy $h^F: I \times A_q \to U_{q+1}$ is defined for each $(q-1)$-dimensional singular simplex $T$ such that the conditions (i)-(iv) hold. Now, let $T: A_q \to U_q$ be a $q$-dimensional simplex. If $T$ is reduced, then define $h^F$ by $h^F = T$ for each $t \in I$. Suppose that $T$ is not reduced. Then, as in 2°, the homotopies $h^F(t) (t = 0, 1, ..., q)$ together with the mapping $T$ yield a mapping $g: I \times A_q \cup \{0\} \times A_q \to U_{q+1}$ which can be extended to a mapping $h^F: I \times A_q \to U_{q+1}$, since $I \times A_q \cup \{0\} \times A_q$ is a retract of $I \times A_q$. The family $\{h^F\}$ of all homotopies obtained by this method satisfies the required conditions.

3.4. LEMMA. If the pair $(X, x_0)$ is approximately $s$-connected for all $s = 0, 1, ..., n+1 (n \geq 2)$, then for any neighbourhood $U$ of $T$ there is a neighbourhood $U_x$ of $X$ and for any $q = 0, 1, ..., n+1$ there is a homomorphism $\tau_q: C_q(U_x) \to C_q(U)$ such that the following conditions are satisfied:

(a) for any $\lambda \in C_q(U_x)$ the singular chain $\tau_q(\lambda)$ is reduced,
(b) if the chain $\lambda \in C_q(U_x)$ is reduced, then $\tau_q(\lambda) = \lambda$,
(c) $\tau_q(\lambda) = \tau_{q-1}(\lambda)$ for any $\lambda \in C_q(U_x)$, $q = 1, 2, ..., n+1$,
(d) if the chain $\alpha \in C_q(U_x)$ is a cycle (mod $x_0$), then the cycles $\alpha$ and $\tau_q(\alpha)$ are homologous in $(U, x_0)$.

PROOF. The set of all singular $q$-dimensional simplices in $U_x$ is a system of generators of the group $C_q(U_x)$; hence it is sufficient to define the function $\tau_q$ on any singular $q$-dimensional simplex $T: A_q \to U_q$, and then to extend it (linearly) to a homomorphism of the whole group $C_q(U_x)$. Let $h^F$ be the homotopy obtained by Lemma 3.3. Define $\tau_q(T) = h^F$. The singular simplex $h^F$ lies in $U_q$; therefore $\tau_q: C_q(U_x) \to C_q(U)$. The conditions (a) and (b) are evidently satisfied. To prove condition (c) it is sufficient to verify it for the chain $\lambda = T$, where $T: A_q \to U_q$ is a singular $q$-simplex. On the one hand,
\[ \partial_q \tau_q(T) = \partial_q h^F = \sum_{t=0}^{q} (-1)^t \partial_q h^F \cdot e^F_t; \]
on the other hand,
\[ \tau_{q-1}(\partial_q T) = \tau_{q-1} \left( \sum_{t=0}^{q} (-1)^t T \cdot e^F_t \right) = \tau_{q-1} \left( \sum_{t=0}^{q} (-1)^t T^6 \right) = \sum_{t=0}^{q} (-1)^t \tau_q(T^6) = \sum_{t=0}^{q} (-1)^t h^F_{7, t}; \]
Condition (ii) of Lemma 3.3 yields the equality $h^F_{7, t} = h^F_{7, t} \cdot e^F_t$, therefore $\partial_q \tau_q(T) = \tau_{q-1}(\partial_q T)$. To verify condition (d), consider a singular $q$-cycle
a = \sum_{j} a_j T_j \in (U, x_0). \text{ For any } j, \text{ the homotopy } h^{(j)} \text{ is a } (q+1)-\text{dimensional prism in } U \text{ (see \cite{5}, chap. 7, \S 6-7) and}
\delta_a h^{(j)} = h^{(j-1)} - h^{(j)} - \sum_{k=0}^{q} (-1)^k h^{(j-k)} \cdot
Therefore
\delta \left( \sum_{i} a_i h^{(j)} \right) = \tau_0(a) - a - \sum_{j} a_j \sum_{k=0}^{q} (-1)^k h^{(j-k)},
which shows that \tau_0(a) and a are homologous in \((U, x_0),\) since the prism-chain \(\sum_{i} a_i h^{(j)}\) lies at \(x_0,\) and the proof is complete.

Consider now a mapping \(T : \Delta_{n} \to S^n\) which maps the boundary \(\Delta_{n}\) of \(\Delta_{n}\) onto \(\Delta_{n}\) and is 1-1 for all other points. \(T\) is a singular n-cycle in \((S^n, x_0)\), moreover, the homology class \(e = [T]_H\) of this cycle is a generator of the n-singular homology group \(H_n(S^n, x_0)\).

Let \(U\) be a subset of the Hilbert-cube \(Q\) with \(x_0 \in U\) and let \(\xi : (S^n, x_0) \to (U, x_0)\) be a mapping. The composition \(\xi \circ T : \Delta_{n} \to U\) is a reduced singular n-simplex; let it be denoted by \(T_\alpha\). Observe that the singular chain \(T_\alpha\) is an n-cycle in \((U, x_0)\) and \(\xi_\alpha(e) = [T_\alpha]\), where \(\xi_\alpha : H_n(S^n, x_0) \to H_n(U, x_0)\) is the homomorphism induced by \(\xi\).

3.5. Lemma. Let \([\xi_1], [\xi_2], \ldots, [\xi_m] \in \pi_n(U, x_0) (n \geq 2)\) be the homotopy classes of the mappings \(\xi_1, \xi_2, \ldots, \xi_m : (S^n, x_0) \to (U, x_0)\). If there exists a reduced \((n+1)-\text{dimensional chain } \lambda \in \pi_n(U, x_0) \text{ such that } \lambda T_\alpha = \sum_{j=1}^{m} a_j T_j, \text{ then } \sum_{j=1}^{m} a_j [\xi_j] = 0 \text{ in the group } \pi_n(U, x_0) \text{ (compare \cite{5}, p. 527).}

Proof. It is sufficient to prove the lemma under the assumption that \(\lambda = T\) (where \(T : \Delta_{n+1} \to U\) is a reduced singular \((n+1)-\text{dimensional simplex),}\) since the group \(\pi_n(U, x_0)\) is Abelian for \(n \geq 2). \text{ Then there exist mappings } \xi_i : (S^n, x_0) \to (U, x_0), \text{ for } i = 0, 1, \ldots, n+1, \text{ such that } T_{\lambda_i} = T^{(0)} \text{ (since } T \text{ is reduced); therefore } \lambda = \sum_{i=0}^{n} (-1)^i T_{\lambda_i}. \text{ Let } dT \text{ denote the restriction } \xi_{\lambda_i} : \Delta_{n+1} \to U, \text{ and let } (dT)_{\lambda_i} = \pi_n(\Delta_{n+1}) \to \pi_n(U, x_0) \text{ be the homomorphism of the homotopy groups induced by } dT. \text{ There is a generator } e' \text{ of the group } \pi_n(\Delta_{n+1}) \approx Z, \text{ such that } (dT)_{\lambda_i}(e') = \sum_{i=0}^{n} (-1)^i [\xi_i] \in \pi_n(U, x_0), \text{ since } T \text{ is reduced. But the inclusion map } j : \Delta_{n+1} \to \Delta_{n+1} \text{ induces the } 0-\text{homomorphism } j_0 \text{ (indeed, } \Delta_{n+1} \text{ is contractible) and } (dT)_{\lambda_i}(e') = T_{\lambda_i}(e') = 0, \text{ where } T_i : \pi_n(\Delta_{n+1}) \to \pi_n(U, x_0) \text{ is the homomorphism induced by } T.

Proof of Theorem 3.2. Since \(g\) is defined as the composition \(\omega \circ \psi\) and \(\omega\) is an isomorphism (see \S 1), it is sufficient to show that \(\psi\) is also an isomorphism.

Let \(U\) be a neighbourhood of \(X\) in \(Q,\) Observe that if \(T : \Delta_{n} \to U\) is a reduced singular n-simplex, then there exists a mapping \(\xi : (S^n, x_0) \to (U, x_0)\) such that \(T = T_\xi.\) Therefore, for any reduced singular n-cycle \(\alpha\) in \((U, x_0)\) there exists a mapping \(\xi_{\alpha} : (S^n, x_0) \to (U, x_0)\) such that \(\xi_{\alpha}(e) = [\alpha].\)

1° We will prove that \(\psi\) is an epimorphism.

Let \(\alpha = \{a_k\}\) be a fundamental n-cycle of \((X, x_0)\). Let \((U^n)\) be a sequence of neighbourhoods of \(X\) such that \(U^n = U^n (\text{see Lemma 3.4})\) and \(0 < a_k < 0.\) There is an infinite singular \((n+1)-\text{chain } \lambda = (\lambda_k)\) such that \(\lambda_k = a_k \circ a_k, \text{ (mod } a_k),\) since \(a_k\) is fundamental. Suppose that \(\lambda_k\) lies in \(U^n.\) The generality of the proof is not reduced by this assumption, since instead of \(\lambda\) we can take a suitable subsequence \(\lambda'\) of \(\lambda_k\) which is always a fundamental n-cycle homologous to \(\lambda\) in \((X, x_0),\) and which satisfies this assumption. Let \(\beta_k = \pi_n(a_k)\) and \(\beta_k = \pi_n(\lambda_k)\) (for \(k = 2, 3, \ldots\)), where \(\tau_k = C_{S^n} \circ \xi_\alpha(e) = C_{S^n} (U^n) (q = n, n+1)\) is the homomorphism defined in 3.4. Clearly, the sequence \(\beta = (\beta_k)\) is a fundamental n-cycle of \((X, x_0),\) since \(\beta_k = \beta - \beta_{k+1}(\text{mod } a_k)\) and, moreover, \(\beta, \alpha, \lambda\) are homologous in \((X, x_0),\)

The singular n-cycle \(\beta_k\) in \((U^n, x_0)\) is reduced (for any \(k\)); hence there exists a sequence of mappings \((\xi_k) : (S^n, x_0) \to (U^n, x_0),\) such that \(\xi_k(e) = \beta_k (k = 2, 3, \ldots).\) We will now show that the mappings \(\xi_k : (S^n, x_0) \to (U^n, x_0)\) forms an approximative map of \((S^n, x_0)\) towards \((X, x_0).\)

Let \((\xi_k - \xi_{k+1})^* (e) = \beta_k - \beta_{k+1}(\text{mod } a_k)\). On the other hand, \(\beta_k - \beta_{k+1}\) is a boundary in \((U^n, x_0)\) of the reduced chain \(a_k.\) This and Lemma 3.5 yield the homotopy \(\xi_k \simeq_{k+1} \xi_{k+1} (\text{in } U^n, x_0).\) Hence \(\xi_\alpha = (\xi_{k}, (S^n, x_0) \to (X, x_0))\) is an approximative map. Moreover, \(\psi([\xi]) = [\alpha],\) which proves that \(\psi\) is an epimorphism.

2° Suppose now that \(\psi([\xi]) = 0,\) where \(\xi = (\xi_1, (S^n, x_0) \to (X, x_0)).\) There exists an infinite singular \((n+1)-\text{chain } \lambda = (\lambda_k)\) of \(X\) such that \(\lambda_k\) is a reduced chain in \(U^n(\text{ mod } a_k)\) and \(\lambda_k = T_{\lambda_k},\) and Lemma 3.5 implies that the mapping \(\xi_{\lambda_k} : (S^n, x_0) \to (U^n, x_0)\) is null-homotopic in \((U^n, x_0).\) Thus \(\xi_\alpha = 0,\) that is \([\xi] = 0,\) which proves that \(\psi\) is a monomorphism.

3.6. Remark. If the pointed compactum \((X, x_0)\) is movable, then the condition \(\psi_\alpha(X, x_0) \neq 0\) for \(q = 0, 1, 2, \ldots\) (see \cite{2}, p. 271 and \cite{4}).

3.7. Corollary. If the pointed compactum \((X, x_0)\) is movable and \(\psi_\alpha(X, x_0) \neq 0\) for \(q = 0, 1, 2, \ldots\) \((n \geq 2),\) then the limit Hurewicz homomorphism \(\psi_\alpha(X, x_0) \to H_\alpha(x_0, x_0)\) is an isomorphism.
Spaces of ANR's

by

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1. Introduction. For a finite dimensional compactum $X$, let $2^X$ denote the hyperspace of ANR's lying in $X$, with the metric $d_h$ introduced and studied by K. Borsuk [3]. Among many results established by Borsuk, we mention here that $2^X$ is complete and separable, and the topology of $2^X$ is characterized by homotopic convergence: a sequence $\{A_i\}$ converges to $A$ in $2^X$ if and only if (1) $\{A_i\}$ converges to $A$ in the Hausdorff sense and (2) for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for each $i$, every subset of $A_i$ of diameter less than $\delta$ is contractible to a point in a subset of $A_i$ of diameter less than $\varepsilon$. Thus two ANR's in $X$ which are "close" relative to the metric $d_h$ have similar homotopy properties.

In particular, as was shown in [3], for each $A \subset 2^X$, all ANR's in $X$ which are sufficiently close to $A$ in $2^X$ are homotopically equivalent to $A$.

The aim of the present paper is to investigate topological properties of the space $2^X$, primarily for $X = S^3$.

It is evident that the subspace $C_x$ of $2^X$ consisting of all connected ANR's in $X$ is open and closed in $2^X$. Our attention will frequently be directed to (this complete) subspace of $2^X$ rather than to the whole space. For notational convenience, $C_x$ will be denoted simply by $C$.

We show that each pair of homotopically equivalent elements of $C$ can be joined by an arc in $2^X$, thus characterizing the components of $C$ as precisely the sets $[C] = \{A \in 2^X | A \notin C\}$, for $C \subset C$. It is clear that $S^3$ is an isolated point of $2^X$, since no ANR properly contained in $S^3$ is homotopically equivalent to $S^3$, but there are no other isolated points in $2^X$. In fact, $2^X$ is infinite dimensional at every point of $2^X - \{S^3\}$, and is not locally compact at any point except $S^3$.

As partial answers to questions posed by Borsuk ([3], p. 201, [4], p. 221), we show that the set of polyhedra properly contained in $S^3$ is dense in $2^X$ and is of the first (Baire) category. On the other hand, the set of topological polyhedra in $S^3$ is of the second category (in fact, residual) in $2^X$.  

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