Now let ϱ be such that for all $\gamma < \varrho$, ω^γ is a $\oplus$-prime. If $\mu < \omega^\varrho$ and $\nu < \omega^\varrho$, there are $\delta < \varrho$ and $M < \omega$ for which $\mu < \omega^\delta \cdot M$ and $\nu < \omega^\delta \cdot M$. Thus

$$\alpha \oplus \beta < \omega^\delta \cdot M \oplus \omega^\delta \cdot M < \omega^\varrho.$$

Consequently, ω^ϱ is a $\oplus$-prime.

COROLLARY. a) *A sufficient condition for a natural sum $\oplus$ to be continuous is for every $\oplus$-prime to be $\oplus$-irreducible.*

b) *Let λ be a prime component. A sufficient condition for a λ-natural sum $\oplus_\lambda$ to be continuous is for every $\oplus_\lambda$-prime to be $\oplus_\lambda$-irreducible.*

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Some remarks on selectors (I)

by

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The Axiom of Choice states that every family of pairwise disjoint non-void sets $\mathfrak{X} = \langle X_i \rangle_{i \in I}$ has a selector, i.e. there is a set S such that $|S \cap X_i| = 1$ for every $i \in I$. The situation is quite different when we consider non-disjoint families. In the present paper we will study the problem of the existence of selectors of the families which have large subfamilies with selectors. (Of course the Axiom of Choice is assumed throughout.) So our problem has rather a "compactness" character.

We say that a family $\mathfrak{X} = \langle X_\alpha \rangle_{\alpha < \varkappa}$ has *partial selectors* if for every $\beta < \varkappa$ the family $\mathfrak{X} \upharpoonright \beta = \langle X_\alpha \rangle_{\alpha < \beta}$ has a selector. $\mathbf{E}(\varkappa, \lambda)$ (or respectively $\mathbf{F}(\varkappa, \lambda)$ (¹)) will denote the following statement: *For every family $\mathfrak{X} = \langle X_\alpha \rangle_{\alpha < \varkappa}$ of sets of powers $< \lambda$ (or $= \lambda$ respectively) if $\mathfrak{X}$ has partial selectors then $\mathfrak{X}$ has a selector.*

It is easy to see that for each infinite cardinal $\varkappa$, the statement $\mathbf{E}(\varkappa, \omega_0)$ is provable in **ZFC**. In [1], P. Erdös and A. Hajnal ask: *Does* $\mathbf{E}(\omega_2, \omega_1)$ *hold?* We give a partial answer (Corollary 4.6) to the question. The main result is contained in § 4 (Theorem 4.4). It states that under the assumption of **GCH**, the property $\mathbf{E}(\varkappa, \varkappa)$ is equivalent to the weak compactness of $\varkappa$.

The paper is arranged as follows: in § 0 we give some neccessary definitions, and in § 1 we prove some simplest properties of the statements **E** and **F**. In particular, from 1.1.5 it follows that the investigations of the statement **F** can be reduced to **E** with respective parameters. In § 2, we give the proof of a part of 4.3, namely that the weak compactness of $\varkappa$ implies $\mathbf{E}(\varkappa, \varkappa)$. In § 3, we study connections between $\mathbf{E}(\varkappa, \varkappa)$ and the *tree property* of $\varkappa$. From the results of §§ 3 and 4 it follows that if we do not assume the strong inaccessibility of $\varkappa$, then the property $\mathbf{E}(\varkappa, \varkappa)$ is a better approximation of the weak compactness than the tree property of $\varkappa$. Finally, in § 4, we prove two theorems which have rather a combinatorial character.

(¹) $\mathbf{F}(\varkappa, \lambda)$ denotes $S(\varkappa, \lambda, 2) \to B(2)$ in the terminology of [1].

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§ 0. Notation and terminology. We define some concepts used frequently in the text. We always identify an ordinal with the set of previous ordinals and a cardinal with the smallest ordinal having that cardinality. The letters $\alpha, \beta, \gamma, \delta, \nu, \sigma, \xi, \eta, \zeta$ will be used for ordinals and the letters $\varkappa, \lambda, \mu$ for infinite cardinals. We write $|X|$ for the cardinality of X and λ^+ for the next cardinal after λ. A cardinal $\varkappa$ is a limit cardinal if $\varkappa \neq \lambda^+$ for each cardinal λ; otherwise $\varkappa$ is a successor cardinal. Recall that $\mathrm{cf}(\alpha)$ is the smallest β such that there is a mapping of β onto a cofinal subset of α. An infinite cardinal $\varkappa$ is *regular* if $\mathrm{cf}(\varkappa) = \varkappa$, and *singular* if $\mathrm{cf}(\varkappa) < \varkappa$. If f is a function from a set I into X, then we use the notation f, $\langle f(i)\rangle_{i\in I}$, and $\langle f_i\rangle_{i\in I}$ equally. If the values of a function $\mathfrak{X} = \langle X_i\rangle_{i\in I}$ are sets, then $\mathfrak{X}$ is called a *family of sets*. We often identify $\mathfrak{X}$ with the set $\{X_i: i \in I\}$. If $\mathfrak{X} = \langle X_\xi\rangle_{\xi<\alpha}$ and $\beta < \alpha$, then $\mathfrak{X}\restriction\beta = \langle X_\xi\rangle_{\xi<\beta}$. The Cartesian product of the sets X and Y is denoted by $X \times Y$ and the Cartesian product of a family $\langle X_i\rangle_{i\in I}$ of sets is denoted by $\prod_{i\in I} X_i$. ${}^XY$ denotes the set of all mappings from X to Y. The cardinality of the set of all subsets of $\varkappa$ will be denoted by $2^\varkappa$; similarly $|{}^\lambda\varkappa| = \varkappa^\lambda$. A cardinal $\varkappa$ is *weakly inaccessible* if it is regular and limit. $\varkappa$ is *strongly inaccessible* if it is regular and for each $\lambda < \varkappa$ we have $2^\lambda < \varkappa$.

GCH denotes the *Generalized Continuum Hypothesis*, i.e., the statement $(\forall\varkappa)(2^\varkappa = \varkappa^+)$. **ZFC** denotes the Zermelo-Fraenkel Set Theory with the Axiom of Choice.

A tree $\boldsymbol{T}$ is a partially ordered set $\boldsymbol{T} = \langle T, \leqslant\rangle$ such that, for every $x \in T$, the set $N_x = \{y \in T: y < x\}$ is well-ordered by $\leqslant$. The order type of N_x is called the *type of* x, denoted by $l(x)$, and the length of $\boldsymbol{T}$ is $\bigcup\{l(x)+1: x \in T\}$; an $\varkappa$-tree is a tree of the power $\varkappa$. The αth level of $\boldsymbol{T}$ is the set U_α of all elements of $\boldsymbol{T}$ whose type is α, i.e., $U_\alpha = \{x \in T: l(x) = \alpha\}$. $T\restriction\alpha$ is the union of all U_β, for $\beta < \alpha$ and $\boldsymbol{T}\restriction\alpha = \langle T\restriction\alpha, \leqslant \cap (T\restriction\alpha)^2\rangle$. Any maximal linearly ordered subset of a tree $\boldsymbol{T}$ is called a *branch of* $\boldsymbol{T}$; a subset of $\boldsymbol{T}$ with pairwise incomparable elements is called an *antichain of* $\boldsymbol{T}$.

We deal with structures having relations, functions and constants. A structure $\mathfrak{A}$ is a family of sets whose domain I contains 0 and such that if A is the image of 0, called the *universum of* $\mathfrak{A}$, then the image of every other element of I is an n-ary relation or an n-ary operation on A for some natural number n, or an element of A. The similarity type of a structure $\mathfrak{A}$ is a function ϱ with domain $I - \{0\}$ which takes elements of $I - \{0\}$ into $2n+2$, $2n+1$ or 0, according as they determine an n-ary relation or an n-ary function or a constant in A. To each similarity type ϱ there corresponds a first order language $\mathfrak{L}(\varrho)$ (or, if ϱ is fixed, $\mathfrak{L}$) with equality which has an n-ary relation symbol for each i such that $\varrho(i) = 2n+2$, an n-ary function symbol for each $i \in I$ such that $\varrho(i) = 2n+1$, or a constant symbol if $\varrho(i) = 0$. The variables of $\mathfrak{L}$ are v_n, $n < \omega$. If ϱ is the similarity type of $\mathfrak{A}$, then $L(\varrho)$ will also be called the *language of* $\mathfrak{A}$. If R is a relation symbol of $\mathfrak{L}$, then $R^{\mathfrak{A}}$ denotes the corresponding relation in $\mathfrak{A}$; similarly for operation and constant symbols. The set of formulas in $\mathfrak{L}(\varrho)$ is defined in the usual inductive way. For each cardinal $\varkappa$ we define an infinitary language $\mathfrak{L}_{\varkappa,\varkappa}$ which has the same relation, function and constant symbols as $\mathfrak{L}$ but which allows conjunctions and (homogenuous) quantifier blocks of all the lengths $<\varkappa$. The variables of $\mathfrak{L}_{\varkappa,\varkappa}$ will be v_α, $\alpha < \varkappa$. We assume that the notion of satisfaction of a formula from $\mathfrak{L}(\varrho)$ $\big($or $\mathfrak{L}_{\varkappa,\varkappa}(\varrho)\big)$ in a structure $\mathfrak{A}$ of the type ϱ is known. $\mathfrak{A} \prec_{\varkappa,\varkappa} \mathfrak{B}$ means that $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$ in the sense of $\mathfrak{L}_{\varkappa,\varkappa}$. Now, let $\mathfrak{L}'$ be an expansion of $\mathfrak{L}$ by adding a new function symbol G and let $\mathfrak{A}$ be a structure for $\mathfrak{L}$. Then we can expand $\mathfrak{A}$ to a structure for $\mathfrak{L}'$ in the following way: we choose a function g in A of the same arity as G and define $\mathfrak{A}' = (\mathfrak{A}, G^{\mathfrak{A}'})$ where $G^{\mathfrak{A}'} = g$.

A cardinal $\varkappa$ is said to be *weakly compact* whenever the following condition is satisfied: if Σ is a set of sentences in $\mathfrak{L}_{\varkappa,\varkappa}$ having the power $\leqslant\varkappa$ and every subset of Σ of the power $<\varkappa$ has a model, then Σ has a model.

§ 1. Simplest properties of E and F. We give first a corollary which follows immediately from the definitions introduced above.

COROLLARY. 1.1. (.1) *If* $\lambda < \mu$ *then* $\mathbf{E}(\varkappa, \mu)$ *implies* $\mathbf{E}(\varkappa, \lambda)$, *and* $\mathbf{E}(\varkappa, \mu)$ *implies* $\mathbf{F}(\varkappa, \lambda)$.

(.2) *If there is an increasing sequence of ordinals* $\langle \alpha_\xi\rangle_{\xi<\lambda}$ *such that* $\alpha_\xi < \varkappa$ *for all* $\xi < \lambda$ *and* $\varkappa = \bigcup\{\alpha_\xi: \xi < \lambda\}$, *then* $\neg\mathbf{F}(\varkappa, \lambda)$. *In particular,* $\neg\mathbf{F}(\varkappa, \varkappa)$ *and* $\neg\mathbf{F}\big(\varkappa, \mathrm{cf}(\varkappa)\big)$.

(.3) *If* $\lambda > \varkappa$ *then* $\neg\mathbf{E}(\varkappa, \lambda)$.

(.4) *For each* $\lambda < \mu$, $\mathbf{F}(\varkappa, \mu)$ *implies* $\mathbf{F}(\varkappa, \lambda)$.

(.5) *For each* λ, $\mathbf{E}(\varkappa, \lambda^+)$ *is equivalent to* $\mathbf{F}(\varkappa, \lambda)$.

Proof. (.1) is obvious. (.3) follows from (.2) and (.1), (.4) is also obvious.

Proof of (.2). Let $\varkappa = \bigcup\{\alpha_\xi: \xi < \lambda\}$, where $\langle\alpha_\xi\rangle_{\xi<\lambda}$ satisfies the hypotheses of (.2). We define the sets X_σ, for $\sigma < \varkappa$, as follows:

$$X_\sigma = \begin{cases} \{\eta < \lambda: \eta \geqslant \xi\} & \text{if } \sigma = \alpha_\xi \text{ for some } \xi < \lambda, \\ \{\lambda+\sigma\} \cup \lambda & \text{otherwise}. \end{cases}$$

We have $|X_\xi| = \lambda$ for all $\xi < \varkappa$. Observe that the family $\mathfrak{X} = \langle X_\xi\rangle_{\xi<\varkappa}$ has no selector since its subfamily $\langle X_{\alpha_\xi}\rangle_{\xi<\lambda}$ obviously has no selector. On the other hand, take an arbitrary $\beta < \varkappa$, then $\beta < \alpha_{\xi_0}$ for some $\xi_0 < \lambda$.

Now, it is easy to see that $\{\xi_0\}$ is a selector of the family $\mathfrak{X}\restriction\beta$; thus $\mathfrak{X}$ has partial selectors. Consequently $\neg\mathbf{F}(\varkappa, \lambda)$.

Proof of (.5). The implication $\mathbf{E}(\varkappa, \lambda^+)\to\mathbf{F}(\varkappa, \lambda)$ is a special case of (.1). It suffices to prove that $\mathbf{F}(\varkappa, \lambda)$ implies $\mathbf{E}(\varkappa, \lambda^+)$. Assume $\mathbf{F}(\varkappa, \lambda)$ and suppose that a family $\mathfrak{X} = \langle X_\alpha\rangle_{\alpha<\varkappa}$ has partial selectors and, for every $\alpha<\varkappa$, we have $|X_\alpha|\leqslant\lambda$. We shall show that $\mathfrak{X}$ has a selector. Consider the family $\mathfrak{Y} = \langle Y_\alpha\rangle_{\alpha<\varkappa}$ where $Y_\alpha = X_\alpha\times\lambda$, for all $\alpha<\varkappa$. Then $|Y_\alpha| = \lambda$ for all $\alpha<\varkappa$. It is easy to see that the family $\mathfrak{Y}$ has partial selectors. Since $\mathbf{F}(\varkappa, \lambda)$ holds, $\mathfrak{Y}$ has a selector, say T. Define

$$S = \{x\colon (\exists y)(\langle x, y\rangle \in T)\}\,.$$

We shall show that S is a selector of $\mathfrak{X}$. The fact that $S\cap X_\alpha \neq 0$ for all $\alpha<\varkappa$ is obvious. Suppose that for some $\alpha<\varkappa$ there are $x_1, x_2\in S\cap X_\alpha$ such that $x_1\neq x_2$. Then for some y_1, y_2 we have $\langle x_1, y_1\rangle$, $\langle x_2, y_2\rangle\in T$, and $y_1, y_2\in\lambda$. Thus $\langle x_1, y_1\rangle$, $\langle x_2, y_2\rangle\in T\cap Y_\alpha$, but $\langle x_1, y_1\rangle\neq\langle x_2, y_2\rangle$ since $x_1\neq x_2$. So T is not a selector of $\mathfrak{Y}$, contrary to the previous assumption. Consequently S is a selector of $\mathfrak{X}$ and $\mathbf{E}(\varkappa, \lambda^+)$ holds.

§ 2. A positive result. In this and the subsequent sections we shall study the property $\mathbf{E}(\varkappa, \varkappa)$. In this section we shall prove the following positive result:

THEOREM 2.1. *If $\varkappa$ is weakly compact then* $\mathbf{E}(\varkappa, \varkappa)$.

Proof. Let $\mathfrak{X} = \langle X_\alpha\rangle_{\alpha<\varkappa}$ be a family of sets satisfying the hypotheses of $\mathbf{E}(\varkappa, \varkappa)$. Since $|\bigcup\{X_\alpha\colon \alpha<\varkappa\}|\leqslant\varkappa$, without loss of generality we can assume that each X_α is a subset of $\varkappa$. Consider a first order language $\mathfrak{L}$ with two binary predicates $\leqslant$ and F and with individual constants c_α, $\alpha<\varkappa$. Let $\mathfrak{A} = \langle\varkappa, \leqslant^{\mathfrak{A}}, F^{\mathfrak{A}}, c_\alpha^{\mathfrak{A}}\rangle_{\alpha<\varkappa}$ be a structure for $\mathfrak{L}$ in which $\leqslant^{\mathfrak{A}}$ is the natural ordering of $\varkappa$, $c_\alpha^{\mathfrak{A}} = \alpha$ for all $\alpha<\varkappa$ and $\{x\in\varkappa\colon \mathfrak{A}\models F(\alpha, x)\} = X_\alpha$. Let $T = \{\varphi\in\mathfrak{L}_{\varkappa\varkappa}\colon \mathfrak{A}\models\varphi\}$. Observe that if $\mathfrak{B}$ is a model of T, then, up to isomorphism, $\mathfrak{A}\precsim_{\varkappa\varkappa}\mathfrak{B}$. Moreover, if $\mathfrak{B} = \langle B, \leqslant^{\mathfrak{B}}, F^{\mathfrak{B}}, c_\alpha^{\mathfrak{B}}\rangle_{\alpha<\varkappa}$ then we have $X_\alpha = \{\beta\in\varkappa\colon \mathfrak{B}\models F(c_\alpha, c_\beta)\}$ for all $\alpha<\varkappa$ and if $\mathfrak{B}\models F(c_\alpha, b)$ for some $b\in B$, then there is a $\beta<\varkappa$ such that $c_\beta^{\mathfrak{B}} = b$.

Expand the language $\mathfrak{L}$ to $\mathfrak{L}'$ by adding a new unary function symbol G and consider the following set of sentences: $T' = T\cup\Sigma_1\cup\Sigma_2$, where

$$\Sigma_1 = \{F(c_\alpha, G(c_\alpha))\colon\ \alpha<\varkappa\}.$$

$$\Sigma_2 = \{F(c_\alpha, G(c_\beta))\to G(c_\alpha) = G(c_\beta)\colon\ \alpha, \beta<\varkappa\}\,.$$

We shall prove that every subset $T_0\subseteq T'$ of the power $<\varkappa$ has a model. Indeed, since for each $\alpha<\varkappa$ we have $|X_\alpha|<\varkappa$ putting $D = \{\alpha<\varkappa\colon c_\alpha$ occurs in $T_0\}$, we infer by the regularity of $\varkappa$ that $|D|<\varkappa$ and consequently $\alpha_0 = \sup\{\alpha\colon \alpha\in D\}<\varkappa$. Consider the model $\mathfrak{A}_0 = \langle\varkappa, \leqslant^{\mathfrak{A}}, F^{\mathfrak{A}}, c_\alpha^{\mathfrak{A}}\rangle_{\alpha<\alpha_0}$ which is the reduct of $\mathfrak{A}$. Then $\mathfrak{A}_0$ is a model of $T\cap T_0$. Consider the family $\mathfrak{X}\restriction\alpha_0$. Since $\mathfrak{X}$ has partial selectors, $\mathfrak{X}\restriction\alpha_0$ has a selector; let S be a selector of $\mathfrak{X}\restriction\alpha_0$. Define a function $g_0\colon\varkappa\to\varkappa$ by $g_0(\xi)\in S\cap X_\xi$ for $\xi<\alpha_0$. The function g_0 is well defined, since S is a selector. Consider the structure $(\mathfrak{A}_0, g_0)$. Then it is easy to see that $(\mathfrak{A}_0, g_0)$ is a model of T_0. Now $|T'| = \varkappa$ and each subset T_0 of T' of the power $<\varkappa$ has a model. Thus, it follows from the weak compactness of $\varkappa$ that T' has a model. Let $\mathfrak{B}' = \langle B, \leqslant^{\mathfrak{B}}, F^{\mathfrak{B}}, c_\alpha^{\mathfrak{B}}, G^{\mathfrak{B}}\rangle_{\alpha<\varkappa}$ be a model of T'. Then $\mathfrak{B}$ which is the reduct of $\mathfrak{B}'$ to $\mathfrak{L}$, is a model of T. Let $S\subseteq\varkappa$ be defined by $S = \{\alpha<\varkappa\colon (\exists\beta<\varkappa)\mathfrak{B}'\models G(c_\beta) = c_\alpha\}$. Since $\mathfrak{B}'$ is a model of Σ_1, for each $\alpha<\varkappa$ we have $S\cap X_\alpha\neq 0$. Similarly, since $\mathfrak{B}'$ is a model of Σ_2, for each $\alpha<\varkappa$ we have $|S\cap X_\alpha| = 1$. Thus S is a selector of the family $\mathfrak{X}$ and $\mathbf{E}(\varkappa, \varkappa)$ holds.

§ 3. Trees. In this section we shall prove a simple connection between $\mathbf{E}(\varkappa, \varkappa)$ and the *tree property* of $\varkappa$.

Let $\varkappa$ be a cardinal. Recall that a tree $\boldsymbol{T}$ is an *Aronszajn $\varkappa$-tree* if it is a $\varkappa$-tree the levels of which are of the power less than $\varkappa$ and which has no branch of the length $\varkappa$. A cardinal $\varkappa$ has the *tree property* if and only if there is no Aronszajn $\varkappa$-tree (in symbols $\varkappa\in\mathbf{TP}$).

The following theorem explains the simplest properties of the class $\mathbf{TP}$.

THEOREM 3.1. (.1) *There is an Aronszajn ω_1-tree.*

(.2) ($\mathbf{GCH}$) *For every regular cardinal $\varkappa$ there is an Aronszajn $\varkappa^+$-tree.*

(.3) *If $\varkappa$ is a singular cardinal then $\varkappa\notin\mathbf{TP}$.*

(.4) *If $\varkappa$ is strongly inaccessible then $\varkappa\in\mathbf{TP}$ if and only if $\varkappa$ is weakly compact.*

Proof. (.1) and (.2) are well known, see e.g., [5]. (.3) is easy to check. For (.4) see e.g., [3].

THEOREM. 3.2. $\mathbf{E}(\varkappa, \varkappa)$ *implies* $\varkappa\in\mathbf{TP}$.

Proof. First of all, remark that $\mathbf{E}(\varkappa, \varkappa)$ implies that $\varkappa$ is regular. Indeed, if $\mathrm{cf}(\varkappa) = \lambda<\varkappa$, then by 1.1.2, we have $\neg\mathbf{F}(\varkappa, \lambda)$; hence, by 1.1.1, we have also $\neg\mathbf{E}(\varkappa, \varkappa)$.

So let $\varkappa$ be a regular cardinal and let $\boldsymbol{T} = \langle T, \leqslant\rangle$ be a $\varkappa$-tree. Using $\mathbf{E}(\varkappa, \varkappa)$ we shall construct a $\varkappa$-branch in $\boldsymbol{T}$.

Since $\varkappa$ is regular, $\boldsymbol{T}$ has the length $\varkappa$ and for each $\alpha<\varkappa$ we have $|T\restriction\alpha|<\varkappa$. Let $\langle U_\alpha\rangle_{\alpha<\varkappa}$ be the family of all levels of $\boldsymbol{T}$, i.e., U_α is the αth level of $\boldsymbol{T}$ for all $\alpha<\varkappa$. By the hypotheses, we have $|U_\alpha|<\varkappa$ for $\alpha<\varkappa$. For every two incomparable elements $a, b\in T$, such that $l(a)\neq l(b)$, fix a maximal antichain in $T\restriction\gamma$ (where $\gamma = \max(l(a), l(b))+1$) which contains a and b. Denote this antichain by $A(a, b)$. We have $|A(a, b)|<\varkappa$ because of $A(a, b)\subseteq T\restriction\gamma$. Now, let $\mathfrak{X} = \langle X_\alpha\rangle_{\alpha<\varkappa}$ be the family of all subsets of T such that X_α is either a level of $\boldsymbol{T}$ or a set of the form $A(a, b)$

for some $a, b \in T$. By the regularity of $\varkappa$, for every $\beta < \varkappa$ there is a $\gamma < \varkappa$ such that $X_\alpha \subseteq T \upharpoonright \gamma$ for all $\alpha < \beta$. Thus for every $\beta < \varkappa$ there is a selector of $\langle X_\alpha \rangle_{\alpha<\varkappa}$. Indeed, choose an arbitrary $x \in U_\gamma$ and consider the set $S = \{y \in T:\ y \leqslant x\}$. Since S is linearly ordered and each X_α is an anti-chain, we have $|S \cap X_\alpha| \leqslant 1$ for all $\alpha < \beta$. We shall show that for $\alpha < \beta$ we have $S \cap X_\alpha \neq 0$. It is easy to see that this is true if X_α is a level of T. Suppose that $A(a, b) = X_\alpha$ and let $\delta = \max(l(a), l(b))$. Take $y \in S \cap U_\delta$. Then $y \in T \upharpoonright (\delta+1)$. If $y \in A(a, b)$ then $S \cap X_\alpha \neq 0$. Suppose that $y \notin A(a, b)$. By the maximality of $A(a, b)$ in $T \upharpoonright (\delta+1)$, there is a $z \in A(a, b)$ which is comparable with y. If $l(z) = l(y)$ then y and z lie on the same level and so are equal, which is impossible since $z \in A(a, b)$ and $y \notin A(a, b)$. The case $l(z) > l(y) = \delta$ is excluded since $l(z) \leqslant \sup\{l(t):\ t \in A(a, b)\} = \delta$. So we have $l(z) < l(y)$ as the only possibility. But using the comparability of z and y, we have $z \leqslant y$; hence $z \in S$ and consequently $A(a, b) \cap S \neq 0$, which means that $|X_\alpha \cap S| = 1$. In other words, we have shown that $\mathfrak{X}$ has partial selectors; thus $\mathfrak{X}$ satisfies all the hypotheses of $\mathbf{E}(\varkappa, \varkappa)$. Now, by $\mathbf{E}(\varkappa, \varkappa)$, $\mathfrak{X}$ has a selector, say S. We shall show that then S will be a $\varkappa$-branch in T. To prove this, we shall show that every two elements of S are comparable. Suppose, on the contrary, that there are two incomparable elements a, b in S. If $l(a) = l(b)$, then a and b are on the same level, and so $|S \cap U_{l(a)}| > 1$, which is impossible. Let $l(a) \neq l(b)$. Then the set $A(a, b)$ belongs to $\mathfrak{X}$ and $a, b \in A(a, b)$. Thus $|A(a, b) \cap S| > 1$, which is a contradiction. Thus S is linearly ordered and since S intersects each level of T, S is a branch of T of the length $\varkappa$. Thus T is not an Aronszajn $\varkappa$-tree. This finishes the proof of 3.2.

§ 4. Some negative results. Since, as the author knows, the problem of the existence of Aronszajn $\varkappa^+$-trees for singular $\varkappa$ is still open (some special cases have been partially solved by Prikry [2]), the results of § 3 do not give the answer to the question whether $\mathbf{E}(\varkappa, \varkappa)$ holds for each $\varkappa$. Now we shall prove two negative results which solve this question (under the assumption of **GCH**) completely.

THEOREM 4.1. *For every cardinal* $\varkappa$, *we have* $\neg\mathbf{E}(2^\varkappa, \varkappa^+)$.

Proof. We shall construct the required family of sets. Let $f \in {}^\varkappa\varkappa$ and let $F(f) = \{(f, \xi):\ \xi < \varkappa\}$. Obviously $|F(f)| = \varkappa$. Define the family of sets $\mathcal{A}(f)$ as follows:

$$(1)\quad \begin{cases} \mathcal{A}(f) = \{F(f)\} \cup \{A^f_{\xi\zeta}:\ \xi, \zeta < \varkappa\}, \quad \text{where} \\ A^f_{\xi\zeta} = \big(F(f) - \{(f, \xi)\}\big) \cup \{\big((f, \xi), \zeta, \eta\big):\ \eta < \varkappa\} \ \text{for } \xi \neq \zeta \quad \text{and} \\ A^f_{\xi\xi} = \big(F(f) - \{(f, \xi)\}\big) \cup \{\big((f, \xi), \xi, \eta\big):\ \eta < \varkappa \text{ and } \eta \neq f(\xi)\}. \end{cases}$$

It is easy to see that for every $X \in \mathcal{A}(f)$ we have $|X| = \varkappa$ and for $f \neq g$ we have $\mathcal{A}(f) \cap \mathcal{A}(g) = 0$.

We claim that the family $\mathcal{A}(f)$ has a selector, and every selector of the family $\mathcal{A}(f)$ is of the form:

$$(2)\quad \begin{cases} S = \{(f, \xi)\} \cup \{\big((f, \xi), \zeta, g(\zeta)\big):\ \zeta < \varkappa\}, \quad \text{where} \\ g \in {}^\varkappa\varkappa \text{ is such that } g(\xi) \neq f(\xi). \end{cases}$$

Indeed, any set S of the form (2) is a selector. Namely: $S \cap F(f) = \{(f, \xi)\}$, $S \cap A^f_{\xi'\zeta} = \{(f, \xi)\}$ for $\xi' \neq \xi$ and $S \cap A^f_{\xi\zeta} = \{\big((f, \xi), \zeta, g(\zeta)\big)\}$. (For $\zeta = \xi$ this holds since $g(\xi) \neq f(\xi)$.)

Conversely, suppose that S is a selector of $\mathcal{A}(f)$. (Without loss of generality we can assume that $S \subseteq \bigcup \mathcal{A}(f)$). Then $S \cap F(f)$ has one element, say $(f, \xi) \in S$. Since $(f, \xi) \in A^f_{\xi'\zeta}$ for $\xi' \neq \xi$, no triple $\big((f, \xi'), \zeta, \eta\big)$ belongs to S. Next, for each $\zeta < \varkappa$, $(f, \xi) \notin A^f_{\xi\zeta}$, and so for each $\zeta < \varkappa$ there is precisely one triple $\big((f, \xi), \zeta, \eta\big)$ in S. Thus we can define a function $g \in {}^\varkappa\varkappa$ by:

$$g(\zeta) = \eta \ \text{ if and only if } \ \big((f, \xi), \zeta, \eta\big) \in S.$$

Finally, remark that $\big((f, \xi), \xi, f(\xi)\big) \notin A^f_{\xi\xi}$; hence also $\big((f, \xi), \xi, f(\xi)\big) \notin S$ and consequently $g(\xi) \neq f(\xi)$, which ends the proof of our claim.

Now, let $f, h \in {}^\varkappa\varkappa$ be given. From (2) it follows that each selector of $\mathcal{A}(f) \cup \mathcal{A}(h)$ is of the form:

$$(3)\quad \begin{cases} \{(f, \xi_1), (h, \xi_2)\} \cup \{\big((f, \xi_1), \zeta, g_1(\zeta)\big):\ \zeta < \varkappa\} \cup \{\big((h, \xi_2), \zeta, g_2(\zeta)\big):\ \zeta < \varkappa\} \\ \text{where } g_1, g_2 \in {}^\varkappa\varkappa \text{ and } g_1(\xi_1) \neq f(\xi_1) \text{ and } g_2(\xi_2) \neq h(\xi_2). \end{cases}$$

We shall define a new family of sets $\mathcal{B}(f, h)$ such that each selector of $\mathcal{A}(f) \cup \mathcal{A}(h) \cup \mathcal{B}(f, h)$ will be of the form (3) with the additional property that $g_1 = g_2$. The family $\mathcal{B}(f, h)$ is defined as follows:

$$(4)\quad \begin{cases} \mathcal{B}(f, h) = \{B_{\xi\eta}:\ \xi, \eta < \varkappa\} \quad \text{where} \\ B_{\xi\eta} = \{\big((f, \zeta), \xi, \eta\big):\ \zeta < \varkappa\} \cup \{\big((h, \zeta), \xi, \nu\big):\ \zeta, \nu < \varkappa \text{ and } \nu \neq \eta\}. \end{cases}$$

It is easy to see that for each $X \in \mathcal{B}(f, h)$ we have $|X| = \varkappa$. Let S be a set of the form (3) with $g = g_1 = g_2$. We will show that S is a selector of the family $\mathcal{A}(f) \cup \mathcal{A}(h) \cup \mathcal{B}(f, h)$. Obviously S, being of the form (3), is a selector of $\mathcal{A}(f) \cup \mathcal{A}(h)$. We wish to show that S is also a selector of $\mathcal{B}(f, h)$. We have two cases: (I) if $g(\xi) = \eta$, then

$$S \cap B_{\xi\eta} = \{\big((f, \xi_1), \xi, g(\xi)\big)\},$$

or (II) if $g(\xi) \neq \eta$, then

$$S \cap B_{\xi\eta} = \{\big((h, \xi_2), \xi, g(\xi)\big)\}.$$

Thus S is a selector of $\mathcal{B}(f, h)$.

Conversely, suppose that S is a selector of the family $\mathcal{A}(f) \cup \mathcal{A}(h) \cup \mathcal{B}(f, h)$. Then, since S is a selector of $\mathcal{A}(f) \cup \mathcal{A}(h)$, S has the form (3).

We will show that $g_1 = g_2$. Suppose the contrary. Then for some $\xi < \varkappa$ we have $\eta = g_1(\xi) \neq g_2(\xi)$. Thus

$$S \cap B_{\xi\eta} = \{((f, \xi_1), \xi, g_1(\xi)), ((h, \xi_2), \xi, g_2(\xi))\};$$

hence S is not a selector of $\mathcal{A}(f) \cup \mathcal{A}(g) \cup \mathcal{B}(f, h)$, which is a contradiction.

Having the families of the form $\mathcal{A}(f)$ and $\mathcal{B}(f, h)$ we can construct a family which realizes $\neg\mathbf{E}(2^\varkappa, \varkappa^+)$. For this purpose let us remark that $|\mathcal{A}(f)| = |\mathcal{B}(f, h)| = \varkappa$. Consider the family $\mathfrak{X} = \bigcup\{\mathcal{A}(f)\colon f \in {}^\varkappa\varkappa\} \cup \bigcup\{\mathcal{B}(f, h)\colon f, h \in {}^\varkappa\varkappa \text{ and } f \neq h\}$. We claim that $\mathfrak{X}$ has no selector. Suppose, on the contrary, that S is a selector of $\mathfrak{X}$. In particular, S is a selector of $\bigcup\{\mathcal{A}(f)\colon f \in {}^\varkappa\varkappa\}$, and so, by (2), S has the form:

$$S = \{(f, \xi_f)\colon f \in {}^\varkappa\varkappa\} \cup \bigcup\Big\{\{((f, \xi_f), \zeta, g_f(\zeta))\colon \zeta < \varkappa\}\colon f \in {}^\varkappa\varkappa\Big\},$$

where $g_f \in {}^\varkappa\varkappa$ and for each $f \in {}^\varkappa\varkappa, f(\xi_f) \neq g_f(\xi_f)$. Moreover, for every $f_1, f_2 \in {}^\varkappa\varkappa$ with $f_1 \neq f_2$, S is a selector of $\mathcal{B}(f_1, f_2)$; thus $g_{f_1} \neq g_{f_2}$, and consequently there is a function $g \in {}^\varkappa\varkappa$ such that $g = g_f$ for all $f \in {}^\varkappa\varkappa$. But this is impossible since it would give $g = g_g$ and $g(\xi_g) \neq g_g(\xi_g)$, which gives a contradiction. So $\mathfrak{X}$ has no selector.

Finally, we shall prove that $\mathfrak{X}$ has partial selectors. Let Z be a proper subset of ${}^\varkappa\varkappa$ and let us consider the family

$$\mathfrak{X}_Z = \bigcup\{\mathcal{A}(f)\colon f \in Z\} \cup \bigcup\{\mathcal{B}(f, h)\colon f, h \in Z \text{ and } f \neq h\}.$$

We claim that $\mathfrak{X}_Z$ has a selector. Indeed, since ${}^\varkappa\varkappa - Z \neq 0$, there is a $g \in {}^\varkappa\varkappa - Z$. For each $f \in Z$, choose $\xi_f < \varkappa$ in such a way that $f(\xi_f) \neq g(\xi_f)$. Then it is clear that the set

$$S = \{(f, \xi_f)\colon f \in Z\} \cup \bigcup\{\{((f, \xi_f), \zeta, g(\zeta))\colon \zeta < \varkappa\}\colon f \in Z\}$$

is a selector of $\mathfrak{X}^Z$. Now, since $\varkappa < 2^\varkappa$, we can enumerate the family $\mathfrak{X}$ in such away that $\mathfrak{X} = \langle X_\alpha\rangle_{\alpha<2^\varkappa}$ and for each $\beta < 2^\varkappa$ there exists a proper subset Z of ${}^\varkappa\varkappa$ such that $\mathfrak{X}\restriction\beta \subseteq \mathfrak{X}_Z$. Thus the family $\mathfrak{X}$ has partial selectors. Since $\mathfrak{X}$ has no selector, $\neg\mathbf{E}(2^\varkappa, \varkappa^+)$ holds and our proof is complete.

For singular cardinals we can obtain a better result:

THEOREM 4.2. *If* $\mathrm{cf}(\varkappa) = \lambda < \varkappa$, *then* $\neg\mathbf{E}(\varkappa^\lambda, \varkappa)$.

Proof. We shall construct, as before, the sets $\mathcal{A}(f)$ and $\mathcal{B}(f, h)$ using some other set of functions instead of ${}^\varkappa\varkappa$.

Let $\{\alpha_\xi\colon \xi < \lambda\}$ be a set of cardinals less than $\varkappa$, such that the set $P = \prod_{\xi<\lambda} \alpha_\xi$ has the power $\varkappa^\lambda$. Define $F(f) = \{(f, \xi)\colon \xi < \lambda\}$ for $f \in P$. Then $|F(f)| = \lambda < \varkappa$. Next, define the family $\mathcal{A}(f)$ as follows:

$$(1)\quad \begin{cases} \mathcal{A}(f) = \{F(f)\} \cup \{A^f_{\xi\zeta}\colon \xi, \zeta < \lambda\}, & \text{where}\\ A^f_{\xi\zeta} = (F(f) - \{(f, \xi)\}) \cup \{((f, \xi), \zeta, \eta)\colon \eta < \alpha_\zeta\} \text{ for } \xi \neq \zeta & \text{and}\\ A^f_{\xi\xi} = (F(f) - \{(f, \xi)\}) \cup \{((f, \xi), \xi, \eta)\colon \eta < \alpha_\xi \text{ and } \eta \neq f(\xi)\}. \end{cases}$$

As before, we have:

(i) for each $X \in \mathcal{A}(f)$, $|X| < \varkappa$;

(ii) if $f, g \in P$ and $f \neq g$ then $\mathcal{A}(f) \cap \mathcal{A}(g) = 0$;

(iii) each selector of the family $\mathcal{A}(f)$ is of the form $S = \{(f, \xi)\} \cup \cup \{((f, \xi), \zeta, g(\zeta))\colon \zeta < \lambda\}$, where $g \in P$ and $g(\xi) \neq f(\xi)$.

Suppose $f, h \in P, f \neq h$. Then we define the family $\mathcal{B}(f, h)$ as follows:

$$(2)\quad \begin{cases} \mathcal{B}(f, h) = \{B_{\xi\eta}\colon \xi < \lambda \text{ and } \eta < \alpha_\xi\}, \quad \text{where}\\ B_{\xi\eta} = \{((f, \zeta), \xi, \eta)\colon \zeta < \lambda\} \cup \{((h, \zeta), \xi, \nu)\colon \zeta < \lambda \text{ and } \nu \in (\alpha_\xi - \{\eta\})\}. \end{cases}$$

Then, as before, we have:

(iv) for each $X \in \mathcal{B}(f, h)$, $|X| < \varkappa$;

(v) each selector of $\mathcal{A}(f) \cup \mathcal{A}(h) \cup \mathcal{B}(f, h)$ is of the form:

$$\{(f, \xi_1), (h, \xi_2)\} \cup \{((f, \xi_1), \zeta, g(\zeta))\colon \zeta < \lambda\} \cup \{((h, \xi_2), \zeta, g(\zeta))\colon \zeta < \lambda\}$$

where $g \in P$ and $f(\xi_1) \neq g(\xi_1)$ and $h(\xi_2) \neq g(\xi_2)$.

Taking, as before,

$$\mathfrak{X} = \bigcup\{\mathcal{A}(f)\colon f \in P\} \cup \bigcup\{\mathcal{B}(f, h)\colon f, h \in P \text{ and } f \neq h\}$$

we see that $|\mathfrak{X}| = \varkappa^\lambda$ and for each $X \in \mathfrak{X}$ we have $|X| < \varkappa$. The proof that $\mathfrak{X}$ has no selector but has partial selectors, using the same argumentation as the proof of Theorem 4.1, follows by the facts (i)–(v). Thus, we get $\neg\mathbf{E}(\varkappa^\lambda, \varkappa)$ as required.

COROLLARY 4.3. (.1) *For each* $\varkappa$, *we have* $\neg\mathbf{E}(2^\varkappa, 2^\varkappa)$.

(.2) *For each* $\varkappa$ *with* $\lambda = \mathrm{cf}(\varkappa) < \varkappa$ *we have* $\neg\mathbf{E}(\varkappa^\lambda, \varkappa^\lambda)$.

Proof. (.1) is a consequence of Theorem 4.1 and 1.1.1, since $2^\varkappa \geqslant \varkappa^+$ (.2) is a consequence of Theorem 4.2 and 1.1.1, since $\varkappa^\lambda \geqslant \varkappa$.

THEOREM 4.4. (**GCH**). $\mathbf{E}(\varkappa, \varkappa)$ *if and only if* $\varkappa$ *is weakly compact.*

Proof. The part "if" (even without using **GCH**) is the assertion of Theorem 2.1, and so we wish to prove the part "only if". Assume $\mathbf{E}(\varkappa, \varkappa)$. Then, by 1.1.1, and 1.1.2, $\varkappa$ cannot be singular. Using **GCH**, we see, by 4.3.1, that $\varkappa$ is not a successor. So $\varkappa$ is weakly inaccessible. By **GCH** again, $\varkappa$ is strongly inaccessible. Finally, by Theorem 3.2, $\varkappa$ has the tree property, and so, by 3.1.4, $\varkappa$ is weakly compact.

COROLLARY 4.5. *If the theory* **ZFC** + *"there is a weakly compact cardinal" is consistent, then also the theory* **ZFC** + *"there exists a cardinal* $\varkappa$ *having the tree property and* $\neg\mathbf{E}(\varkappa, \varkappa)$*" is consistent.*

This means that the converse to Theorem 3.2 is unprovable in **ZFC**.

Proof. In [4], Silver proved that if **ZFC** + "there is a weakly compact cardinal" then **ZFC** + $2^{\omega_0} = 2^{\omega_1} = \omega_2$ + "ω_2 has the tree property" is consistent. But, by 4.3.1, $\neg\mathbf{E}(\omega_2, \omega_2)$ is a theorem of the last theory.

Thus, the theory $\mathsf{ZFC}+\neg\mathsf{E}(\omega_2,\omega_2)+$"$\omega_2$ has the tree property" is consistent.

The next corollary is a partial answer to the question of Erdös and Hajnal stated in the Introduction.

COROLLARY 4.6. *$\mathsf{E}(\omega_2,\omega_1)$ is unprovable in* ZFC.

Proof. By Theorem 4.1, $\mathsf{E}(\omega_2,\omega_1)$ does not hold in any model of ZFC in which $2^{\omega_0}=\omega_2$.

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