A property of stable theories

by

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In this paper is proved the following theorem about models $\mathcal{A}$, $\mathcal{B}$ of a countable stable theory. Let $\boldsymbol{p}$ be a unary predicate symbol. If $\mathcal{A}$ is an elementary submodel of $\mathcal{B}$ (written $\mathcal{A} \precsim \mathcal{B}$ below), $\mathcal{A} \neq \mathcal{B}$, and $\boldsymbol{p}(\mathcal{A}) = \boldsymbol{p}(\mathcal{B})$, then there exists $\mathcal{C}$ such that $\mathcal{B} \precsim \mathcal{C}$, $\mathcal{B} \neq \mathcal{C}$, and $\boldsymbol{p}(\mathcal{A}) = \boldsymbol{p}(\mathcal{C})$. We also construct a counterexample to show that the theorem fails in general for unstable countable theories even if we make the additional assumption that $\mathcal{A}$ and $\mathcal{B}$ are countable.

Vaught has proved [5], p. 55 that under the hypotheses of our theorem, without the requirement of stability, there exists $\mathcal{C} \equiv \mathcal{A}$ such that $|\mathcal{C}|$, $\boldsymbol{p}(\mathcal{C})$ have cardinalities $\aleph_1, \aleph_0$ respectively. This paper was motivated by the question as to how far Vaught's theorem admits a naive proof. The main idea of our proof, namely using a definition of rank based on only a finite number of formulas comes from Shelah [6] as indeed does the notion of stable theory. The study of stable theories arose from a generalization of the work of Morley [4] on totally transcendental theories. Morley used the concept of a totally transcendental theory in his proof of the Łoś conjecture. Recently, Shelah has used the notion of stability to prove the Łoś conjecture for uncountable languages as well as many other striking theorems. The reader will find Shelah [7] an invaluable source of further information.

The plan of the paper is as follows. In the first section we give some definitions and the necessary background material concerning the ranks of formulas in first order theories. In the second section we prove the main theorem and in the third section we construct the counterexample.

1. Preliminaries. We use the notation and terminology of Shoenfield [8] with some exceptions which are noted below. When a, b, c are members of the universe of a structure their names will be $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ respectively and similarly when a, b, c have superscripts or subscripts attached to them. We shall assume that the underlying first order language is L, and that L is countable except at the end of § 2 where we discuss the possibility of extending the main theorem to uncountable languages. The set of L-formulas is denoted by S. The variables are $z_1, z_2, \ldots$ and for $n \in \omega$, S_n denotes the set of L-formulas containing at most $z_1, \ldots, z_n$ free. By $S(X)$ we mean the set of $L(X)$-formulas where $L(X)$ is obtained by adjoining to L names for the members of X; $S_n(X)$ is defined similarly. If $\mathcal{A}$ is a structure and $\boldsymbol{A}$ is a formula, by an *$\mathcal{A}$-instance of $\boldsymbol{A}$* we mean any formula obtained from $\boldsymbol{A}$ by substituting names of members of $|\mathcal{A}|$ for variables. If $\boldsymbol{A} \in S_1(|\mathcal{A}|)$ then $\boldsymbol{A}(\mathcal{A})$ denotes the set of $a \in |\mathcal{A}|$ such that $\boldsymbol{A}[i]$ is true; $\boldsymbol{p}z_1(\mathcal{A})$ is abbreviated to $\boldsymbol{p}(\mathcal{A})$ when $\boldsymbol{p}$ is a unary predicate symbol. Finally $\mathrm{Th}(\mathcal{A}, X)$, where $X \subseteq |\mathcal{A}|$, denotes the theory of the structure obtained from $\mathcal{A}$ by adjoining the names of the members of X.

We require the following generalization of Morley's notion of transcendental rank. This kind of generalization was first considered in Shelah [6]. For an arbitrary subset Δ of S define $\Delta^b(X)$ to consist of those formulas in $S_1(X)$ which can be obtained from the formulas in Δ by a finite number of applications of the operations of conjunction, disjunction, negation, and substitution of the name of a member of X for a variable. If $\boldsymbol{A} \in S_1(|\mathcal{B}|)$ and $\Gamma = \{\boldsymbol{C}_1, \ldots, \boldsymbol{C}_n\} \subseteq S_1(|\mathcal{B}|)$ then Γ is said to *partition* $\boldsymbol{A}$ if $\boldsymbol{C}_1 \vee \ldots \vee \boldsymbol{C}_n \vee \neg \boldsymbol{A}$ is true in $\mathcal{B}$, $\boldsymbol{C}_i \,\&\, \boldsymbol{C}_j \cdot\!\rightarrow \neg \boldsymbol{A}$ is true in $\mathcal{B}$ for all i and j with $i \neq j$, and for all i $\exists z_1(\boldsymbol{A} \,\&\, \boldsymbol{C}_i)$ is true in $\mathcal{B}$.

For every structure $\mathcal{A}$ and ordinal α we define $S^\alpha(\mathcal{A}, \Delta)$ and $\mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$ as follows. Firstly let $S^0(\mathcal{A}, \Delta)$ consist of all $\boldsymbol{A}$ in $S_1(|\mathcal{A}|)$ such that $\exists z_1 \boldsymbol{A}$ is true in $\mathcal{A}$. If $S^\alpha(\mathcal{B}, \Delta)$ has been defined for all $\mathcal{B}$ let $\boldsymbol{A} \in \mathrm{Tr}^\alpha(\boldsymbol{A}, \Delta)$ if and only if $\boldsymbol{A} \in S^\alpha(\mathcal{A}, \Delta)$ and there exists $k \in \omega$ such that for any $\mathcal{B} \succeq \mathcal{A}$ and any finite $\Gamma \subseteq \Delta^b(|\mathcal{B}|)$ which partitions $\boldsymbol{A}$ we have $\boldsymbol{A} \,\&\, \boldsymbol{C} \in S^\alpha(\mathcal{B}, \Delta)$ for at most k members $\boldsymbol{C}$ of Γ. (If $\boldsymbol{A} \in \mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$ then the least such k is called the *Δ-degree* of $\boldsymbol{A}$ in $\mathcal{A}$, while α is called the *Δ-rank* of $\boldsymbol{A}$ in $\mathcal{A}$.) To complete the definition let $S^{\alpha+1}(\mathcal{A}, \Delta) = S^\alpha(\mathcal{A}, \Delta) - \mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$ and if α is a limit ordinal let $S^\alpha(\mathcal{A}, \Delta) = \bigcap_{\beta<\alpha} S^\beta(\mathcal{A}, \Delta)$.

It is quite easy to show that if Γ is a 1-type in $\mathrm{Th}(\mathcal{A}, |\mathcal{A}|)$ its Morley rank is the least α such that $\Gamma \cap \mathrm{Tr}^\alpha(\mathcal{A}, S) \neq \emptyset$, and its Morley degree is the minimum of the S-degrees of the formulas in $\Gamma \cap \mathrm{Tr}^\alpha(\mathcal{A}, S)$. We say that $\boldsymbol{A}$ is *minimal in* $\mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$ if $\boldsymbol{A} \in \mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$ and there is no $\mathcal{A}$-instance $\boldsymbol{D}$ of a member of Δ such that $\boldsymbol{A} \,\&\, \boldsymbol{D}$ and $\boldsymbol{A} \,\&\, \neg \boldsymbol{D}$ are both in $\mathrm{Tr}^\alpha(\mathcal{A}, \Delta)$.

Since the main result of this paper was discovered the author has received a preprint of [7]. For several of the lemmas in this section there are similar results in [7] which will be noted. The latter part of the paper is concerned only with Δ-rank where Δ is finite. However, to provide a wider perspective we have included in this section some results concerning Δ-rank which hold for arbitrary Δ. From these it follows that

if Γ is a 1-type in $\mathrm{Th}(\mathcal{A}, |\mathcal{A}|)$ which has a Morley rank then the Morley degree of Γ is 1. This answers a question which as far as we know was first raised by Lenore Blum. The following lemma lists some of the more obvious properties of rank:

LEMMA 1. (i) *If $\boldsymbol{A}$, $\boldsymbol{B}$ are in $S_1(|\mathcal{A}|)$, $\boldsymbol{A}(\mathcal{A}) \subseteq \boldsymbol{B}(\mathcal{A})$, and $\boldsymbol{A} \in S^\alpha(\mathcal{A}, \Delta)$ then $\boldsymbol{B} \in S^\alpha(\mathcal{A}, \Delta)$.*

(ii) *If $\boldsymbol{A} \in S_1(\mathcal{A})$ and $\mathcal{B} \succeq \mathcal{A}$, then $\boldsymbol{A} \in S^\alpha(\mathcal{A}, \Delta)$ if and only if $\boldsymbol{A} \in S^\alpha(\mathcal{B}, \Delta)$.*

(iii) *If $\boldsymbol{A}_0$, $\boldsymbol{A}_1$, and $\boldsymbol{A}$ are in $S_1(|\mathcal{A}|)$, $(\boldsymbol{A}_0 \vee \boldsymbol{A}_1)(\mathcal{A}) \supseteq \boldsymbol{A}(\mathcal{A})$, and $\boldsymbol{A} \in S^\alpha(\mathcal{A}, \Delta)$ then one of $\boldsymbol{A}_0$ and $\boldsymbol{A}_1$ is in $S^\alpha(\mathcal{A}, \Delta)$.*

The proof is by induction on α and is straightforward so we omit it. We call $\Gamma \subseteq S(|\mathcal{A}|)$ *satisfiable in* $\mathcal{A}$ if there is a substitution of names for the variables which transforms every member of Γ into a formula true in $\mathcal{A}$. We call $\Gamma \subseteq S(|\mathcal{A}|)$ *weakly satisfiable in* $\mathcal{A}$ if no finite disjunction of negations of formulas in Γ is valid in $\mathcal{A}$. Clearly Γ is weakly satisfiable in $\mathcal{A}$ if and only if Γ is satisfiable in some $\mathcal{B} \succeq \mathcal{A}$. For any $\Delta \subseteq S$ let Δ^- consist of the negations of the formulas in Δ.

LEMMA 2. *Let $\boldsymbol{A} \in S^\alpha(\mathcal{A}, \Delta)$. Let Γ consist of all those formulas which can be obtained by substituting variables for variables in the members of $\{\boldsymbol{A}\} \cup \Delta \cup \Delta^-$. There exists $\Gamma^* \subseteq \Gamma$ weakly satisfiable in $\mathcal{A}$ such that if $\mathcal{B}$ is a structure, $\boldsymbol{B} \in S_1(|\mathcal{B}|)$, and $\Gamma^{\#}$ is weakly satisfiable in $\mathcal{B}$, where $\Gamma^{\#}$ is obtained from Γ^* by replacing each instance of $\boldsymbol{A}$ by the corresponding instance of $\boldsymbol{B}$, then $\boldsymbol{B} \in S^\alpha(\mathcal{B}, \Delta)$.*

Proof. For convenience we suppose that $\boldsymbol{A}$ contains no names and that $\boldsymbol{B}$ is $\boldsymbol{A}$ so that $\Gamma^{\#}$ is Γ^*. The lemma is certainly true for $\alpha = 0$. Suppose the lemma is true for all $\alpha < \beta$ where $\beta > 0$, and let $\boldsymbol{A} \in S^\beta(\mathcal{A}, \Delta)$. Let us first consider the case in which β is a limit ordinal. By the induction hypothesis for each $\gamma < \beta$ there exists $\Gamma_\gamma \subseteq \Gamma$ weakly satisfiable in $\mathcal{A}$ such that if $\mathcal{B}$ is any structure such that $\boldsymbol{A} \in S_1(|\mathcal{B}|)$ and Γ_γ is weakly satisfiable in $\mathcal{B}$ then $\boldsymbol{A} \in S^\gamma(\mathcal{B}, \Delta)$. Further if $\gamma, \delta < \beta$ and $\gamma \neq \delta$, we may suppose that Γ_γ and Γ_δ have no variable in common. To obtain the conclusion of the lemma with $\alpha = \beta$ we take $\Gamma^* = \bigcup \{\Gamma_\gamma \mid \gamma < \beta\}$. Now suppose that $\beta = \gamma + 1$. From the definition of $S^\beta(\mathcal{A}, \Delta)$ and Lemma 1 it is easy to see that there exists $\mathcal{B} \succeq \mathcal{A}$ and a sequence $\langle \boldsymbol{A}_n \mid n \in \omega \rangle$ of members of $\Delta^b(\mathcal{B})$ such that $(\boldsymbol{A}_m \,\&\, \boldsymbol{A}_n)(\mathcal{B}) = \emptyset$ whenever $m \neq n$ and such that $\boldsymbol{A} \,\&\, \boldsymbol{A}_n \in S^\gamma(\mathcal{B}, \Delta)$. Let Γ^m consist of those formulas obtainable from members of $\{\boldsymbol{A} \,\&\, \boldsymbol{A}_n\} \cup \Delta \cup \Delta^-$ by substituting variables for variables. By the induction hypothesis there exists $\Gamma_n \subseteq \Gamma^m$ weakly satisfiable in $\mathcal{B}$ such that if $\boldsymbol{A} \,\&\, \boldsymbol{A}_n \in S_1(|\mathfrak{C}|)$ and Γ_n is weakly satisfiable in $\mathfrak{C}$ then $\boldsymbol{A} \,\&\, \boldsymbol{A}_n \in S^\gamma(\mathfrak{C}, \Delta)$. Without loss we may suppose that $\boldsymbol{A}_0, \boldsymbol{A}_1, \ldots$ have been chosen such that $\neg \boldsymbol{A}_m \vee \neg \boldsymbol{A}_n$ is a tautology whenever $m \neq n$, and such that for each n, A_n is a conjunction of instances of formulas in $\Delta \cup \Delta^-$. This last simplification is possible by Lemma 1 (iii). We may also assume that no variable appears in both Γ_m and Γ_n if $m \neq n$, and that there are infinitely many variables which do not occur in any Γ_n. Now let Γ_n^* be obtained from Γ_n by substituting for each name $\boldsymbol{i}$ appearing in A_n a new variable unique to $\boldsymbol{i}$ and independent of n. Let $\Gamma^* \subseteq \Gamma$ be equivalent to $\bigcup \{\Gamma_n |\ n \in \omega\}$. It is clear that Γ^* exists and is weakly satisfiable in $\mathcal{A}$. Suppose that Γ^* is weakly satisfiable in a structure $\mathcal{B}$ such that $\boldsymbol{A} \in S_1(|\mathcal{B}|)$ we have to show that $\boldsymbol{A} \in S^\beta(\mathcal{B}, \Delta)$. There exists $\mathcal{B}_0 \succeq \mathcal{B}$ such that Γ^* is satisfiable in $\mathcal{B}_0$. By an appropriate interpretation of the names occurring in A_n in $|\mathcal{B}_0|$ we have Γ_n satisfiable in $\mathcal{B}_0$ for each $n \in \omega$, whence $\boldsymbol{A}\ \&\ A_n \in S^\gamma(\mathcal{B}_0, \Delta)$. It follows that $\boldsymbol{A} \in S^\beta(\mathcal{B}_0, \Delta)$ whence $\boldsymbol{A} \in S^\beta(\mathcal{B}, \Delta)$ by Lemma 1 (ii).

When Δ is finite we have the following stronger result. Let Γ be defined as in the statement of Lemma 2.

Lemma 3. *Let $n \in \omega$, Δ be finite, and $\boldsymbol{A}$ be a formula in $S(X)$ containing at most z_1 free and possibly some names for the elements of the universe of a model. There exists $\Gamma^* \subseteq S(X)$ depending only on n, $\boldsymbol{A}$, and Δ such that for any $\mathcal{A}$, if $\boldsymbol{A} \in S_1(|\mathcal{A}|)$, then $\boldsymbol{A} \in S^n(\mathcal{A}, \Delta)$ if and only if Γ^* is weakly satisfied in $\mathcal{A}$.*

Proof. The proof is by induction. For $n = 0$ the conclusion is clear. Suppose the result is true for $n = m$ where $m \geqslant 0$. For $\boldsymbol{B} \in S^{m+1}(\mathcal{B}, \Delta)$ let $\Theta(\boldsymbol{B})$ consist of all $\boldsymbol{D}$ in $\Delta \cup \Delta^-$ such that $\boldsymbol{B}\ \&\ \boldsymbol{D}' \in S^{m+1}(\mathcal{B}', \Delta)$ and $\boldsymbol{B}\ \&\ \neg \boldsymbol{D}' \in S^m(\mathcal{B}', \Delta)$ for some $\mathcal{B}' \succeq \mathcal{B}$ and some $\mathcal{B}'$-instance $\boldsymbol{D}'$ of $\boldsymbol{D}$. Given $\boldsymbol{A} \in S^{m+1}(\mathcal{A}, \Delta)$ choose $\mathcal{B} \succeq \mathcal{A}$ and $\boldsymbol{B} \in S^{m+1}(\mathcal{B}, \Delta)$ such that $\boldsymbol{B}$ implies $\boldsymbol{A}$ and $\Theta(\boldsymbol{B})$ is minimal. Let $\boldsymbol{D} \in \Theta(\boldsymbol{B})$ then we can form sequences $\langle \mathcal{B}_i |\ i \in \omega \rangle$ and $\langle \boldsymbol{D}_i |\ i \in \omega \rangle$ such that for all $i \in \omega$, $\mathcal{B}_{i+1} \succeq \mathcal{B}_i \succeq \mathcal{B}$, $\boldsymbol{D}_i$ is a $\mathcal{B}_i$-instance of $\boldsymbol{D}$, $\boldsymbol{B}\ \&\ \boldsymbol{D}_0\ \&\ \ldots\ \&\ \boldsymbol{D}_i \in S^{m+1}(\mathcal{B}_i, \Delta)$, and $\boldsymbol{B}\ \&\ \boldsymbol{D}_0\ \&\ \ldots$ $\ldots\ \&\ \boldsymbol{D}_{i-1}\ \&\ \neg \boldsymbol{D}_i \in S^m(\mathcal{B}_i, \Delta)$. Let Γ_i be the class of formulas corresponding to $\boldsymbol{A}\ \&\ \boldsymbol{D}_0\ \&\ \ldots\ \&\ \boldsymbol{D}_{i-1}\ \&\ \neg \boldsymbol{D}_i$ under the induction hypothesis. Adjoin infinitely many names for each member of $\bigcup \{|\mathcal{B}_i|\ |\ i \in \omega\}$. Without loss suppose that for each $i \in \omega$ no name occurs twice in $\boldsymbol{D}_i$, that no name occurring in $\boldsymbol{D}_i$ occurs in $\boldsymbol{A}$ or in $\boldsymbol{D}_j$ for $j \neq i$, and that if $i \neq j$ then Γ_i and Γ_j have no variable in common. Next we substitute new variables for the names in $\boldsymbol{D}_0, \boldsymbol{D}_1, \ldots$ choosing different variables for different names. This converts Γ_i to Γ_i' for each $i \in \omega$. Notice that the names originating in $\boldsymbol{A}$ are left unchanged. Now let $\Gamma^*(\boldsymbol{D})$ be $\bigcup \{\Gamma_i' |\ i \in \omega\}$ and notice that $\Gamma^*(\boldsymbol{D})$ is unique to within an isomorphism of the set of variables occurring in its members. Given any sets $\Gamma_1^*, \ldots, \Gamma_n^*$ we can construct Γ^* such that Γ^* will be weakly satisfiable if and only if one of $\Gamma_1^*, \ldots, \Gamma_n^*$ is weakly satisfiable. Since $\boldsymbol{D} \in \Delta \cup \Delta^-$ and Δ is finite the lemma follows.

The above lemma is analogous to Lemma 2.7 of [7]. An important application is:

Lemma 4. *Let Δ be finite, $n \in \omega$, and let $\boldsymbol{A} \in \mathrm{Tr}^n(\mathcal{A}, \Delta)$ have Δ-degree 1. For each $\boldsymbol{B} \in (\Delta \cup \Delta^-) \cap S_{k+1}$ there exists $\boldsymbol{C} \in S_k(|\mathcal{A}|)$ such that if $\mathcal{B} \succeq \mathcal{A}$ then $\boldsymbol{A}\ \&\ \boldsymbol{B}[z_1, \boldsymbol{j}_1, \ldots, \boldsymbol{j}_k] \in \mathrm{Tr}^n(\mathcal{B}, \Delta)$ if and only if $\boldsymbol{C}[\boldsymbol{j}_1, \ldots, \boldsymbol{j}_k]$ is true in $\mathcal{B}$ where $b_1, \ldots, b_k$ are arbitrary in $|\mathcal{B}|$.*

Proof. Let Γ^* be the set of formulas corresponding to

$$\boldsymbol{A}\ \&\ \neg \boldsymbol{B}[z_1, \boldsymbol{j}_1, \ldots, \boldsymbol{j}_k]$$

by Lemma 3. Consider the set of all ordered pairs $(\mathcal{B}, \langle b_1, \ldots, b_k \rangle)$ such that $\mathcal{B} \succeq \mathcal{A}$, $b_1, \ldots, b_k \in |\mathcal{B}|$, and Γ^* is not weakly satisfiable in $\mathcal{B}$. For every such pair some particular finite subset of Γ^* is not satisfiable in $\mathcal{B}$. It follows that there is a finite subset Γ^0 of Γ^* such that for any of the pairs considered Γ^0 is not satisfiable in $\mathcal{B}$. Otherwise by the compactness theorem for some $\mathcal{B} \succeq \mathcal{A}$ and $b_1, \ldots, b_k \in |\mathcal{B}|$ we should have both $\boldsymbol{A}\ \&\ \neg \boldsymbol{B}[z_1, \boldsymbol{j}_1, \ldots, \boldsymbol{j}_k]$ and $\boldsymbol{A}\ \&\ \boldsymbol{B}[z_1, \boldsymbol{j}_1, \ldots, \boldsymbol{j}_k]$ in $S^n(\mathcal{B}, \Delta)$ which would contradict $\boldsymbol{A}$ having Δ-degree 1. Let $\boldsymbol{C} \in S_k(|\mathcal{A}|)$ be chosen such that $\boldsymbol{C}[\boldsymbol{j}_1, \ldots, \boldsymbol{j}_k]$ is true if and only if Γ^0 is not satisfiable in $\mathcal{B}$.

Shelah observed in [6] that when Δ was a singleton the Δ-rank of a 1-type could only be finite; in [7], Lemma 2.8 this is proved for arbitrary finite Δ. It is easily shown by induction that our Δ-rank is less than Shelah's, which yields:

Lemma 5. *If Δ is finite then $\mathrm{Tr}^\alpha(\mathcal{A}, \Delta) = \emptyset$ if $\alpha \geqslant \omega$.*

Another way of looking at this lemma is to see it as an analogue of the result proved in [3] which says that $\mathrm{Tr}^\alpha(\mathcal{A}, S) = \emptyset$ if $\alpha \geqslant \omega_1$. Let $\boldsymbol{A} \in S_1(|\mathcal{A}|)$, we say that $\boldsymbol{A}$ is *Δ-stable in $\mathcal{A}$* if $\boldsymbol{A}$ has a Δ-rank in $\mathcal{A}$, and $\boldsymbol{A}$ is *stable in $\mathcal{A}$* if $\boldsymbol{A}$ is Δ-stable in $\mathcal{A}$ for every finite Δ. Let $\boldsymbol{A} \in S_1$ then $\boldsymbol{A}$ is called *stable* in the complete theory T if $\boldsymbol{A}$ is stable in some and hence every model of T. A complete theory T is called *stable* if $z_1 = z_1$ is stable in T; a thorough discussion of stability can be found in [7]. The next lemma is equivalent to Theorem 3.1 of [7]; we came on this result via a similar one of Baldwin [1], Theorem 12.

Lemma 6. *Let $\mathcal{A} \preceq \mathcal{B}$, $\boldsymbol{A}$ be a formula in $S_1(|\mathcal{A}|)$ which is Δ-stable, $\boldsymbol{B} \in \Delta^b(\mathcal{B})$, then there exists $\boldsymbol{C}$ in $S_1(|\mathcal{A}|)$ such that $\boldsymbol{A}(\mathcal{A}) \cap \boldsymbol{B}(\mathcal{B}) = \boldsymbol{C}(\mathcal{A})$.*

Lemma 7. *If $\boldsymbol{A}$ is minimal in $\mathrm{Tr}^\alpha(\mathcal{A}, S)$ then the S-degree of $\boldsymbol{A}$ is 1.*

Proof. Suppose for contradiction that $\boldsymbol{A}$ is minimal in $\mathrm{Tr}^\alpha(\mathcal{A}, S)$ but that the S-degree of $\boldsymbol{A}$ in $\mathcal{A}$ is >1. Then there exists $\mathcal{B} \succeq \mathcal{A}$ and $\boldsymbol{D} \in S$ such that for some $\mathcal{B}$-instance $\boldsymbol{D}'$ of $\boldsymbol{D}$, $\boldsymbol{A}\ \&\ \boldsymbol{D}'$ and $\boldsymbol{A}\ \&\ \neg \boldsymbol{D}'$ are both in $S^\alpha(\mathcal{B}, S)$. Since $\boldsymbol{A}$ has an S-rank it is certainly stable in $\mathcal{A}$ whence by Lemma 6 there exists $\boldsymbol{B} \in S_1(|\mathcal{A}|)$ such that $\boldsymbol{B}(\mathcal{A}) = \boldsymbol{A}(\mathcal{A}) \cap \boldsymbol{D}'(\mathcal{B})$.

Now one of B, A & $\neg B$ has S-rank $<\alpha$ in $\mathcal{A}$. Without loss suppose that B has S-rank $<\alpha$, then we can replace A by A & $\neg B$ to obtain $(A \,\&\, D')(\mathcal{B}) \cap |\mathcal{A}| = \emptyset$. Suppose that D' is $D[z_1, j_1, \ldots, j_n]$ where $b_1, \ldots, b_n \in |\mathcal{B}|$ and without loss suppose that $b_1, \ldots, b_n \notin |\mathcal{A}|$. Let Γ^* be the set of formulas corresponding to A & D' by Lemma 2. For $m \in \omega$ let D^m be $D[z_1, j_1^m, \ldots, j_n^m]$ where $b_1^m, \ldots, b_n^m$ are elements of a model yet to be found. Let A^m be $A \,\&\, D^m \,\&\, \neg D^{m-1} \,\&\, \ldots \,\&\, \neg D^0$ where A^0 is $A \,\&\, D^0$. Let Γ_m^* be obtained from Γ^* by first replacing every substitution instance of A & D' by the corresponding substitution instance of A^m, and then substituting $z_1^m, z_2^m, \ldots$ for $z_1, z_2, \ldots$ respectively. The variables z_k^m are chosen so that no two of them are the same.

We claim that it is consistent to suppose that there is an elementary extension $\mathcal{B}^*$ of $\mathcal{A}$ such that $b_1^m, \ldots, b_n^m \in |\mathcal{B}^*|$ for each $m \in \omega$ and such that $\bigcup\{\Gamma_m^* |\ m \in \omega\}$ is satisfiable in $\mathcal{B}^*$. If not, then by the compactness theorem, for some $k \in \omega$ and some finite $\Gamma^0 \subset \Gamma^*$ it is impossible to choose $b_1^0, \ldots, b_n^0, \ldots, b_1^{k-1}, \ldots, b_n^{k-1}$ in $|\mathcal{B}|$ such that $\bigcup\{\Gamma_m^0 |\ m < k\}$ is satisfiable in $\mathcal{B}$, where Γ_m^0 is formed from Γ^0 in exactly the same way as Γ_m^* is formed from Γ^*. Fix Γ^0 and consider the least such k. Now Γ_0^* is weakly satisfiable in $\mathcal{B}$ with $b_1^0 = b_1, \ldots, b_n^0 = b_n$ because A^0 is $A \,\&\, D^0$. Thus $k > 1$, say $k = l+1$. Since $b_1^0, \ldots, b_n^0, \ldots, b_1^{l-1}, \ldots, b_n^{l-1}$ can be chosen in $|\mathcal{B}|$ such that $\bigcup\{\Gamma_m^0 |\ m < l\}$ is satisfiable in $\mathcal{B}$, it follows that $b_1^0, \ldots, b_n^0, \ldots, b_1^{l-1}, \ldots, b_n^{l-1}$ can be chosen in $|\mathcal{A}|$ such that $\bigcup\{\Gamma_m^0 |\ m < l\}$ is satisfiable in $\mathcal{A}$. Call this latter choice the "$\mathcal{A}$-choice" and let a_i^m denote the value of b_i^m. Let Γ_m' be obtained from Γ_m^0 by replacing each instance of A^m by the corresponding instance of A^m & $\neg D'$ and substituting $z_1^{m+1}, z_2^{m+1}, \ldots$ for $z_1^m, z_2^m, \ldots$ respectively. Since $(A \,\&\, D')(\mathcal{B}) \cap |\mathcal{A}| = \emptyset$, for the $\mathcal{A}$-choice $\bigcup\{\Gamma_m' |\, m < l\}$ is satisfiable in $\mathcal{B}$, whence $\Gamma^0 \cup \bigcup\{\Gamma_m' |\ m < l\}$ is satisfiable in $\mathcal{B}$. Notice that substituting j_i^0 for j_i and j_i^{m+1} for j_i^m, for $1 \leqslant i \leqslant n$ and $m < l$, transforms $\Gamma^0 \cup \bigcup\{\Gamma_m' |\, m < l\}$ into $\Gamma_0^0 \cup \bigcup\{\Gamma_{m+1}^0 |\ m < l\}$. Thus if $b_1^0 = b_1, \ldots, b_n^0 = b_n$ and $b_i^{m+1} = a_i^m$, $\bigcup\{\Gamma_m^0 |\ m < l+1\}$ is satisfiable in $\mathcal{B}$. This contradicts the choice of l and shows that $\mathcal{B}^* \succeq \mathcal{A}$ exists such that for a suitable choice of $b_i^m \in |\mathcal{B}^*|$, $1 \leqslant i \leqslant n$ and $m < \omega$, $\bigcup\{\Gamma_m^* |\ m < \omega\}$ is satisfiable in $\mathcal{B}^*$. From Lemma 2 $A \,\&\, D^0$, $A \,\&\, D^1 \,\&\, \neg D^0, \ldots$ are all in $S^\alpha(\mathcal{B}^*, S)$, whence $A \in S^{\alpha+1}(\mathcal{B}^*, S)$. Since this contradicts the assumption that $A \in \mathrm{Tr}^\alpha(\mathcal{A}, S)$ the lemma is proved.

In Morley's terminology the last lemma says that all transcendental points in the Stone space of a model have degree 1. We now derive the corresponding result for Δ-rank. The analogous result in [7] is Theorem 3.3.

LEMMA 8. *If Δ is finite and A is minimal in* $\mathrm{Tr}^n(\mathcal{A}, \Delta)$ *then the Δ-degree of A in $\mathcal{A}$ is* 1.

Proof. Suppose for contradiction that A is minimal in $\mathrm{Tr}^n(\mathcal{A}, \Delta)$ but that the Δ-degree of A in $\mathcal{A}$ is >1. To simplify the argument we

suppose that in fact the Δ-degree of A is 2. Then there exists $\mathcal{B} \succeq \mathcal{A}$ and $D \in \Delta$ such that for some $\mathcal{B}$-instance D' of D, $A \,\&\, D'$ and $A \,\&\, \neg D'$ both have Δ-rank n and Δ-degree 1. Since A is Δ-stable in $\mathcal{A}$ we can apply Lemma 6 obtaining $B \in S_1(|\mathcal{A}|)$ such that $B(\mathcal{A}) = A(\mathcal{A}) \cap D'(\mathcal{B})$. If either $A \,\&\, B \,\&\, D'$ or $A \,\&\, \neg B \,\&\, \neg D'$ has Δ-rank $<n$ we can argue as in Lemma 7. Thus we assume that $A \,\&\, B \,\&\, D'$ and $A \,\&\, \neg B \,\&\, \neg D'$ both have Δ-rank n. Let $D \in S_{m+1}$. From Lemma 4 there exist formulas C and C^- in $S_m(|\mathcal{A}|)$ such that for any $\mathcal{B}' \succeq \mathcal{A}$ and $b_1, \ldots, b_m \in |\mathcal{B}'|$, $A \,\&\, B \,\&\, D[z_1, j_1, \ldots, j_m]$ has Δ-rank n if and only if $C[j_1, \ldots, j_m]$ is true in $\mathcal{B}'$, while $A \,\&\, \neg B \,\&\, \neg D[z_1, j_1, \ldots, j_m]$ has Δ-rank n if and only if $C^-[j_1, \ldots, j_m]$ is true. Since $A \,\&\, B \,\&\, D'$ and $A \,\&\, \neg B \,\&\, \neg D'$ both have Δ-rank n, $\exists z_1 \ldots \exists z_n (C \,\&\, C^-)$ is true in $\mathcal{B}$ and hence in $\mathcal{A}$. Hence for some $\mathcal{A}$-instance D'' of D we have $A \,\&\, B \,\&\, D''$ and $A \,\&\, \neg B \,\&\, \neg D''$ both with Δ-rank n which contradicts A being minimal in $\mathrm{Tr}^n(\mathcal{A}, \Delta)$.

2. The model extension theorem. We now prove the main result of the paper:

THEOREM. *Let $\mathcal{A}$, $\mathcal{B}$ be models of a countable stable theory and suppose that $|\mathcal{A}| \neq |\mathcal{B}|$, $\mathcal{A} \preceq \mathcal{B}$, and $p(\mathcal{A}) = p(\mathcal{B})$ where p is a unary predicate symbol. There exists $\mathcal{C} \succeq \mathcal{B}$ such that $|\mathcal{C}| \neq |\mathcal{B}|$ and $p(\mathcal{C}) = p(\mathcal{B})$.*

Proof. We first prove that there exists $\mathcal{C}_0 \succeq \mathcal{B}$ and $c \in |\mathcal{C}_0| - |\mathcal{B}|$ such that for any $A \in S_1(|\mathcal{B}| \cup \{c\})$ if $\exists z_1(A \,\&\, pz_1)$ is true in $\mathcal{C}_0$ then so is $A[i]$ for some $a \in p(\mathcal{A})$. The remainder of the proof is concerned with extending the subset $|\mathcal{B}| \cup \{c\}$ to the universe of a model $\mathcal{C} \succeq \mathcal{B}$ such that $p(\mathcal{C}) = p(\mathcal{B})$.

Let $\Delta_0, \Delta_1, \ldots$ be a strictly increasing sequence of finite subsets of S whose union is S. Such a sequence can be found because the language in question is countable. Choose $b \in |\mathcal{B}| - |\mathcal{A}|$. For each $n \in \omega$ choose $A_n \in S_1(|\mathcal{A}|)$ such that A_n has minimal Δ_n-rank subject to $A_n[j]$ being true in $\mathcal{B}$ and such that the Δ_n-degree of A_n is 1. From Lemma 8 this choice is possible. Let the Δ_n-rank of A_n be α_n. We define $\Gamma \subseteq S_1(|\mathcal{B}|)$ as follows: let $B \in S_{n+1}$, $b_1, \ldots, b_n \in |\mathcal{B}|$, and m be chosen such that $B \in \Delta_m$, then $B[z_1, j_1, \ldots, j_n] \in \Gamma$ if and only if $B[z_1, j_1, \ldots, j_n] \,\&\, A_m$ has Δ_m-rank α_m. We want to demonstrate both that Γ is well defined and that Γ is weakly satisfiable in $\mathcal{B}$. If not, there exist $B^1, \ldots, B^k \in S_{n+1}$, and $b_1^1, b_2^1, \ldots, b_n^k \in |\mathcal{B}|$ such that for each i in $1 \leqslant i \leqslant k$, $B^i \in \Delta_{m_i}$ and $B^i[z_1, j_1^i, \ldots, j_n^i] \,\&\, A_{m_i}$ has Δ_{m_i}-rank α_{m_i}, and such that

$$\neg\exists z_1(A_{m_1} \,\&\, \ldots \,\&\, A_{m_k} \,\&\, B^1[z_1, j_1^1, \ldots, j_n^1] \,\&\, \ldots \,\&\, B^k[z_1, j_1^k, \ldots, j_n^k])$$

is true in $\mathcal{B}$. For notational convenience suppose that $n = 1$ and let C^i be the formula corresponding to A_{m_i} and B^i by Lemma 4. It follows that

$$\exists z_2 \exists z_3 \ldots \exists z_{k+1} \neg\exists z_1(A_{m_1} \,\&\, \ldots \,\&\, A_{m_k} \,\&\, B^1[z_1, z_2] \,\&\, \ldots \\ \ldots \,\&\, B^k[z_1, z_{k+1}] \,\&\, C^1[z_2] \,\&\, \ldots \,\&\, C^k[z_{k+1}])$$

is true in $\mathcal{B}$ and hence in $\mathcal{A}$. Thus we may suppose that above $b_1^1, b_2^1, \ldots, b_n^k \in |\mathcal{A}|$. Since $A_{m_i}[j]$ is true for $1 \leqslant i \leqslant k$ there exists i such that $B^i[j, j_1^i]$ is false whence both $A_{m_i} \,\&\, B^i[z_1, j_1^i]$ and $A_{m_i} \,\&\, \neg B^i[z_1, j_1^i]$ have Δ_{m_i}-rank α_{m_i}, the latter because otherwise there would be a formula containing b whose Δ_{m_i}-rank is $< \alpha_{m_i}$. This contradicts the Δ_{m_i}-degree of A_{m_i} being 1. Thus Γ is well defined and consistent. Let $\mathfrak{C}_0 \succeq \mathcal{B}$ be chosen and $c \in |\mathfrak{C}_0|$ be chosen such that c realizes Γ, then $c \notin |\mathcal{B}|$. We claim that if $A \in S_1(|\mathcal{B}| \cup \{c\})$ and $\exists z_1(A \,\&\, pz_1)$ is true in $\mathfrak{C}_0$ then $A[i]$ is true in $\mathfrak{C}_0$ for some $a \in p(\mathcal{A})$. If not, let A be $B[k, z_1, j_1, \ldots, j_n]$ where $b_1, \ldots, b_n \in |\mathcal{B}|$, and let m be chosen such that B and $\exists z_2(B \,\&\, pz_2)$ are both in $\Delta_m \cap S_{n+2}$. Then by assumption $\exists z_2(B[z_1, z_2, j_1, \ldots, j_n] \,\&\, pz_2) \,\&\, A_m$ has Δ_m-rank α_m while for each $a \in p(\mathcal{A})$, $A_m \,\&\, B[z_1, i, j_1, \ldots, j_n]$ has Δ_m-rank $< \alpha_m$. Let $C_0 \in S_{n+1}(|\mathcal{A}|)$ be the formula corresponding to A_m and $\exists z_2(B \,\&\, pz_2)$ by Lemma 4, and let $C_1 \in S_{n+1}(|\mathcal{A}|)$ be the formula corresponding to A_m and B by Lemma 4, then

$$\exists z_2 \ldots \exists z_{n+1}\big(\exists z_1 C_0 \,\&\, \forall z_1(pz_1 \to \neg C_1)\big)$$

is true in $\mathcal{B}$ and hence in $\mathcal{A}$. Again we may suppose that $b_1, \ldots, b_n \in |\mathcal{A}|$. From the way A_m was chosen $\exists z_2(B[j, z_2, j_1, \ldots, j_n] \,\&\, pz_2)$ is true in $\mathcal{B}$ whence for some a in $p(\mathcal{A})$, $B[j, i, j_1, \ldots, j_n]$ is true in $\mathcal{B}$. But since $A_m \,\&\, B[z_1, i, j_1, \ldots, j_n]$ has Δ_m-rank $< \alpha_m$, $\neg B[j, i, j_1, \ldots, j_n]$ is true in $\mathcal{B}$. From this contradiction the claim follows.

If $\mathcal{B}$ is countable we can obtain the desired elementary extension $\mathfrak{C}$ of $\mathcal{B}$ by applying Ehrenfeucht's theorem ([8], p. 90) to $\mathrm{Th}(\mathfrak{C}_0, |\mathcal{B}| \cup \{c\})$. In the general case we can proceed as follows. We construct a transfinite sequence $\langle(\mathfrak{C}_\gamma, X_\gamma)\rangle_{\gamma<\beta}$ such that $X_0 = |\mathcal{B}| \cup \{c\}$, $\mathfrak{C}_{\gamma+1} \succeq \mathfrak{C}_\gamma$ and $X_{\gamma+1} = X_\gamma \cup \{c_\gamma\} \subseteq |\mathfrak{C}_{\gamma+1}|$ for each $\gamma < \beta$, and $\mathfrak{C}_\delta = \bigcup\{\mathfrak{C}_\gamma |\ \gamma < \delta\}$ and $X_\delta = \bigcup\{X_\gamma |\ \gamma < \delta\}$ for each limit ordinal $\delta \leqslant \beta$. Further, for each γ, $(\mathfrak{C}_\gamma, X_\gamma)$ is to have the property that has already been established for $\gamma = 0$, that is, for any $A \in S_1(X_\gamma)$ if $\exists z_1(A \,\&\, pz_1)$ is true in $\mathfrak{C}_\gamma$ then so is $A[i]$ for some $a \in p(\mathcal{A})$. Further our construction will be such that for each $\gamma < \beta$ there exists $A \in S_1(X_\gamma)$ such that $\exists z_1 A$ is true in $\mathfrak{C}_\gamma$, $A[i]$ is not true for any $a \in X_\gamma$, and $A[k_\gamma]$ is true in $\mathfrak{C}_{\gamma+1}$.

Suppose that $(\mathfrak{C}_\gamma, X_\gamma)$ has been defined without any of the conditions stipulated above being violated. Choose $A \in S_1(X_\gamma)$ such that $\exists z_1 A$ is true in $\mathfrak{C}_\gamma$ but such that $A[i]$ is not true for any $a \in X_\gamma$. (If no such A can be found the construction ceases.) We now define formulas $A_0, A_1, \ldots$ in $S_1(X_\gamma)$ as follows. Let A_0 be A and suppose that A_m has been defined such that $\exists z_1 A_m$ is true in $\mathfrak{C}_\gamma$. Choose $B \in (\Delta_m)^b(X_\gamma)$ such that $\exists z_1(A_m \,\&\, B)$ is true in $\mathfrak{C}_\gamma$ and there is no partition of $A_m \,\&\, B$ by formulas from $(\Delta_m)^b(X_\gamma)$. Such a B can certainly be found because $\mathrm{Th}(\mathfrak{C}_\gamma) = \mathrm{Th}(\mathcal{A})$ is stable by hypothesis. Let A_{m+1} be $A_m \,\&\, B$. We let $\mathfrak{C}_{\gamma+1}$ be any elementary extension of $\mathfrak{C}_\gamma$ containing an element c_γ such that $A_m[k_\gamma]$ is true in $\mathfrak{C}_{\gamma+1}$ for every m.

Such an elementary extension of $\mathfrak{C}_\gamma$ clearly exists and certainly $A[k_\gamma]$ is true in $\mathfrak{C}_{\gamma+1}$. As stated above we let $X_{\gamma+1} = X_\gamma \cup \{c_\gamma\}$. It only remains to show that if $A' \in S_1(X_{\gamma+1})$ and $\exists z_1(A' \,\&\, pz_1)$ is true in $\mathfrak{C}_{\gamma+1}$ then so is $A'[i]$ for some $a \in p(\mathcal{A})$. If not let A' be $A''[k_\gamma, z_1, k^1, \ldots, k^n]$ where $A'' \in S_{n+2}$ and $c^1, \ldots, c^n \in X_\gamma$. Let m be chosen such that A'' and $\exists z_2(A'' \,\&\, pz_2)$ are both in Δ_m. From the choice of B we see that

$$A_{m+1} \to \exists z_2(A''[z_1, z_2, k^1, \ldots, k^n] \,\&\, pz_2)$$

is valid in $\mathfrak{C}_\gamma$ and

$$A_{m+1} \to \neg A''[z_1, i, k^1, \ldots, k^n]$$

is valid in $\mathfrak{C}_\gamma$ for every a in $p(\mathcal{A})$. But now letting $A^{\#}$ be

$$\exists z_2(A_{m+1}[z_2] \,\&\, A''[z_2, z_1, k^1, \ldots, k^n])$$

we have $\exists z_1(A^{\#} \,\&\, pz_1)$ true in $\mathfrak{C}_\gamma$ and $A^{\#}[i]$ false in $\mathfrak{C}_\gamma$ for every a in $p(\mathcal{A})$. This contradicts our hypothesis, hence $(\mathfrak{C}_{\gamma+1}, X_{\gamma+1})$ does satisfy the conditions set out above.

If $(\mathfrak{C}_0, X_0)$, $(\mathfrak{C}_1, X_1)$, ... are now generated in the manner prescribed we shall eventually reach $(\mathfrak{C}_\gamma, X_\gamma)$ such that for any A in $S_1(X_\gamma)$, if $\exists z_1 A$ is true in $\mathfrak{C}_\gamma$ then $A[i]$ is true in $\mathfrak{C}_\gamma$ for some $a \in X_\gamma$. (This is obvious when one regards the union of the sets X_γ as being the Skolem closure of X_0.) We let β be the first such γ. It is clear that the substructure $\mathfrak{C}$ of $\mathfrak{C}_\beta$ defined by $|\mathfrak{C}| = X_\beta$ is an elementary submodel of $\mathfrak{C}_\beta$, and that $p(\mathfrak{C}) = p(\mathcal{A})$. This completes the proof of the theorem.

Shelah has isolated the principle used to form $\mathfrak{C}$ from the pair $(\mathfrak{C}_0, X_0)$ in [6], § 0, and his discovery of it was independent of the present work. A slightly different rendering is the following where we call a theory *essentially countable* if the set of nonlogical symbols which are not constants is countable.

LEMMA 9. *If T is an essentially countable stable theory T has a model $\mathcal{A}$ with the following property. If $a \in |\mathcal{A}|$ and $A \in S_{n+1}(T)$ there exists $B \in S_1(T)$ such that $B[i]$ is true in $\mathcal{A}$ and such that for any constants $c_1, \ldots, c_n$ of $L(T)$, $A[i, c_1, \ldots, c_n]$ is true in $\mathcal{A}$ only if $\vdash_T B \to A[z_1, c_1, \ldots, c_n]$.*

The proof of this lemma should be clear from the proof of the theorem. Unfortunately this lemma is false in general for uncountable stable theories and even for uncountable superstable theories. A counter-example for the case of stable theories is as follows. The language L consists of unary function symbols f_γ, $\gamma < \omega_1$, and constants $c_{\gamma,n}$, $\gamma < \omega_1$ and $n < \omega$. For any $X \subseteq \omega_1$, X of power $\aleph_0$, let

$$L_X = \{f_\gamma |\ \gamma \in X\} \cup \{c_{\gamma,n} |\ \gamma \in X \,\&\, n < \omega\}.$$

Let $\mathcal{A}_X$ be the structure for L_X such that

$$|\mathcal{A}_X| = (X \times \omega) \cup \{g |\, g\colon X \to \omega \,\&\, g \text{ injective}\},$$

(γ, n) is the interpretation of $c_{\gamma,n}$ and the interpretation f_γ of f_γ is defined by $f_\gamma((\delta, n)) = (\delta, n)$ and $f_\gamma(g) = (\gamma, g(\gamma))$. It is easy to show that there is a complete theory T for L such that for every X of the kind considered $\mathrm{Th}(\mathcal{A}_X) \subseteq T$. If A is $z_1 \neq z_2$ then the conclusion of the lemma fails, because for any model $\mathcal{A}$ of T and $\gamma \in \omega_1$ there exists a in $\mathrm{rng} f_\gamma$ which is not the interpretation of any $c_{\gamma,n}$. For such an a there is no corresponding B.

Another negative result which is pertinent here is the fact that in general Ehrenfeucht's theorem fails for essentially countable theories even for those which are superstable.

3. A counterexample. It is easy to find $\mathcal{A}$, $\mathcal{B}$ such that $\mathcal{A} \preceq \mathcal{B}$, $|\mathcal{A}| \neq |\mathcal{B}|$, $p(\mathcal{A}) = p(\mathcal{B})$ and no $\mathcal{C}$ exists satisfying the conclusion of our theorem. For example let $|\mathcal{B}|$ be the real numbers, $p_{\mathcal{B}}$ be the rationals. Let the only other nonlogical symbol be a binary predicate symbol q and let $q_{\mathcal{B}} = \{(x, y) |\ x \in |\mathcal{B}|, y \in p(\mathcal{B}), \text{ and } x < y\}$. Let $\mathcal{A}$ be any proper elementary submodel of $\mathcal{B}$ such that $p(\mathcal{A}) = p(\mathcal{B})$. (In fact we can make $\mathcal{A}$ countable if we wish.)

However counterexamples in which $|\mathcal{B}|$ is countable seem to be scarce and we had to go to some pains to find the one which is presented below. Let the language L consist of a ternary predicate symbol q, binary predicate symbols $q^0, q^1, \ldots$, a unary predicate symbol p, constants $\boldsymbol{\omega}, \boldsymbol{0}, \boldsymbol{1}, \ldots$ naming $\omega, 0, 1, \ldots$ respectively, and m_n naming (m, n) for each $(m, n) \in \omega \times \omega$. A structure $\mathcal{A}$ is called *admissible* if its universe is the union of $\omega \cup \{\omega\}$ and a finite subset F of $\omega \times \omega$, the language of $\mathcal{A}$ being L with the names of the members of $\omega \times \omega - F$ omitted and if the following conditions are satisfied:

(i) the following formulas are valid in $\mathcal{A}$

$$qz_1z_2z_4 \,\&\, qz_1z_3z_4 \cdot\to\cdot \neg z_1 = z_4 \,\&\, z_2 = z_3 \,\&\, \neg pz_1 \,\&\, pz_2 \,\&\, \neg pz_4,$$
$$\neg z_1 = z_4 \,\&\, \neg pz_1 \,\&\, \neg pz_4 \cdot\to\cdot \exists z_2\, qz_1z_2z_4,$$

and $q\,\boldsymbol{\omega}\, m\, n_m$ for each $(n, m) \in F$,

(ii) $p_{\mathcal{A}} = \omega$ and for each $n \in \omega$, $q^n_{\mathcal{A}} = \{(m, (m, n)) |\ (m, n) \in F\}$.

An *admissible pair* is an ordered pair $(\mathcal{A}, R)$ where $\mathcal{A}$ is an admissible structure and R is an equivalence relation on $(F \cup \{\omega\}) \times \omega$ such that all the equivalence classes under R are finite and at most a finite number have cardinality >1, and such that if (a, n) and (a', n') are R-equivalent then $qinz_1(\mathcal{A}) = qi'n'z_1(\mathcal{A})$.

An admissible pair $(\mathcal{A}', R')$ is said to be an *extension* of an admissible pair $(\mathcal{A}, R)$ if $\mathcal{A}'$ is an extension of $\mathcal{A}$ and if R-equivalence implies R'-equivalence. Let $(\mathcal{A}, R)$ be an admissible pair and $A \in S_0(L)$ be interpreted in $\mathcal{A}$, we define $(\mathcal{A}, R)$ *forces* A, written $(\mathcal{A}, R) \Vdash A$ by induction on the length of A as follows. If A is quantifier free, $(\mathcal{A}, R) \Vdash A$ if $\mathcal{A}(A) = T$. Also $(\mathcal{A}, R) \Vdash \neg A$ if there is no extension $(\mathcal{A}', R')$ of $(\mathcal{A}, R)$ such that $(\mathcal{A}', R') \Vdash A$; $(\mathcal{A}, R) \Vdash A_0 \vee A_1$ if $(\mathcal{A}, R) \Vdash A_0$ or $(\mathcal{A}, R) \Vdash A_1$; and $(\mathcal{A}, R) \Vdash \exists zA$ if $(\mathcal{A}, R) \Vdash A_z[c]$ for some constant c in L. It is easy to see that this definition is consistent. It is easy to prove that if $(\mathcal{A}, R) \Vdash A$ and $(\mathcal{A}', R')$ is an extension of $(\mathcal{A}, R)$ then $(\mathcal{A}', R') \Vdash A$, and that for any admissible $(\mathcal{A}, R)$ and $A \in S_0(L)$ there is an extension $(\mathcal{A}', R')$ of $(\mathcal{A}, R)$ such that either $(\mathcal{A}', R') \Vdash A$ or $(\mathcal{A}', R') \Vdash \neg A$. Let $(\mathcal{A}_0, R_0)$, $(\mathcal{A}_1, R_1), \ldots$ be chosen so that each pair is extended by its successor, and such that for each $A \in S_0(L)$ there exists n such that $(\mathcal{A}_n, R_n)$ forces either A or $\neg A$. Let L' have the same predicate symbols as L but no constants and let $\mathcal{B}' = (\bigcup \{\mathcal{A}_n |\ n \in \omega\})|L'$. Let $\mathcal{B}$ be the substructure of $\mathcal{B}'$ whose universe is $|\mathcal{B}'| - \{\omega\} = \omega \cup (\omega \times \omega)$.

Suppose for contradiction that $\mathcal{B}''$ is a proper elementary extension of $\mathcal{B}'$ and that $p(\mathcal{B}') = p(\mathcal{B}'')$. Let $b \in |\mathcal{B}''| - |\mathcal{B}'|$. From (i) above which must hold of $\mathcal{B}'$ with $F = \omega \times \omega$ we see that for some m in ω we have $q\omega mj$ true in $\mathcal{B}''$. Also from (i) and (ii) the formulas

$$q\omega m z_1 \to \exists z_2 (q^m z_2 z_1 \,\&\, pz_2), \qquad q^m z_2 z_1 \,\&\, q^m z_2 z_3 \cdot\to z_1 = z_3$$

are valid in each $\mathcal{A}_n$, and hence in $\mathcal{B}'$, and hence in $\mathcal{B}''$. Thus for some $n \in \omega$, $q^m nj$ is true, whence $b = (n, m)$ since $q^m n n_m$ is also true. Thus $\mathcal{B}'$ has no proper elementary extension $\mathcal{B}''$ such that $p(\mathcal{B}') = p(\mathcal{B}'')$.

It only remains to show that $\mathcal{B} \preceq \mathcal{B}'$. For this we need some more definitions. Let A be in $S_k(L)$. We associate with A the language $L(A)$ which has predicate symbols p, q, together with those q^n's which occur in A, all the constants which occur in A and also each n such that q^n occurs in A. Let $(\mathcal{A}, R)$, $(\mathcal{A}', R')$ be admissible pairs with $L(\mathcal{A})$ and $L(\mathcal{A}')$ both extending $L(A)$ then a bijective map $f\colon |\mathcal{A}| \to |\mathcal{A}'|$ is said to induce an *$L(A)$-isomorphism of $(\mathcal{A}, R)$ and $(\mathcal{A}', R')$* if

(i) f is an isomorphism between $\mathcal{A}|L(A)$ and $\mathcal{A}'|L(A)$,

(ii) f induces an isomorphism between R and R', and

(iii) (ω, n) is R'-equivalent to $(f(\omega), n)$ for each n such that q^n is in $L(A)$.

LEMMA 10. *Let A be in $S_k(L)$ and $f\colon |\mathcal{A}| \to |\mathcal{A}'|$ induce an $L(A)$-isomorphism of $(\mathcal{A}, R)$ and $(\mathcal{A}', R')$. If $f(a_n) = a'_n$ for $1 \leqslant n \leqslant k$, then $(\mathcal{A}, R) \Vdash A[i_1, \ldots, i_k]$ if and only if $(\mathcal{A}', R') \Vdash A[i'_1, \ldots, i'_k]$.*

Proof. We proceed by induction on A. The only step which is not obvious consists in showing that some extension of $(\mathcal{A}, R)$ forces $A[i_1, \ldots, i_k]$ if and only if some extension of $(\mathcal{A}', R')$ forces $A[i'_1, \ldots, i'_k]$. Suppose that $(\mathcal{A}_0, R_0)$ is an extension of $(\mathcal{A}, R)$ such that $(\mathcal{A}_0, R_0) \Vdash A[i_1, \ldots, i_k]$. We extend f to an injective mapping f_0 with domain $|\mathcal{A}_0|$ as follows. If $a_0 \in |\mathcal{A}_0| - |\mathcal{A}|$ let m be the unique member of ω such that $(f^{-1}(\omega), m, a_0) \in q_{\mathcal{A}_0}$. If $q^m \in L(A)$ and a_0 is (n, m) then we define $f_0(a_0) = (f(n), m)$. Otherwise, that is, if $q^m \notin L(A)$ we let $f_0(a_0)$ be any pair $(n', f(m))$ subject to f_0 being injective. Let $|\mathcal{A}'_0|$ be the range of f_0 and let the predicate symbols be interpreted on $|\mathcal{A}_0'|$ so as to make $\mathcal{A}_0$ admissible and f_0 an isomorphism of $\mathcal{A}_0|L(A)$ and $\mathcal{A}_0'|L(A)$. Let R_0' be the unique equivalence relation on $(|\mathcal{A}_0'|-p(\mathcal{A}_0'))\times\omega$ such that R_0 is isomorphic to R_0' via f_0. Then f_0 induces an $L(A)$-isomorphism between $(\mathcal{A}_0, R_0)$ and $(\mathcal{A}_0', R_0')$. By the induction hypothesis $(\mathcal{A}_0', R_0')\Vdash A[i_1', \ldots, i_k']$ since $(\mathcal{A}_0, R_0)\Vdash A[i_1, \ldots, i_k]$. Thus some extension of $(\mathcal{A}', R')$ forces $A[i_1', \ldots, i_k']$. Since f^{-1} induces an $L(A)$-isomorphism between $(\mathcal{A}', R')$ and $(\mathcal{A}, R)$, if some extension of $(\mathcal{A}', R')$ forces $A[i_1', \ldots, i_k']$ then some extension of $(\mathcal{A}, R)$ forces $A[i_1, \ldots, i_k]$. This completes the proof of the lemma.

Using the lemma we can now prove that $\mathcal{B}\lesssim\mathcal{B}'$. We first observe that if $A\in S_0(L)$ then $\mathcal{B}'(A)=T$ if and only if $(\mathcal{A}_n, R_n)\Vdash A$ for some $n\in\omega$. Let $A\in S_1(L)$ and suppose that $(\mathcal{A}, R)\Vdash A[\omega]$. We define an extension $(\mathcal{A}', R')$ of $(\mathcal{A}, R)$ as follows. Let $|\mathcal{A}'|=|\mathcal{A}|\cup\{a\}$ where $a=(n, m)$ is chosen in $\omega\times\omega-|\mathcal{A}|$ so that $q_m\notin L(A)$. Let the predicate symbols be interpreted on $|\mathcal{A}'|$ so that $\mathcal{A}'$ is admissible and so that $f\colon |\mathcal{A}'|\to|\mathcal{A}'|$ is an automorphism of $\mathcal{A}'|L(A)$, where $f(\omega)=a$, $f(a)=\omega$, and f is the identity on $|\mathcal{A}'|-\{a, \omega\}$. Let R' be the least equivalence relation on $(|\mathcal{A}'|-p(\mathcal{A}'))\times\omega$ which extends R and which is such that (a, m) and (ω, m) are R'-equivalent for each m such that either the R-equivalence class of (ω, m) has power >1 or $q^m\in L(A)$. Now f induces an $L(A)$-automorphism of $(\mathcal{A}', R')$. It follows from the lemma that $(\mathcal{A}', R')\Vdash A[i]$ since $(\mathcal{A}', R')\Vdash A[\omega]$. Thus if $A[\omega]$ is true in $\mathcal{B}'$ so is $A[i]$ for some $a\neq\omega$. This demonstrates that $\mathcal{B}\lesssim\mathcal{B}'$.

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An isomorphism theorem of the Hurewicz-type in Borsuk's theory of shape

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Introduction. In Hurewicz's well-known paper [6] is a homomorphism φ defined from the nth homotopy group $\pi_n(X)$ into the nth singular homology group $H_n(X)$ with integral coefficients, for any compact, pathwise-connected space X, and it is proved there (for $n\geqslant 2$) that if the space X is $(n-1)$-connected (that is, if $\pi_1(X)\approx\pi_2(X)\approx\ldots\ldots\approx\pi_{n-1}(X)\approx 0$), then the homomorphism φ is an isomorphism.

In this note an analogous homomorphism with similar properties will be constructed on the ground of Borsuk's theory of shape (introduced in [1]).

The singular homology groups of a pointed compactum (X, x_0) will be replaced by the Vietoris–Čech homology groups of (X, x_0), denoted by $\check{H}_q(X, x_0)$, and the homotopy groups $\pi_q(X, x_0)$ will be replaced by the so called fundamental groups $\underline{\pi}_q(X, x_0)$, defined by K. Borsuk (see also [1]). In the general case, Hurewicz's assumption of the $(n-1)$-connectednes of X will be replaced by approximative q-connectedness (for $q=0,1,\ldots\ldots,n-1$) of (X, x_0) (see for instance [2], p. 266, or Definition 3.1 in this paper). But if the pointed compactum (X, x_0) is connected and movable (see [2], § 4) then the assumption of the approximative q-connectedness (for $q=0,1,\ldots,n-1$) is equivalent to $\underline{\pi}_1(X, x_0)\approx\underline{\pi}_2(X, x_0)\approx\ldots\ldots\approx\underline{\pi}_{n-1}(X, x_0)$.

§ 1 of this paper contains a modified definition of the homology groups $\check{H}_q(X, x_0)$ and a proof of the equivalence of this definition to the original Vietoris definition. § 2 contains a construction of a homomorphism $\underline{\varphi}\colon \underline{\pi}_n(X, x_0)\to\check{H}_n(X, x_0)$, called the limit Hurewicz homomorphism. In § 3 the following theorem is proved:

If the pointed compactum (X, x_0) is approximatively q-connected for $q=0,1,\ldots,n-1$ $(n\geqslant 2)$, then the limit Hurewicz homomorphism $\underline{\varphi}$ is an isomorphism.

§ 1. The groups $\Gamma_q(X, x_0)$. Let Q denote the Hilbert-cube, X — a non-empty closed subset of Q and x_0 — a point lying in X. For any positive