The theorem of Miss Mullikin-Mazurkiewicz-van Est for unicoherent Peano spaces

by

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1. Introduction. In [6] Miss Mullikin proved the following theorem:

If X is the plane and $M_1, M_2, \ldots$ is a sequence of disjoint closed subsets of X no one of which separates X, then $\bigcup_{i=1}^{\infty} M_i$ does not separate X.

In [5] Mazurkiewicz considerably simplified Miss Mullikin's proof, and in [9] van Est used a homological argument to extend the theorem to the n-dimensional Euclidean spaces. On the other hand, while the theorem can easily be shown to hold for a unicoherent Peano continuum X (see [2]), in [2] the second-named author gave an example of a unicoherent Peano space X for which the theorem does not hold. In view of this, he raised the question in [2] of finding a class of unicoherent Peano spaces in which the theorem holds, and which is sufficiently wide to include the Euclidean spaces. In this note we answer this question. In fact, we show that the theorem holds for a Peano space X having a covering by unicoherent regions $U_1 \subset \overline{U}_1 \subset U_2 \subset \overline{U}_2 \subset \ldots$ with compact closures. (Such a space is necessarily unicoherent, by Theorems 3 and 8 of § 3, Chap. I of [1].)

2. The theorem. A *Peano space* is a locally compact, connected and locally connected metric space. A connected space X is said to be *unicoherent* if for each pair of closed and connected subsets A, B such that $X = A \cup B$, $A \cap B$ is connected. A subset Q of a space X is said to separate two points a, b *irreducibly in* X if it separates a, b in X but none of its proper subsets separates a, b in X. We then have the following simple lemma.

LEMMA. *Let M be a subset of a locally connected and completely normal space X which separates two points a, b in X. Then M contains a closed subset Q of X which separates a, b irreducibly in X.*

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Proof. Let $X-M = S\cup T$, where S, T are separated sets containing a, b, respectively. By the complete normality of X, there are two disjoint open sets U, V containing S, T, respectively. Let C be the component of $X-\overline{U}$ containing b, and let D be the component of $X-\overline{C}$ containing a. Then $\operatorname{Fr} D\subset \operatorname{Fr} C\subset M$, and so $Q=\operatorname{Fr} D$ meets the requirements of the lemma.

The proof of our theorem relies on two ideas introduced by Mazurkiewicz in [4] and [5]. Namely, we use an irreducible separation of the space, as Mazurkiewicz did in [5], and we use Baire's category theorem, as Mazurkiewicz did in the proof of Lemma 3 of [4]. (The latter proof is credited to Kuratowski in [4].)

THEOREM. *Let X be a Peano space having a covering by unicoherent regions $U_1\subset \overline{U}_1\subset U_2\subset \overline{U}_2\subset \ldots$ with compact closures. If $M_1, M_2, \ldots$ is a sequence of disjoint closed subsets of X no one of which separates X, then $M=\bigcup_{i=1}^{\infty} M_i$ does not separate X.*

Proof. Suppose on the contrary that M separates some pair of points a, b in X. By virtue of the lemma we may suppose that M is a closed subset of X which separates a, b irreducibly in X. We may also suppose without loss of generality that $a, b\in U_1$.

Now $M\cap U_n$ separates a, b in the unicoherent subspace U_n of X. Thus by Theorem 1 (vi), p. 429 of [8] a component L_n of $M\cap U_n$ separates a, b in U_n. By Sierpiński's theorem on continua in [7], the continuum $\overline{L}_n$ cannot be expressed as the union of a non-degenerate countable collection of non-empty disjoint closed sets. Since $\overline{L}_n=\bigcup_{i=1}^{\infty}\overline{L}_n\cap M_i$, it follows that $\overline{L}_n=\overline{L}_n\cap M_{i_n}$, for some i_n, and so $L_n\subset M_{i_n}\cap U_n$. Thus $M_{i_n}\cap U_n$ separates a, b in U_n.

Let $K=\limsup M_{i_n}$. Since we are assuming that M is closed $K\subset M$. We claim that $M=K$. To prove this it suffices to show that K separates a, b in X, because we are assuming that M separates a, b irreducibly in X. Therefore suppose that K does not separate a. b in X and let ab be an arc in $X-K$ joining a, b. Then $ab\subset U_k$, for some k. Since $\limsup(M_{i_n}\cap \overline{U}_k)\subset K\cap \overline{U}_k$, it follows from Theorem 7.2, p. 12 of [10] that $M_{i_N}\cap \overline{U}_k\subset \overline{U}_k-ab$, for some $N\geqslant k$. Thus $M_{i_N}\cap U_k\subset U_k-ab$. Consequently $ab\subset U_N-M_{i_N}$; i.e., $M_{i_N}\cap U_N$ does not separate a, b in U_N, which is false. Thus $M=K$.

Finally, it follows from Baire's category theorem that $M_j-(\overline{M-M_j})\neq\emptyset$, for some j, because M is locally compact. Since M_j does not separate a, b in X, there is an arc ab in $X-M_j$ joining a, b. Then $ab\subset U_k$, for some k. Now $M_{i_n}\neq M_j$ for $n\geqslant k$, because $M_{i_n}\cap U_n$ separates a, b in U_n

and hence, for $n\geqslant k$, $M_{i_n}\cap U_k$ — unlike $M_j\cap U_k$ — separates a, b in U_k. Consequently,

$$(\limsup M_{i_n})\cap\big(M_j-(\overline{M-M_j})\big)=\emptyset;\qquad \text{i.e.,}\qquad K\cap\big(M_j-(\overline{M-M_j})\big)=\emptyset .$$

Thus $M\neq K$, and this contradiction proves the theorem.

In conclusion we remark that the theorem holds if $M_1, M_2, \ldots$ are merely *mutually separated sets* (i.e., each is disjoint from the closure of each other), because Sierpiński's theorem on continua (loc. cit.) can be proved without change for a decomposition of a continuum into a sequence of mutually separated sets.

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