# Total and absolute paracompactness

by

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A Hausdorff space $X$ is *totally paracompact* [4] if every open base for $X$ contains a locally finite cover. If $X$ is a closed subspace of $Y$ such that every open base for $Y$ contains a locally finite (in $Y$) cover of $X$, we say that $X$ is *totally paracompact relative to* $Y$ [8]. Clearly, this implies that $X$ is totally paracompact. A paracompact space $X$ is *absolutely paracompact* [9] if for every closed imbedding $h\colon X \to Y$ into a paracompact space $Y$, $h(X)$ is totally paracompact relative to $Y$. Thus absolute paracompactness implies total paracompactness. The space of rationals is totally paracompact but not absolutely paracompact [5]; the space of irrationals is not totally paracompact [1], [3].

We show in § 1 that absolute paracompactness is weakly hereditary, and is characterized by a vanishing non-locally-absolutely paracompact kernel. A countable sum theorem is obtained, from which it follows that $F_\sigma$-subsets of totally paracompact spaces, and paracompact countable sums of closed absolutely paracompact spaces, are totally paracompact. In § 2 we consider a modification of absolute paracompactness suitable for metric spaces, and with applications particularly to $\sigma$-locally separable metric spaces. It is shown in § 3 that every Hurewicz space is totally paracompact.

## 1. Absolute paracompactness.

1.1. THEOREM. *Every closed subspace of an absolutely paracompact space is absolutely paracompact.*

Proof. Let $A$ be a closed subspace of an absolutely paracompact space $X$, and $h\colon A \to Y$ a closed imbedding into a paracompact space $Y$. It is easily verified that the adjunction space $Z = X \cup_h Y$ is paracompact. We regard $X$ and $Y$ as closed subspaces of $Z$, with $X \cap Y = A$. Let $\mathfrak{U}$ be an open base for $Y$, and let $\mathfrak{V}$ be an open base for $Z$ such that $V \in \mathfrak{V}$ and $V \cap Y \neq \emptyset$ imply $V \cap Y \in \mathfrak{U}$. Then there exists a locally finite (in $Z$) subcollection $\mathfrak{V}_0$ of $\mathfrak{V}$ covering $X$, and $\mathfrak{U}_0 = \{V \cap Y\colon V \in \mathfrak{V}_0,\ V \cap Y \neq \emptyset\}$ is a locally finite subcollection of $\mathfrak{U}$ covering $A$.

1.2. LEMMA. *Every paracompact locally absolutely paracompact space is absolutely paracompact.*

Proof. Let $X$ be such a space, and suppose $X$ is a closed subspace of a paracompact space $Y$. Let $\{G_\alpha\}$ be a locally finite open cover of $Y$ such that $\bar{G}_\alpha \cap X$ is absolutely paracompact (or empty) for each $\alpha$. We may shrink $\{G_\alpha\}$ to an open cover $\{G'_\alpha\}$ such that $\bar{G}'_\alpha \subset G^\alpha$ for each $\alpha$. Let $\mathcal{U}$ be an open base for $Y$. Then there exists for each $\alpha$ a locally finite subcollection $\mathcal{U}_\alpha$ of $\mathcal{U}$ covering $\bar{G}'_\alpha \cap X$, with each member of $\mathcal{U}_\alpha$ contained in $G_\alpha$. It follows that $\bigcup_\alpha \mathcal{U}_\alpha$ is a locally finite cover of $X$.

Both of the above results hold also for total paracompactness. In the case of 1.1 this is well-known and easy to see, and in the case of 1.2 was proved by Telgársky [8], whose argument we have virtually repeated.

1.3. LEMMA. *If a paracompact space $X$ contains a closed absolutely paracompact subspace $A$ such that $X \backslash A$ is locally absolutely paracompact, then $X$ is absolutely paracompact.*

Proof. Let $X = A \cup (X \backslash A)$ be such a space, and suppose $X$ is a closed subspace of a paracompact space $Y$. Let $\mathcal{U}$ be an open base for $Y$. There exists a locally finite cover $\mathcal{U}'$ of $A$; set $V = |\mathcal{U}'|$. Then the closed subspace $X \backslash V$ contained in $X \backslash A$ is locally absolutely paracompact, hence absolutely paracompact, and there exists a locally finite cover $\mathcal{U}''$ of $X \backslash V$. Thus $\mathcal{U}' \cup \mathcal{U}''$ is a locally finite cover of $X$.

1.4. THEOREM. *If every nonempty closed subspace of a paracompact space $X$ is locally absolutely paracompact at some point, then $X$ is absolutely paracompact.*

Proof. Define the derivative $A^*$ of a closed subspace $A$ of $X$ as the set of all points of $A$ which have no absolutely paracompact neighborhood in $A$. Clearly, $A^*$ is a closed subspace of $A$, and by hypothesis $A^* \neq A$ if $A \neq \emptyset$. Set $X^{(0)} = X$, and $X^{(\beta)} = \bigcap_{\alpha<\beta} (X^{(\alpha)})^*$ for every ordinal $\beta$. Then $X^{(\alpha)}$ is a closed subspace of $X$ for each ordinal $\alpha$, and for $\alpha < \beta$, $X^{(\beta)}$ is a proper subspace of $X^{(\alpha)}$, unless $X^{(\alpha)} = \emptyset$. Thus eventually $X^{(\alpha)} = \emptyset$. We show by transfinite induction on $\alpha$ that any paracompact space with this property is absolutely paracompact. Suppose that any paracompact space $X$ for which $X^{(\alpha)} = \emptyset$ for some $\alpha < \beta$ is absolutely paracompact, and let $Y$ be a paracompact space such that $Y^{(\beta)} = \emptyset$. If $\beta$ is a limit ordinal, then $Y = \bigcup_{\alpha<\beta} (Y \backslash Y^{(\alpha)})$ is locally absolutely paracompact, hence absolutely paracompact. If $\beta = \alpha+1$, then $Y = Y^{(\alpha)} \cup (Y \backslash Y^{(\alpha)})$ is the union of a closed absolutely paracompact subspace and a locally absolutely paracompact subspace, and by 1.3 is absolutely paracompact.

The technique used above of forming the transfinite iterates of the non-locally-$P$ derivative of a space, with respect to a weakly hereditary property $P$, is well-known and is described, for example, in [7]. The eventually constant closed subspace $X^{(\alpha)} = X^{(\alpha+1)}$ is called the *non-locally-$P$ kernel* of $X$, and is the largest closed subspace which is nowhere-locally-$P$. It should be noted that the theorem and preceding lemma do not hold for total paracompactness. This is shown by the example of [5]: a paracompact but not totally paracompact space $S$ containing a closed copy $D$ of the rationals such that $S \backslash D$ is discrete.



A space is said to be *C-scattered* if it has an empty non-locally-compact kernel. It follows naturally from Theorem 1.4 that every paracompact $C$-scattered space is absolutely paracompact (this has been proved previously by Telgársky [9]). A completely regular space is *topologically complete* (in the sense of Čech) if it is a $G_\delta$-subset in its Stone-Čech compactification. The following corollary is another direct consequence of Theorem 1.4.

1.5. COROLLARY. *Every topologically complete paracompact space which is the countable union of closed absolutely paracompact subspaces is absolutely paracompact.*

Proof. Topological completeness is weakly hereditary, and implies the Baire property. Thus the hypothesis of 1.4 is satisfied.

1.6. THEOREM. *Every paracompact space $X$ which is the countable union of closed subspaces totally paracompact relative to $X$ is totally paracompact.*

Proof. Let $X = \bigcup_1^\infty A_i$, with each $A_i$ totally paracompact relative to $X$, and let $\mathcal{U}$ be an open base for $X$. There exists a locally finite subcollection $\mathcal{A}_1$ of $\mathcal{U}$ covering $A_1$. Let $\mathcal{A}_1^*$ be an open star-refinement of the open cover $\mathcal{A}_1 \cup \{X \backslash A_1\}$ of $X$, and let $\mathcal{U}_1$ be the largest subcollection of $\mathcal{U}$ refining $\mathcal{A}_1^*$. Inductively, suppose a locally finite subcollection $\mathcal{A}_i$ of $\mathcal{U}_{i-1}$ covering $A_i$ has been chosen. Let $\mathcal{A}_i^*$ be an open star-refinement of the open cover $\mathcal{A}_i \cup \{X \backslash A_i\}$ of $X$ and let $\mathcal{U}_i$ be the largest subcollection of $\mathcal{U}_{i-1}$ refining $\mathcal{A}_i^*$. There exists a locally finite subcollection $\mathcal{A}_{i+1}$ of $\mathcal{U}_i$ covering $A_{i+1}$. For each $j$, set $\mathcal{V}_j = \{V \in \mathcal{A}_j\colon\ V \backslash \bigcup_{i<j} |\mathcal{A}_i| \neq \emptyset\}$. Then $\mathcal{V} = \bigcup_1^\infty \mathcal{V}_j$ covers $X$, and we claim that $\mathcal{V}$ is locally finite. Every $x \in X$ is in some $A_i$, and in some $W \in \mathcal{A}_i^*$. Suppose $V \cap W \neq \emptyset$ for $V \in \mathcal{A}_j \subset \mathcal{U}_{j-1}$ for $j > i$. Since $\mathcal{U}_{j-1}$ refines $\mathcal{A}_i^*$, $V \subset \operatorname{st}(W, \mathcal{U}_{j-1}) \subset \operatorname{st}(W, \mathcal{A}_i^*) \subset U \in \mathcal{A}_i$, and thus $V \notin \mathcal{V}_j$.

1.7. COROLLARY. *Every paracompact space which is the countable union of closed absolutely paracompact subspaces is totally paracompact.*

A space is *σ-locally compact* if it is the countable union of closed locally compact subspaces. (Of course, every perfectly normal space which is the countable union of locally compact subspaces is σ-locally compact.)

1.8. COROLLARY (this answers Problem 3.2 in [9]). *Every σ-locally compact paracompact space is totally paracompact.*

1.9. LEMMA (cf. [8], Theorem 1). *If $X \subset Y \subset Z$, $Z$ is totally paracompact and $X$ is closed in $Z$, then $X$ is totally paracompact relative to $Y$.*

Proof. Let $\mathcal{U}$ be an open base for $Y$. There exists an open base $\mathcal{V}$ for $Z$ such that for each $V \in \mathcal{V}$, either $V \cap X = \emptyset$ or $V \cap Y \in \mathcal{U}$. Let $\mathcal{V}_0 \subset \mathcal{V}$ be a locally finite cover of $Z$. Then $\mathcal{U}_0 = \{V \cap Y\colon V \in \mathcal{V}_0$ and $V \cap X \neq \emptyset\} \subset \mathcal{U}$ is a locally finite cover of $X$.

1.10. COROLLARY. *Every $F_\sigma$-subset of a totally paracompact space is totally paracompact.*

1.11. PROBLEM. *Is every topologically complete totally paracompact space absolutely paracompact? $C$-scattered?*

Since every nonempty nowhere-locally-compact complete metric space contains a closed copy of the irrationals, every totally paracompact complete metric space is $C$-scattered. It is known that every topologically complete Hurewicz space is $C$-scattered [10].

1.12. PROBLEM. *Is every paracompact space which is the countable union of closed totally paracompact subspaces totally paracompact?*

**2. $\mathcal{M}$-absolute paracompactness.** We say that a metric space $X$ is *$\mathcal{M}$-absolutely paracompact* if for every closed imbedding $h\colon X \to Y$ into a metric space $Y$, $h(X)$ is totally paracompact relative to $Y$. It is easily seen that the analogues of Theorems 1.1 and 1.4 for $\mathcal{M}$-absolute paracompactness are true. The analogue of Corollary 1.7 can be strengthened:

2.1. THEOREM. *Every metric space which is the countable union of closed $\mathcal{M}$-absolutely paracompact subspaces is $\mathcal{M}$-absolutely paracompact.*

Proof. Let $X = \bigcup_1^\infty A_i$, with each $A_i$ a closed $\mathcal{M}$-absolutely paracompact subspace, and suppose $X$ is a closed subspace of a metric space $Y$. Then $X = \bigcap_1^\infty G_i$, with each $G_i$ open in $Y$ and $\overline{G}_{i+1} \subset G_i$. Let $\mathcal{U}$ be an open base for $Y$. The proof of (1.6) is now repeated, with the following modification: $\mathcal{A}_i^*$ is an open star-refinement of the open cover $\mathcal{A}_i \cup \{G_i \setminus A_i\} \cup \{Y \setminus X\}$ of $Y$. Then the locally finite cover $\mathcal{V} \subset \bigcup_1^\infty \mathcal{A}_j$ is selected as before, subject to the condition that $V \cap X \neq \emptyset$ for each $V \in \mathcal{V}$.

2.2. COROLLARY. *Every $F_\sigma$-subset of an $\mathcal{M}$-absolutely paracompact space is $\mathcal{M}$-absolutely paracompact.*

2.3. COROLLARY. *Every σ-locally compact metric space is $\mathcal{M}$-absolutely paracompact.*

Thus the rationals, while not absolutely paracompact, are $\mathcal{M}$-absolutely paracompact. The class of $\mathcal{M}$-absolutely paracompact spaces is



quite extensive; we show that every σ-locally separable totally metacompact metric space is $\mathcal{M}$-absolutely paracompact. (A space is totally metacompact if every open base contains a point-finite cover.) Lelek [6] showed that every separable totally metacompact metric space is totally paracompact. In proving the following lemma we recast and extend his proof.

2.4. LEMMA. *Every separable totally metacompact metric space is $\mathcal{M}$-absolutely paracompact.*

Proof. Let $X$ be a closed separable totally metacompact subspace of a metric space $Y$, and let $\mathcal{U}$ be an open base for $Y$. We may assume that $Y$ has a metric whose restriction to $X$ is totally bounded, since $X$ has a totally bounded metric which can be extended to a metric on $Y$ [2]. For each $i$, let $X_i$ be a finite $1/i$-net in $X$, and let $\mathcal{R}_i = \{(V, W)\colon V, W \in \mathcal{U},\ \operatorname{diam} V,\ \operatorname{diam} W \leqslant 1/i,\ d(V, W) \leqslant 1/i,\ V \cap X_i \neq \emptyset\}$. Here, $d(V, W) = \inf\{d(v, w)\colon v \in V, w \in W\}$. Then, for each $i$, $\mathcal{G}_i = \{(V \cup W) \cap X\colon (V, W) \in \mathcal{R}_i\}$ is an open cover of $X$, with mesh $\mathcal{G}_i \leqslant 3/i$, and $\mathcal{G} = \bigcup_1^\infty \mathcal{G}_i$ is a base for $X$. Thus there exists a point-finite cover $\mathcal{F} \subset \mathcal{G}$; since each $X_i$ is finite, the collection $\mathcal{U}_i = \{V, W\colon (V, W) \in \mathcal{R}_i,\ (V \cup W) \cap X \in \mathcal{F}\}$ is finite. Set $\mathcal{V}_i = \{V \in \mathcal{U}_i\colon V \subset U \in \mathcal{U}_j$ only if $i \leqslant j\}$, and $\mathcal{V} = \bigcup_1^\infty \mathcal{V}_i$. Since mesh $\mathcal{U}_i \leqslant 1/i$ and $\bigcup_1^\infty \mathcal{U}_i$ covers $X$, $\mathcal{V}$ is a locally finite cover of $X$.

2.5. LEMMA. *Every σ-locally separable metric space is the countable union of closed locally separable subspaces.*

Proof. Let $A$ be a locally separable subspace of a metric space $X$. Set $\tilde{A} = \{x \in \overline{A}\colon \overline{A}$ is locally separable at $x\}$. Then $A \subset \tilde{A}$, $\tilde{A}$ is locally separable, and $\tilde{A}$ is open in $\overline{A}$. Thus $\tilde{A}$ is an $F_\sigma$-set in $X$, and the lemma follows.

2.6. THEOREM. *Every σ-locally separable totally metacompact metric space is $\mathcal{M}$-absolutely paracompact.*

Proof. It follows from 2.4 and the local character of $\mathcal{M}$-absolute paracompactness that every locally separable totally metacompact metric space is $\mathcal{M}$-absolutely paracompact. The theorem then follows from 2.1 and 2.5.

A regular space $X$ is said to be *Hurewicz* if, for every sequence $\gamma_1, \gamma_2, \ldots$ of open covers of $X$, there exist finite subcollections $\alpha_i \subset \gamma_i$ for each $i$ such that $\alpha_1 \cup \alpha_2 \cup \ldots$ covers $X$. Lelek [6] has shown that a separable metric space is totally paracompact if and only if it is Hurewicz. This characterization, together with Theorems 2.1 and 2.6, leads to the following corollaries.

2.7. COROLLARY. *Every σ-locally separable metric space which is the countable union of totally paracompact subspaces is totally paracompact.*

Proof. It suffices to consider the separable case, which (as Lelek has noted) is a direct consequence of the Hurewicz characterization, a countable union of Hurewicz spaces being Hurewicz.

2.8. COROLLARY. *Every product of a σ-locally separable totally paracompact metric space with a σ-locally compact metric space is totally paracompact.*

Proof. We need only consider the product of a separable totally paracompact metric space with a compact metric space. A standard-type argument shows that the product of a Hurewicz space with a compact space is a Hurewicz space [10].

2.9. PROBLEM. *Is every totally paracompact metric space $\mathcal{M}$-absolutely paracompact? σ-locally separable?*

Not every totally paracompact metric space is σ-locally compact, since there exist Hurewicz metric spaces which are not σ-compact (see [6]).

**3. Total paracompactness for Hurewicz spaces.** Lelek's theorem (see above) may be generalized in one direction:

3.1. THEOREM. *Every Hurewicz space is totally paracompact.*

Proof. Let $\mathcal{U}$ be an open base for a Hurewicz space $X$. There exists a countable subcollection $\mathcal{G}_1 = \{G_{1i}\}$ of $\mathcal{U}$ covering $X$, which by paracompactness can be shrunk to a locally finite open cover $\mathcal{F}_1 = \{F_{1i}\}$ such that $\overline{F}_{1i} \subset G_{1i}$ for each $i$. Let $\mathcal{U}_1 = \{U \in \mathcal{U}$: for each $i$, $U \subset G_{1i}$ or $U \cap F_{1i} = \emptyset\}$. Clearly, $\mathcal{U}_1$ is a base. Inductively, suppose $\mathcal{G}_n = \{G_{ni}\}$ is a countable cover of $X$ from the base $\mathcal{U}_{n-1}$, and $\mathcal{F}_n = \{F_{ni}\}$ a locally finite open cover such that $\overline{F}_{ni} \subset G_{ni}$ for each $i$. Then $\mathcal{U}_n = \{U \in \mathcal{U}_{n-1}$: for each $i$, $U \subset G_{ni}$ or $U \cap F_{ni} = \emptyset\}$ is a base, and contains a countable cover $\mathcal{G}_{n+1}$ of $X$. Consider the sequence $\{\mathcal{F}_n\}$ of open covers. There exist finite subcollections $\mathcal{F}'_n \subset \mathcal{F}_n$ such that $\mathcal{F} = \bigcup_1^\infty \mathcal{F}'_n$ covers $X$. Let $\mathcal{G} = \{G_{ni}: F_{ni} \in \mathcal{F}\}$, and let $\mathcal{V} = \{G_{ni} \in \mathcal{G}: G_{ni} \subset G_{pj} \in \mathcal{G}$ only if $n \leqslant p\}$. Then $\mathcal{V} \subset \mathcal{U}$ is a locally finite cover of $X$, since each member of $\mathcal{F}$ meets only finitely many members of $\mathcal{V}$.

3.2. PROBLEM. *Is every Lindelöf totally paracompact space Hurewicz?*

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