Characterizations of σ-spaces

by

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1. Introduction. The class of σ-spaces, first introduced by Okuyama in [8], is defined in terms of a sequence of locally finite collections. Under the assumption of regularity Nagata and Siwiec [7] showed that σ-spaces can be characterized in terms of a sequence of closure preserving collections and also in terms of a sequence of discrete collections. In this paper we obtain several characterizations of σ-spaces in terms of a sequence of open covers. It should be noted that our method of proof yields another proof of the Nagata-Siwiec results. We give two applications of our characterizations. In § 3 we answer affirmatively a question raised by Reed in [9], and in § 4 we prove Heath's result [4] that every stratifiable space is a σ-space.

A collection $\mathcal{F}$ of subsets of a topological space X is a *net* for X if for each point p in X and each open neighborhood U of p there is a F in $\mathcal{F}$ such that $p \in F \subseteq U$. A space with a σ-locally finite net is called a *σ-space* [8].

Unless otherwise stated no separation axioms are assumed. The set of natural numbers will be denoted by N.

2. The characterizations. Let $(X, \mathfrak{I})$ be a topological space and let g be a function from $N \times X$ into $\mathfrak{I}$ such that for each x in X, $x \in \bigcap_{n=1}^{\infty} g(n, x)$. Notice that if we let $\mathfrak{I}_n = \{g(n, x): x \text{ in } X\}$ then $\mathfrak{I}_1, \mathfrak{I}_2, \ldots$ is a sequence of open covers of X. Consider the following properties of the function g.

(A) If $y \in g(n, x)$ then $g(n, y) \subseteq g(n, x)$.

(B) If $p \in g(n, x_n)$ for $n = 1, 2, \ldots$ then the sequence $\langle x_n \rangle$ converges to p.

(C) If $p \in g(n, y_n)$ and $y_n \in g(n, x_n)$ for $n = 1, 2, \ldots$ then the sequence $\langle x_n \rangle$ converges to p.

(D) If $\{p, x_n\} \subseteq g(n, y_n)$ and $y_n \in g(n, x_n)$ for $n = 1, 2, \ldots$ then the sequence $\langle x_n \rangle$ converges to p.

(E) If $\{p, x_n\} \subseteq g(n, y_n)$ and $y_n \in g(n, p) \cap g(n, x_n)$ for $n = 1, 2, \ldots$ then the sequence $\langle x_n \rangle$ converges to p.

Remark. Semi-stratifiable space can be characterized in terms of a function g satisfying (B) and every developable space has a function g satisfying (D). (See [1], [2], [5], and [6].)

THEOREM. *The following are equivalent for a regular space* $(X, \mathfrak{I})$:

(1) *X has a σ-discrete net,*

(2) *X is a σ-space,*

(3) *X has a σ-closure preserving net,*

(4) *X has a function g satisfying* (A) *and* (B),

(5) *X has a function g satisfying* (C),

(6) *X has a function g satisfying* (D),

(7) *X has a function g satisfying* (B) *and* (E).

Proof. The following implications are easy: $(1)\Rightarrow(2)$, $(2)\Rightarrow(3)$, $(4)\Rightarrow(5)$, $(5)\Rightarrow(6)$, and $(6)\Rightarrow(7)$. To complete the proof we need only show that $(3)\Rightarrow(4)$ and $(7)\Rightarrow(1)$.

$(3)\Rightarrow(4)$: Let $\mathcal{F}_1, \mathcal{F}_2, \ldots$ be a sequence of closure preserving collections in X such that $\bigcup_{n=1}^{\infty}\mathcal{F}_n$ is a net for X. We may assume that each $\mathcal{F}_n$ covers X, and by the regularity of X we may also assume that each $\mathcal{F}_n$ is a closed collection. (This is the only place where regularity is used.) For $n = 1, 2, \ldots$ let $\mathfrak{I}_n = \bigwedge_{i=1}^{n}\mathcal{F}_i$. Note that each $\mathfrak{I}_n$ is a closure preserving closed cover of X and that the following property (*) is satisfied: if p is a point of X and U is an open neighborhood of p, then there is a k in N such that for each $n \geqslant k$ there is a G_n in $\mathfrak{I}_n$ such that $p \in G_n \subseteq U$.

For x in X and n in N let $g(n, x) = X - \bigcup\{G \text{ in } \mathfrak{I}_n\colon x \notin G\}$. Clearly $x \in g(n, x)$ and since $\mathfrak{I}_n$ is a closure preserving closed collection it follows that $g(n, x)$ is an open set. Finally, it is straightforward to check that the function g satisfies (A) and (B). (Use property (*) to prove (B).)

$(7)\Rightarrow(1)$: This is the most difficult implication. The key to the proof is Heath's technique [4] of showing that stratifiable spaces are σ-spaces. Let g be a function satisfying (B) and (E). We may assume that for all n in N and x in X, $g(n+1, x) \subseteq g(n, x)$. Let $\leqslant$ be a well ordering on X. For x in X and i, n in N let

$$H(x, i, n) = X - [(\bigcup\{g(n, y)\colon y \notin g(i, x)\}) \cup (\bigcup\{g(i, y)\colon y < x\})]$$

and let $\mathcal{H}(i, n) = \{H(x, i, n)\colon x \in X\}$. One can show that $H(x, i, n) \subset g(i, x)$ and that $\mathcal{H}(i, n)$ is a discrete collection. For $m = 1, 2, \ldots$ let $F(x, i, n, m) = \{y \in H(x, i, n)\colon x \in g(m, y)\}$ and let $\mathcal{F}(i, n, m) = \{F(x, i, n, m)\colon x \text{ in } X\}$. Since $\mathcal{H}(i, n)$ is a discrete collection, so is $\mathcal{F}(i, n, m)$. Let $\mathcal{F} = \bigcup\{\mathcal{F}(i, n, m)\colon i, n, m \text{ in } N\}$. To complete the proof it suffices to show that $\mathcal{F}$ is a net for X.

Let p be a point of X and let U be an open neighborhood of p. For $i = 1, 2, \ldots$ let x_i be the smallest element of X such that $p \in g(i, x_i)$. Since g satisfies (B) it follows that $x_i \to p$. Consider these assertions.

(a) For each m in N there is an index $I(m)$ such that for $i \geqslant I(m)$, $x_i \in U \cap g(m, p)$.

(b) For each i in N there is an index $K(i)$ such that if $n \geqslant K(i)$ and $y \notin g(i, x_i)$ then $p \notin g(n, y)$.

Assertion (a) follows from the fact that $x_i \to p$, and (b) can be proved using the fact that the function g satisfies (B).

Now let $\{i_m\colon m = 1, 2, \ldots\}$ be an increasing sequence of positive integers such that $x_{i_m} \in U \cap g(m, p)$ for $m = 1, 2, \ldots$, and let $\{n_m\colon m = 1, 2, \ldots\}$ be an increasing sequence of positive integers such that if $y \notin g(i_m, x_{i_m})$ then $p \notin g(n_m, y)$. (Such sequences can be constructed using (a) and (b).) Then for $m = 1, 2, \ldots$, $p \in F(x_{i_m}, i_m, n_m, m)$. Now let us show that for some m in N, $F(x_{i_m}, i_m, n_m, m) \subseteq U$. Suppose not. Then there is a sequence $\langle y_m\rangle$ such that $y_m \in F(x_{i_m}, i_m, n_m, m)$ and $y_m \notin U$ for $m = 1, 2, \ldots$ Now $\{p, y_m\} \subseteq F(x_{i_m}, i_m, n_m, m)$ implies that $x_{i_m} \in g(m, p) \cap g(m, y_m)$, and since $F(x_{i_m}, i_m, n_m, m) \subseteq H(x_{i_m}, i_m, n_m) \subseteq g(i_m, x_{i_m}) \subseteq g(m, x_{i_m})$ it follows that $\{p, y_m\} \subseteq g(m, x_{i_m})$. Hence by (E), $y_m \to p$ and this gives a contradiction.

3. An application. In this section we answer affirmatively the following question raised by Reed in [9]: *Is a regular $w\delta$-space with a G_δ^*-diagonal a σ-space?* We begin by introducing a new class of topological spaces.

DEFINITION. A topological space $(X, \mathfrak{I})$ is a *MN-space* if there is a function g from $N \times X$ into $\mathfrak{I}$ with $x \in \bigcap_{n=1}^{\infty} g(n, x)$ for all x in X satisfying this condition: if $\{p, x_n\} \subseteq g(n, y_n)$ and $g(n, p) \cap g(n, x_n) \neq \emptyset$ for $n = 1, 2, \ldots$ then the sequence $\langle x_n\rangle$ converges to p.

It is easy to see that MN-spaces satisfy (7) in § 2 and so every MN-space is a σ-space. Moreover it follows from Reed's Theorem 2.6 that every $w\delta$-space with a G_δ^*-diagonal is a MN-space. Thus Reed's question is answered affirmatively.

Remark. Every Moore space and every Nagata space is a MN-space. (See [2] and [3].) On the other hand, MN-spaces are semi-metric spaces [2]. These implications can be summarized in a diagram as follows.

Moore space ⇓ Nagata space ⇓

MN-space

⇓ semi-metric space ⇓ σ-space

4. Stratifiable spaces are σ-spaces. In this section we use one of our characterizations to give a short proof of Heath's result [4] that every stratifiable space is a σ-space. The following Lemma is due to Heath [4].

LEMMA (Heath). *Let $(X, \mathfrak{I})$ be a stratifiable space. Then there is a function $g\colon N\times X\to\mathfrak{I}$ satisfying these conditions.*

(i) *$x\in\bigcap_{n=1}^{\infty} g(n,x)$ for all x in X;*

(ii) *$g(n+1,x)\subseteq g(n,x)$ for all n in N and x in X;*

(iii) *if $p\in g(n,x_n)$ for $n=1,2,\ldots$ then the sequence $\langle x_n\rangle$ converges to p;*

(iv) *if H is a closed subset of X and $p\notin H$ then there is a n in N such that $p\notin\overline{\bigcup\{g(n,x)\colon x\in H\}}$.*

THEOREM (Heath). *Every stratifiable space is a σ-space.*

Proof. Let X be a stratifiable space and let g be a function satisfying conditions (i)–(iv) in the above Lemma. To show that X is a σ-space it suffices to show that g satisfies condition (C) in § 2. Thus, let $p\in g(n,y_n)$ and $y_n\in g(n,x_n)$ for $n=1,2,\ldots$, and let us show that $x_n\to p$. Let W be an open nghd. of p. Then $p\notin(X-W)$ so by (iv) there is a positive integer n_0 and an open nghd. V of p such that $V\cap(\bigcup\{g(n_0,x)\colon x\in X-W\})=\emptyset$. Now $p\in g(n,y_n)$ for $n=1,2,\ldots$ so by (iii) $y_n\to p$. Hence there is a positive integer $n_1\geqslant n_0$ such that if $n\geqslant n_1$ then $y_n\in V$. It is now easy to check that if $n\geqslant n_1$ then $x_n\in W$.

Remark. As previously stated in § 2, the construction used in proving (7) ⇒ (1) is the same as that used by Heath in proving that every stratifiable space is a σ-space. Thus it is not surprising that Heath's result is easy to recover from one of our characterizations.

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