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Reçu par la Rédaction le 4. 10. 1971

Ultrafilters over measurable cardinals

by

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0. Definitions. The notation and terminology in this paper is that of the most recent set-theoretic literature. For less well-known items we urge the reader to consult A. Mathias (1969). We shall now define our fundamental notions. Unless otherwise mentioned, all the ultrafilters discussed are assumed to be nonprincipal, $\varkappa$-complete over a fixed measurable cardinal $\varkappa$.

0.1. Definition. Given two ultrafilters D, U, we say $D \leqslant U$ if there is a function $f\colon \varkappa \to \varkappa$ so that

$$x \in D \longleftrightarrow f^{-1}(x) \in U .$$

In this case we also denote

$$D = f^*(U) .$$

If $D \leqslant U$ and $U \leqslant D$ we say: D *is isomorphic to* U: In symbols, $D \cong U$.

For more on this order, see K. Kunen [2] and J. Ketonen [1]. The above definition is due to H. J. Keisler.

0.2. Definition. Given an ultrafilter D, functions $f, g\colon \varkappa \to \varkappa$ we say: f, g *are isomorphic* (mod D), in symbols $f \sim g$, if there is a one-to-one function φ so that

$$f = \varphi \circ g \,(\mathrm{mod}\, D) .$$

In this case $f_*(D) \cong g_*(D)$. Another way of describing the above situation is to describe f and g in terms of the partitions $\{f^{-1}(\{a\})|\ a \in \varkappa\}$, $\{g^{-1}(\{a\})|\ a \in \varkappa\}$ they induce. Then $f \sim g$ if and only if there is a set $X \in D$ and a permutation of the labels of the g-partitioning so that the ath part of the f-partitioning intersected with $X =$ ath part of the permuted g-partitioning intersected with X for every $a < \varkappa$.

The following notions are extensions of the concepts of W. Rudin [1956]:

0.3. Definition. If D an ultrafilter. $f\colon \varkappa \to \varkappa$, then D is an f-*P-point* if for every partitioning $\{X_\alpha|\ \alpha<\varkappa\}$ of $\varkappa$ into pieces $\notin D$ there is a set $X\in D$ so that for every $\alpha<\varkappa$

$$|f''(X\cap X_\alpha)|<\varkappa .$$

D is a *P-point* iff D is an id-P-point where id denotes the identity function on $\varkappa$.

0.4. Definition. An ultrafilter D is a *Q-point* if for every partitioning $\{X_\alpha|\ \alpha<\varkappa\}$ of $\varkappa$ into pieces of cardinality $<\varkappa$ there is a $X\in D$ so that

$$|X\cap X_\alpha|\leqslant 1$$

for every $\alpha<\varkappa$.

Let us say that a function $f\colon \varkappa\to\varkappa$ is *almost one-to-one* if the inverse image of each point has cardinality $<\varkappa$. Thus, D is a *Q-point* if and only if every almost one-to-one function is one-to-one on a set $\in D$.

0.5. Definition. An ultrafilter D is *selective* if every non-constant function $(\mathrm{mod}\,D)$ is one-to-one on a set $\in D$.

Thus, D is *selective* iff D is isomorphic to a normal ultrafilter.

0.6. Proposition. (1) *D is selective iff D is $\leqslant$-minimal.*

(2) *If D is a Q-point and a P-point, D is selective.*

(3) *If f is a function $\varkappa\to\varkappa$ not constant* $(\mathrm{mod}\,D)$, *the ultrafilter $U=f_*(D)$ is a P-point if D is a f-P-point.*

Proof of (3). Let $\{X_\alpha|\ \alpha<\varkappa\}$ be a partitioning of $\varkappa$ into pieces not $\in U$ then, for every $\alpha<\varkappa$,

$$Y_\alpha=f^{-1}(X_\alpha)\notin D .$$

Hence, there is a $Z\in D$ so that

$$|f''(Z\cap Y_\alpha)|<\varkappa$$

for every $\alpha<\varkappa$. Thus, for every $\alpha<\varkappa$

$$|f''(Z)\cap X_\alpha|<\varkappa .$$

1. Q-points. Given an ultrafilter D, let i_D denote the elementary embedding of V into the transitive submodel isomorphic to the ultrapower V^D.

1.1. Proposition. *If D,U ultrafilters, $D\leqslant U$, then*

$$i_D(\varkappa)\leqslant i_U(\varkappa)<(2^\varkappa)^+ .$$

Proof. See J. Ketonen [1]: If $\psi\colon \varkappa\to\varkappa$ so that $D=\psi_*(U)$, then the map $f\to f\circ\psi$ yields an elementary embedding $V^D\to V^U$.

1.2. Proposition. *If D,U ultrafilters, $D\leqslant U$ and $i_D(\varkappa)=i_U(\varkappa)$, then for any projection ψ from U to D (i.e. any function ψ s.t. $D=\psi_*(U)$) there is a $Z\in U$ so that for every $\alpha<\varkappa$:*

$$|Z\cap\psi^{-1}(\{\alpha\})|<\varkappa .$$

Proof. Given any such ψ, the map $f\to f\circ\psi$ yields an order-preserving map $i_D(\varkappa)\to i_U(\varkappa)=i_D(x)$. Thus, there is a $f\colon \varkappa\to\varkappa$ s.t.

$$[f\circ\psi]_U\geqslant[\mathrm{id}]_U .$$

As a corollary, we obtain:

1.3. Proposition. *If $D<U$ and U is a Q-point, $i_D(\varkappa)<i_U(\varkappa)$. The Keisler order on Q-points is well-founded.*

1.4. Theorem. *An ultrafilter D is a Q-point if and only if there is a $E\simeq D$ s.t. E extends the closed unbounded filter on $\varkappa$.*

Proof. Assume that D is a Q-point. Let φ be the D-least one-to-one function $\varkappa\to\varkappa$. Define E by:

$$X\in E\leftrightarrow\varphi^{-1}(X)\in D .$$

Then, if C is closed unbounded, $C\in E$: For otherwise we would have the function

$$f(\alpha)=\sup(C\cap\alpha) ,$$

defined on $X=-C$ being 1-1 and $<$id on a set $Y\in E$, violating the minimality of φ. Hence, E extends the closed unbounded filter.

Conversely, assume that D is an extension of the closed unbounded filter. Let $f\colon \varkappa\to\varkappa$ so that $f<\mathrm{id}$ on a set $X\in D$. Then X is Mahlo. Hence, there is a set $Y\subseteq X$ of cardinality $\varkappa$ so that f is constant on Y. Thus, f cannot be almost 1-1. Now, assume that $f>\mathrm{id}$. Let

$$C=\{\delta|\ \gamma<\delta\to f(\gamma)\leqslant\delta\} .$$

Then C is closed unbounded on $\varkappa$, hence $C\in D$. The function f is obviously 1-1 on C.

Let $S(\varkappa)$ denote the following statement: Every $\varkappa$-complete filter over $\varkappa$ can be extended to a $\varkappa$-complete ultrafilter.

1.5. Proposition. *If $S(\varkappa)$, there are $2^{2^\varkappa}$ Q-points. For every $D\varkappa$-complete over $\varkappa$, there is a Q-point $U\geqslant D$.*

Proof. Let $\{X_\alpha|\ \alpha<\varkappa\}$ be a partitioning of $\varkappa$ into Mahlo sets. Given an ultrafilter D over $\varkappa$, let

$$F=C\cup\{\bigcup_{\alpha\in A}X_\alpha|\ A\in D\} ,$$

where ζ is the closed unbounded filter on $\varkappa$. If U is an extension of F, U is a Q-point $\geqslant D$.

The rest follows from K. Kunen (1970): Every ultrafilter over $\varkappa$ has at most $2^\varkappa$ ultrafilters below it in the Keisler order.

1.6. Definition. If R, M are ultrafilters over $\varkappa$, we define their *product* $R\times M$ to be the set of all subsets X of $\varkappa\times\varkappa$ s.t.

$$\{\alpha|\ X|\alpha\in M\}\in R ,\quad \text{where}\quad X|\alpha=\{\beta|\ (\alpha,\beta)\in X\} .$$

Trivially, $R \times M$ is a $\varkappa$-complete ultrafilter over $\varkappa \times \varkappa$. For more on products, see K. Kunen [2], [3] and J. Ketonen [1].

1.7. Proposition. *If D is a product, D is neither a P-point nor a Q-point.*

Proof. For the projection π_1 to the first coordinate on $\varkappa \times \varkappa$ would yield a function which cannot be refined to an almost 1-1 function, and the projection π_2 to the 2nd coordinate yields a function which is almost one-to-one but not one-to-one. This follows from the fact that

$$\{(\alpha, \beta) \mid \alpha < \beta < k\} \in R \times M .$$

Thus, in particular, as Kenneth Kunen independently noted, Kunen's classification of ultrafilters in universes constructed from one normal ultrafilter (see Kunen [2]) combined with 1.4 and 1.7 yield Jech's result on extensions of the closed unbounded filter: In this universe there is only one extension of the closed unbounded filter.

Given a non-Q-point D, there is a canonical Q-point below it:

1.8. Proposition. *If D an ultrafilter, and f is the least almost 1-1 function* (mod D), *$f_*(D)$ is an extension of the closed unbounded filter and hence is a Q-point.*

Proof. For if $f_*(D)$ does not extend the closed unbounded filter, there is an almost 1-1 function $\psi < \mathrm{id}\big(\mathrm{mod} f_*(D)\big)$. Thus, $\psi \circ f < f$ and $\psi \circ f$ is almost 1-1; violating the minimality of f.

This Q-point is in fact the only one given by an almost 1-1 function (mod D):

1.9. Proposition. *Let D be an ultrafilter, g an almost 1-1 function $\varkappa \to \varkappa$ s.t. $g_*(D)$ extends the closed unbounded filter. Then, if $f < g$* (mod D), *f is not almost 1-1.*

Proof. Suppose that for every $\alpha < \varkappa$, $f(\alpha) < g(\alpha)$ and f defined a.e. For $\alpha < \varkappa$, let

$$S_\alpha = \{\gamma \mid f^{-1}(\{\gamma\}) \cap g^{-1}(\{\alpha\}) \neq 0\} .$$

Hence, $S_\alpha \subseteq \alpha$ and

$$f^{-1}(\varkappa) \cap g^{-1}(\{\alpha\}) \subseteq f^{-1}(S\alpha) .$$

Then:

$$T = \{\alpha \mid S_\alpha \neq 0\} \in g_*(D) ,$$

hence T is Mahlo. Define a pressing down function p on T by:

$$p(\alpha) = \mu\beta[\beta \in S_\alpha] .$$

Then there is a $T' \subseteq T$ of cardinality $\varkappa$ s.t. $p =$ constant γ on T'. Hence, for $\alpha \in T$:

$$f^{-1}(\{\gamma\}) \cap g^{-1}(\{\alpha\}) \neq 0 ;$$

i.e.

$$|f^{-1}(\{\gamma\})| = \varkappa .$$

1.10. Proposition. *If D an ultrafilter, g its least almost 1-1 function $\varkappa \to \varkappa$, then for any other function h: $\varkappa \to \varkappa$: h is almost 1-1 iff $h \geqslant g$ iff there is a function φ, φ almost 1-1, so that*

$$\varphi \circ h = g (\mathrm{mod} D) .$$

Proof. Firstly, if $h \geqslant g$, h is almost 1-1: For every $\alpha < \varkappa$:

$$h^{-1}(\{\alpha\}) \subseteq \bigcup \{g^{-1}(\{\gamma\}) \mid \gamma \leqslant \alpha\} .$$

Assume that h is almost 1-1. Let φ be the least almost 1-1 function of $h_*(D)$. Then $(\varphi \circ h)_*(D)$ extends the closed unbounded filter. Hence, by 1.9 $\varphi \circ h = g (\mathrm{mod} D)$.

1.11. Proposition. *No two distinct $\varkappa$-complete extensions of the closed unbounded filter $\varkappa$ ultrafilters can be isomorphic.*

2. Products of ultrafilters. Given two ultrafilters U, V over $\varkappa$ we defined their product $U \times V$ to be the ultrafilter

$$\{X \subseteq \varkappa \times \varkappa \mid \{\alpha \mid X|\alpha \in V\} \in U\} .$$

In the proof of Proposition 1.7 we found that if $\pi_i =$ projection onto the ith coordinate ($i = 1$ or 2), $\pi_{1*}(U \times V) = U$, and $\pi_{2*}(U \times V) = V$ and there is a $Z \in U \times V$ so that for every $\alpha < \varkappa$:

$$|\pi_2^{-1}(\{\alpha\}) \cap Z| \leqslant |\alpha|$$

and therefore $\pi_1 < \pi_2 (\mathrm{mod}\, U \times V)$.

We shall in the following establish some converses to the above facts.

2.1. Proposition. *Let D be an ultrafilter f, g: $\varkappa \to \varkappa$ and h: $\varkappa \to \varkappa \times \varkappa$ be defined by $h(\alpha) = \big(f(\alpha), g(\alpha)\big)$. Then, if for every $Z \in D$*

$$\big\{\alpha \mid \{\beta \mid Z \cap g^{-1}(\{\beta\}) \cap f^{-1}(\{\alpha\}) \neq 0\} \in U\big\} \in V ,$$

U, V being ultrafilters over $\varkappa$, then $h_*(D) = U \times V$.

Proof. If $X \in V \times U$,

$$\{\alpha \mid X|\alpha = \{\beta \mid (\alpha, \beta) \in X\} \in U\} \in V .$$

Thus, if

$$Z = \bigcup_{\alpha < \varkappa} f^{-1}(\{\alpha\}) \cap \Big(\bigcup_{\beta \in X|\alpha} g^{-1}(\{\beta\})\Big) ,$$

then $Z \in D$ since for every $Y \in D$, $Y \cap Z \neq 0$. But

$$Z = \{\gamma \mid \big(f(\gamma), g(\gamma)\big) \in X\} .$$

2.2. Proposition. *Suppose D an ultrafilter, f, g: $\varkappa \to \varkappa$ so that both f, g non-constant, $f < g$ $g_*(D)$ is normal and g is not 1-1 on a set $\in D$. Then*

$$D \geqslant f_*(D) \times g_*(D) = h_*(D)$$

where $h(\alpha) = (f(\alpha), g(\alpha))$ *for every* $\alpha < \varkappa$.

Proof. Let $U = g_*(D)$. If the conditions of Proposition 2.1 were not satisfied, we can find a $Z \in D$ so that for no $\alpha < \varkappa$

$$X_\alpha = \{\beta |\ Z \cap g^{-1}(\{\beta\}) \cap f^{-1}(\{\alpha\}) \neq 0\} \in U$$

we know that $X_\alpha \subseteq (\alpha, k)$, since

$$g^{-1}(\{\alpha\}) \subseteq \bigcup \{f^{-1}(\{\gamma\})|\ \gamma < \alpha\} .$$

Thus, there exists a set $X \in U$ so that the sets $X_\alpha \cap X$ arc disjoint singletons. We can without aloss of generality assume that there is a 1-1 function t so that $X_\alpha \subseteq \{t(\alpha)\}$; i.e.

$$t \circ f = g (\mathrm{mod}\, D) .$$

Hence, there is a 1-1 function I so that $f = I \circ g$. But $f < g$; hence I must be $< \mathrm{id} (\mathrm{mod}\, U)$. But this contradicts the normality of U.

As a corollary, we obtain

2.3. Proposition. *Suppose D is an ultrafilter, $g\colon \varkappa \to \varkappa$ almost 1-1 so that $g_*(D)$ is normal. Then: Either D is a P-point or there is a $U \leqslant D$ so that*

$$U \times g_*(D) \leqslant D .$$

2.4. Proposition. *Suppose U, D ultrafilters, $f\colon \varkappa \to \varkappa$ so that $U = f_*(D)$ is normal $< D$. Then there is a $V \leqslant D$ so that $D \cong V \times U$ if and only if $\exists X \in D$ so that for $\alpha < \varkappa$*

$$|f^{-1}(\{\alpha\}) \cap X| \leqslant |\alpha| .$$

Proof. For assume that for every $\alpha < \varkappa$

$$|f^{-1}(\{\alpha\})| \leqslant |\alpha| .$$

Then there is a partitioning $\{X_\alpha|\ \alpha < \varkappa\}$ of $\varkappa$ into pieces $\notin D$ so that

(1) For every $\alpha < \varkappa$ $f|X_\alpha$ is 1-1

(2) For every $\alpha < \varkappa$,

$$f^{-1}(\{\alpha\}) \subseteq \bigcup \{X_\gamma|\ \gamma < \alpha\} .$$

Thus, if we define $g = \gamma$ on X_γ, the function $t(\alpha) = (h(\alpha), f(\alpha))$ is 1-1. It remains to show that $g < f (\mathrm{mod}\, D)$. This follows from the fact that g is not almost 1-1: If $Y \in D$ define

$$h(\alpha) = \mu\gamma [f^{-1}(\{\alpha\}) \cap X_\gamma \cap Y \neq 0] .$$

Then h is a pressing down function, hence a constant $\gamma (\mathrm{mod}\, U)$. This implies that

$$|g^{-1}(\{\gamma\}) \cap Y| = \varkappa .$$

We shall in the following briefly sketch the theory of the Keisler-order on products.

2.5. Proposition. *If D, E, U ultrafilters, then $U \leqslant D \times E$ iff either $U \leqslant D$ or there are ultrafilters $E_\alpha \leqslant E$ s.t.*

$$X \in U \longleftrightarrow \{\beta |\ X \in E_\beta\} \in D .$$

Proof. Let $\{X_\alpha|\ \alpha < \varkappa\}$ partition $\varkappa \times \varkappa$ so that

$$S \in U \longleftrightarrow \bigcup_{\alpha \in S} X_\alpha \in D \times E .$$

Let $Y^\beta_\alpha = X_\alpha | \beta$.

Case 1. For a.e. $\alpha (\mathrm{mod}\, D)$ there is one, and hence exactly one $f(\alpha)$ so that $Y^{f(\alpha)}_\alpha \in E$. Obviously, in this case $U = f_*(D)$.

Case 2. Not Case 1. Without a loss of generality assume that for every α, β: $Y^\beta_\alpha \notin E$. Define E_β by:

$$X \in E_\beta \longleftrightarrow \bigcup_{\alpha \in X} Y^\beta_\alpha \in E .$$

Obviously, our claim is then satisfied.

Using the reasoning of Proposition 2.5 it is then easy to prove:

2.6. Proposition. *If both U, V are normal, then $U \times V$ has exactly 3 isomorphism classes of non-constant functions: Those isomorphic to π_1, those isomorphic to π_2 and the 1-1 functions. More generally, if $U_1, \ldots, U_n$ $(1 \leqslant n < \omega)$ normal, then the only normal ultrafilters $\leqslant U_1 \times \ldots \times U_n$ are $U_1, \ldots, U_n$ and $U_1 \times \ldots \times U_n$ has exactly $2^n - 1$ isomorphism classes of functions.*

2.7. Proposition. *Let U, E_α $(\alpha < \varkappa)$, D be ultrafilters, U normal s.t.*

$$X \in U \longleftrightarrow \{\beta |\ X \in E_\beta\} \in D .$$

Then $D \times U$ is the D-sum of E_α's: i.e.

$$D \times U = \sum_D E_\alpha = \{X \subseteq \varkappa \times \varkappa |\ \{\alpha |\ X|\alpha \in E_\alpha\} \in D\} .$$

Proof. Let

$$T_\gamma = \{(\gamma, \beta)|\ \gamma < \beta\} \quad \text{and} \quad X_\gamma = \{(\beta, \gamma)|\ \beta < \gamma\} .$$

Then

$$X \in U \longleftrightarrow \bigcup_{\gamma \in X} X_\gamma \in \sum_D E_\alpha \quad \text{and} \quad X \in D \longleftrightarrow \bigcup_{\gamma \in X} T_\gamma \in \sum_D E_\alpha .$$

Now use the reasoning of 2.3.

As a corollary, we obtain:

2.8. PROPOSITION. *If D, E ultrafilters, $U \leqslant D \times E$ is normal, then either $U \leqslant D$ or there are $E_a \leqslant E$ so that*

$$D \times U = \sum_D E_a .$$

In particular, if E is normal, $U \leqslant D$ or $U = E$.

3. P-points. Let D be an ultrafilter. Using Proposition 1.9 we obtain:

3.1. THEOREM. *Let f be the least non-constant function* $(\mathrm{mod} D)$ $\varkappa \to \varkappa$. *Then is a P-point if and only if f is almost 1-1 $(\mathrm{mod} D)$ if and only if for every non-constant function h there is a almost 1-1 φ so that $f = \varphi \circ h (\mathrm{mod} D)$ i.e. there is a set $X \in D$ so that for $\alpha, \beta \in X$:*

$$f(\alpha) \neq f(\beta) \to h(\alpha) \neq h(\beta) .$$

By Proposition 0.6 we also have the following observation:

3.2. PROPOSITION. *If D is a P-point, $U \leqslant D$, U is a P-point. There is at most one normal ultrafilter $\leqslant D$.*

3.3. PROPOSITION. *Let D be a P-point. Then there is a P-point $\bar{D} \leqslant D$ so that if φ is the least non-constant function $\varkappa \to \varkappa$* $(\mathrm{mod} \bar{D})$,

$$g < \mathrm{id} \to \exists f\colon f \circ \varphi = g$$

and there is a disjointed sequence $\{I_a\}_{a<\varkappa}$ of intervalls $\subseteq \varkappa$ whose end points form a strictly increasing sequence so that $\varphi = a$ on $\bar{I}_a$ for $a < \varkappa$.

Proof. Let φ be the D-least non-constant function not expressible in the form $f \circ \bar{\varphi}$, where $\bar{\varphi}$ is the least non-constant function of D. Let $\bar{D} = \varphi_*(D)$. Then the first condition is automatically satisfied. Let $U = \varphi_*(\bar{D})$. Then U is normal. Let $S_a = \varphi^{-1}(\{a\}) \subseteq (a, k)$. Define a map $F\colon [\varkappa]^2 \to \{0, 1\}$ by

$$F(\{\alpha, \beta\}) = \begin{cases} 1 & \text{if } \sup S_a \geqslant \min S_\beta , \\ 0 & \text{otherwise } (\alpha < \beta) . \end{cases}$$

Using the fact that the S_a's have cardinality $< \varkappa$ we find a $X \in U$ so that $F''[X]^2 = \{0\}$. The claim follows.

3.4. DEFINITION. An ultrafilter D is *atomic* if D has exactly two isomorphism classes of non-constant functions; those which are 1-1 and those which are isomorphic to the least non-constant function $\varkappa \to \varkappa$.

3.5. PROPOSITION. *If D is a non-normal P-point, either there is a $U \leqslant D$ so that U is atomic or the Keisler-order on ultrafilters $\leqslant D$ is not well-founded.*

Proof. For assume that there are no atomic P-points below D. Let φ be the least non-1-1 function s.t. φ is not isomorphic to the least function of D. Then, if $D_1 = \varphi_*(D)$, $V < D_1 < D$, and we can apply the same process to D_1 etc.

In the following, we shall construct an atomic P-point. Note that any atomic ultrafilter is either a P-point or a Q-point, dependending on whether the least function is almost 1-1 or not. In the next chapter, we shall construct an atomic Q-point using the same method.

3.6. THEOREM. *If there is a normal ultrafilter containing the set*

$$M = \{\alpha|\ \alpha < \varkappa \ \&\ \alpha \text{ is measurable}\}$$

then there is an atomic, non-normal P-point.

Note that this happens in particular, if $\varkappa$ is supercompact or if $S_\varkappa(2^\varkappa)$ carries a k-complete normal ultrafilter.

Proof of Theorem 3.6. Let $U \ni M$ be normal. For $\alpha \in M$, let $\alpha^* =$ the next measurable cardinal and let D_a be normal over α^*. Define D by:

$$X \in D \longleftrightarrow \{\alpha|\ X \cap \alpha^* \in D_a\} \in U .$$

Firstly,

$$\bigcup_{\alpha \in M} (\alpha, \alpha^*) \in D .$$

This is a non-Mahlo set. Hence, D is non-normal. Define $\varphi = \alpha$ on (α, α^*). Then φ is the first function of D: For let $t < \varphi$. Then for one $\alpha (\mathrm{mod} D)$

$$\{\gamma < \alpha^*|\ t(\gamma) < \varphi(\gamma)\} \cap \alpha^* \in D_a .$$

Since D_a is α^*-complete we find $p(\alpha) < \alpha$ so that $t(\gamma) = p(\alpha)$ for a.e. $\gamma (\mathrm{mod} D_a)$. But p is constant $(\mathrm{mod}\, U)$.

Thus, the first function of D is almost 1-1 and therefore D is a P-point. Now, given any non-constant non 1-1 $(\mathrm{mod} D)$ function $f\colon \varkappa \to \varkappa$, by Proposition 3.1 we can without aloss of generality assume that

$$\alpha < \beta \to f''(\varphi^{-1}(\{\alpha\})) \cap f''(\varphi^{-1}(\{\beta\})) = 0 .$$

Since f is not 1-1,

$$\{\alpha|\ f|(\alpha, \alpha^*) \text{ not 1-1 } (\mathrm{mod} D_a)\} \in U .$$

Thus,

$$\{\alpha|\ f|(\alpha, \alpha^*) \text{ is constant } (\mathrm{mod} D_a)\} \in U$$

by the normality of D_a. Hence, there is a g so that $f = g \circ \varphi$.

Selecting instead of D_a's non-normal ultrafilters we can construct P-points having lots of ultrafilters below them. Note that for the above ultrafilter D:

$$i_D(k) = \text{o.t.}\Big(\prod_U \prod_{D_a} \langle \varkappa, < \rangle\Big)$$
$$= \text{o.t.}\Big(\prod_U \langle \varkappa, < \rangle\Big) = i_U(\varkappa) .$$

4. More on Q-points. In the following, let ζ denote the closed unbounded filter on $\varkappa$.

4.1. Proposition. *Let D be an ultrafilter. Then the set*

$$\Gamma(D) = \{|g|_D|\ g_*(D) \supseteq \zeta\}$$

represents a set of ordinals which is λ-closed for every $\lambda < \varkappa$; i.e. if $\{\alpha_i|\ i < \lambda\}$ is an increasing sequence in $\Gamma(D)$,

$$\sup\{\alpha_i|\ i < \lambda\} \in \Gamma(D)\,.$$

Proof. Let $\{g_i|\ i < \lambda\}$ be a strictly increasing sequence of functions in $\Gamma(D)$, and let $g = \sup(g_i)$. Given any $c \in \zeta$, for every $i < \lambda$ $g_i(\alpha) \in C$ a.e. (mod D). Hence

$$g(\alpha) = \sup_{i<\lambda} g_i(\alpha) \in c \text{ a.e. } (\operatorname{mod} D)\,.$$

Note that if D, U are Q-points and $D = \varphi_*(U)$, then the map $f \to f \circ p$ maps $\Gamma(D)$ into $\Gamma(U)$ and if $D < U$,

$$\text{ordertype}\,(\Gamma(D)) < \text{ordertype}\,(\Gamma(U))\,.$$

As a corollary to Proposition 4.1 we obtain

4.2. Proposition. *If the filter $\zeta \cup \{\alpha|\ cf(\alpha) = \omega\}$ cannot be extended to a $\varkappa$-complete ultrafilter, then every $\varkappa$-complete ultrafilter has at most finitely many normal ultrafilters (and other Q-points) below it in the Keisler-order.*

Proof. For if D is such that $|\Gamma(D)| \geqslant \omega$, we can find an increasing sequence $\{g_i|\ i < \omega\}$ in $\Gamma(D)$. Then $g = \sup g_i \in \Gamma(D)$ and every α: $cf(g(\alpha)) = \omega$; hence

$$\{\alpha|\ cf(\alpha) = \omega\} \in g_*(D)\,.$$

More generally, for any regular cardinal $\lambda < \varkappa$:

4.3. Proposition: *If the filter $\zeta \cup \{\alpha|\ cf(\alpha) = \lambda\}$ cannot be extended to a $\varkappa$-complete ultrafilter, then any $\varkappa$-complete ultrafilter over $\varkappa$ has $<\lambda$ normal ultrafilters below it.*

4.4. Proposition. *If $q \in \Gamma(D)$ and $p < q$, then for any 1-1 function φ: $\varphi \circ p < q(\operatorname{mod} D)$.*

Proof. If not, suppose $\varphi \circ p \geqslant q(\operatorname{mod} D)$. Now,

$$q^{-1}(\{\alpha\}) \subseteq \bigcup\{p^{-1}(\{\gamma\})|\ \gamma < \alpha\}$$

for every α, and each $p^{-1}(\{\gamma\})$ intersects at most $\varphi(\gamma)\, q^{-1}(\{\alpha\})$'s. Let

$$S_\gamma = \{\alpha|\ q^{-1}(\{\alpha\}) \cap p^{-1}(\{\gamma\}) \neq 0\}\,.$$

Thus $|S_\gamma| < \varkappa$ and $S_\gamma \subseteq (\gamma, \varkappa)$. But then

$$X = \bigcup_{\gamma<\varkappa} S_\gamma \in q_*(D)\,.$$

Thus, we can define a function f by: $t(\alpha) = \mu\gamma[\alpha \in S_\gamma]$ a.e. on X. But $t(\alpha) < \alpha$; hence there is a Y of cardinality $\varkappa$ so that t is a constant γ on Y. But then $|S_\gamma| = \varkappa$: a contradiction.

4.5. Proposition. *Let $f < \text{id}$ s.t. there is a almost 1-1 function ψ s.t. $\psi \circ f = p \in \Gamma(D)$ (such a ψ always exists: Let ψ be the least almost 1-1 function of $f_*(D)$). Then there is a 1-1 function φ so that*

$$f < \varphi \circ p\,(\operatorname{mod} D)\,.$$

Proof. Define φ to be an 1-1 function satisfying

$$\varphi(\alpha) \geqslant \sup \psi^{-1}(\{\alpha\}) \qquad (\alpha < k)\,.$$

Then

$$\varphi \circ p = \varphi \circ \psi \circ f \geqslant f\,.$$

These two propositions suggest a method for finding some elements of $\Gamma(D)$:

If $p \in \Gamma(D)$, and p is not almost one-to-one,

$$p^* = \sup_D\{[\varphi \circ p]|\ \varphi \text{1-1}\}$$

is the successor of p in $\Gamma(D)$. Using Proposition 4.1 at limit stages and the above at successorstages, starting from the least function of $\Gamma(D)$ we get the first $\varkappa$ elements of $\Gamma(D)$.

Now, let us assume that $\zeta \cup \{\alpha|\ cf(\alpha) = \lambda\}$ (λ a regular cardinal $<\varkappa$) can be extended to an ultrafilter. By the well-foundedness of the Keisler-order on Q-points, we can pick a $\leqslant$-minimal extension D of $\zeta \cup \cup \{\alpha|\ cf(\alpha) = \lambda\}$.

4.6. Proposition. *The ordertype of $\overline{\Gamma}(D)$ is $\leqslant \lambda$. In particular $|\overline{\Gamma}(D)| \leqslant \lambda$. Here*

$$\overline{\Gamma}(D) = \{g \in \Gamma(D)|\ g < \text{id}\}\,.$$

Proof. For otherwise, there exists an increasing sequence $\{g_i|\ i < \lambda\} \subseteq \overline{\Gamma}(D)$ so that $g = \sup_{i<\lambda} g_i < \text{id}$. But $g_*(D)$ would extend $\zeta \cup \{\alpha|\ cf(\alpha) = \lambda\}$, violating minimality.

4.7. Proposition. *The ordertype of $\Gamma(D)$ is exactly λ; $|\overline{\Gamma}(D)| = \lambda$.*

Proof. If not, pick $\{f_\alpha\}\ \alpha < \lambda$ cofinal in $\text{id}(\operatorname{mod} D)$. Then there is a fixed $p \in \overline{\Gamma}(D)$, $p < \text{id}$, so that for every $\alpha < \lambda$ there is a 1-1 ψ_α so that

$$f_\alpha \leqslant \psi_\alpha \circ p \qquad (\alpha < \lambda)\,.$$

Let $\psi = \sup_{\alpha<\lambda} \psi_\alpha (\operatorname{mod} p_*(D))$. Then $\psi \circ p \geqslant \text{id}(\operatorname{mod} D)$, hence $\psi \circ p$ is 1-1 on a set $\in D$; p is 1-1; a contradiction.

4.8. Corollary. *If $\zeta \cup \{\alpha|\ cf(\alpha) = \lambda\}$ can be extended to a $\varkappa$-complete ultrafilter, then for every $\mu < \lambda$ $\zeta \cup \{\alpha|\ cf(\alpha) = \mu\}$ can be extended to a $\varkappa$-complete ultrafilter.*

4.9. Proposition. *If $U < D_1$ then there are only $<\lambda$ functions $p \in \bar{\Gamma}(D)$ so that if $U = f_*(D)$, there is a φ so that $p = \varphi \circ f (\mathrm{mod} D)$. In particular, $|\bar{\Gamma}(U)| < \lambda$.*

Proof. For if not, there is an increasing sequence $\{p_i|\ i < \lambda\}$, $X \in D$ so that for $i < \lambda$, $\alpha, \beta \in X$

$$p_i(\alpha) \neq p_i(\beta) \rightarrow f(\alpha) \neq f(\beta) .$$

We can without a loss of generality assume that

$$\sup_{i<\lambda} p_i(\alpha) = \alpha \qquad (\alpha \in X) ,$$

and

$$p_1(\alpha) < p_2(\alpha) < \ldots \qquad (\alpha \in X) .$$

Thus:

$$\alpha \neq \beta \rightarrow \exists i_0\colon\ i \geqslant i_0 \rightarrow p_i(\alpha) \neq p_i(\beta) \rightarrow f(\alpha) \neq f(\beta) ,$$

i.e. f is 1-1.

The above observations also yield:

4.10. Proposition. *D is a $\leqslant$-minimal extension of $\zeta \cup \{\alpha|\ cf(\alpha) = \lambda\}$ iff $\Gamma(D)$ has ordertype λ.*

Let $\gamma(D) =$ ordertype of $\Gamma(D) =$ ordertype of $\Gamma(D)$. Can $\gamma(D)$ of a Q-point be a successor? The answer is yes, under certain conditions:

4.11. Proposition. *Suppose that there are $\varkappa$ distinct normal ultrafilters over $\varkappa$. Then, for any Q-point D, there is a Q-point $D^+ > D$ so that $\gamma(D)+1$, the greatest function of $\Gamma(D^+)$ yielding D.*

Proof. Let $\{D_\alpha|\ \alpha < \varkappa\}$ be a sequence of distinct normal ultrafilters $\{A_\alpha|\ \alpha < \varkappa\}$ a partitioning of $\varkappa$ into Mahlo sets A_α so that

$$A_\alpha \in D_\alpha \qquad (\alpha < \varkappa) .$$

Define D^+ by:

$$X \in D^+ \longleftrightarrow \{\alpha|\ X \cap A_\alpha \in D_\alpha\} \in D .$$

If we define $f = \alpha$ on A_α, we find that $f_*(D^+) = \{f\} \cup \{\varphi \circ f|\ \varphi \in \Gamma(D)\}$.

Actually, the map $\varphi \rightarrow \varphi \circ f$ yields an onto map from $\{[\varphi]_D | \varphi\colon \varkappa \rightarrow \varkappa\}$ to $\{[\psi]_D | \psi\colon \varkappa \rightarrow \varkappa$ and $\psi < \mathrm{id}\}$.

In particular, if D is normal, $U = D^+$ is an atomic Q-point, so that

$$\prod_{U\alpha<k} (\alpha, <) \cong \prod_{D} (\varkappa, <) .$$

Note that D^+ is a D-sum of D_α's. This situation should be compared with 2.7 and 1.7.

4.12. Definition. D is a *hereditary Q-point* if for each function $f < \mathrm{id}$, $f_*(D)$ is a Q-point.

Thus, if D is a hereditary Q-point, for every $f < \mathrm{id}$ there is a $p \in \Gamma(D)$ so that $p \sim f$. In particular, D has exactly $|\Gamma(D)|$ isomorphism classes of functions.

If U is normal, then U, U^+, U^{++}, ... are hereditary Q-points, having exactly $1, 2, \ldots$ isomorphism classes of functions.

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Reçu par la Rédaction le 18. 10. 1971