A fixed-point theorem for homeomorphisms of R^{2n}

by

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§ 1. Introduction. In this note we give a proof for the following:

THEOREM. *Let $h\colon R^{2n}\to R^{2n}$ be a stable homeomorphism which is an involution on some non-empty, invariant $(2n-3)$-connected subset $X\subset R^{2n}$. Then $hx=x$ for some $x\in R^{2n}$.*

COROLLARY. *Let $h\colon R^{2n}\to R^{2n}$ be an orientation-preserving diffeomorphism (or, if $2n\neq 6$, a stable homeomorphism) which takes some differentiable $(2n-2)$-sphere $\Sigma\subset R^{2n}$ into itself. Then $hx=x$ for some $x\in R^{2n}$.*

Remark. Let h and X or Σ be as above, and let $Y\subset R^{2n}$, with $\pi_{2n-2}(Y)=0$, be a set containing X or Σ. The conclusions of the theorem and corollary may be strengthened to read: any cell containing $Y\cup hY$ contains a fixed point $x=hx$ of h.

The results of Kirby show that, in dimensions other than four, any orientation preserving homeomorphism is stable [3]. In the case that the set X of the theorem is acyclic mod 2, Smith has shown that $hx=x$ for some $x\in X$ [4].

§ 2. Definitions. We shall use the term *map* for continuous functions. A map $f\colon A\to B$ is an *involution* on a subset $X\subset A$ if $f^2|_X=\mathrm{id}_X$, and X is *invariant* (under f) if $fX\subset X$. For a given space A, let $\mathcal{H}(A)$ be the space of homeomorphisms $h\colon A\to A$ with the compact-open topology; a homeomorphism $h\colon A\to A$ is *stable* if it lies in the connected component $S\mathcal{H}(A)\subset\mathcal{H}(A)$ of the identity $\mathrm{id}_A\colon A\to A$ (cf. Theorem 2, [3]). A space X is *k-connected* if it is path-connected and $\pi_i(X)=0$, $i=1,\ldots,k$.

We write R for the real line, R^{2n} for $2n$-dimensional space, and make the identification

$$R^k=\{(t_1,\ldots,t_k,0,\ldots,0)\colon t_i\in R\}\subset R^{2n}\quad\text{for}\quad k\leqslant 2n\,.$$

For $k<2n$ we let

$$S^k=\{x\in R^{k+1}\colon \|x\|=1\}\,,\qquad B^{k+1}=\{x\in R^{k+1}\colon \|x\|\leqslant 1\}\,,$$
$$D^k_N=\{(t_1,\ldots,t_{k+1},0,\ldots,0)\in S^k\colon t_{k+1}\geqslant 0\}\,,$$
$$D^k_S=\{(t_1,\ldots,t_{k+1},0,\ldots,0)\in S^k\colon t_{k+1}\leqslant 0\}\,.$$

The degree of a map $f\colon S^k \to S^k$ is written df. Of particular importance are the antipodal map $a\colon S^k \to S^k$ and the reflection $r\colon S^k \to S^k$, defined respectively by $x \to -x$ and

$$(t_1, \ldots, t_k, t_{k+1}, 0, \ldots, 0) \to (t_1, \ldots, t_k, -t_{k+1}, 0, \ldots, 0).$$

§ 3. Proof of Theorem. We assume in this section that h and X satisfy the hypotheses of the theorem.

LEMMA 1. *There is a map $f\colon S^{2n-2} \to X$ such that $fa = hf$.*

Proof. Suppose that, for some $k \leqslant 2n-2$, we have a map $f_{k-1}\colon S^{k-1} \to X$ such that $f_{k-1}a = hf_{k-1}$. (This is true vacuously for $k = 0$.) Since X is $(k-1)$-connected we can extend f_{k-1} to a map $F_k\colon D^k_N \to X$. Define $f_k\colon S^k \to X$ by $f_k|_{D^k_N} = F_k$, $f_k|_{D^k_S} = hF_k a$. The restriction of f_k to D^k_N or D^k_S is a well-defined map, and on $D^k_N \cap D^k_S = S^{k-1}$ we have $hF_k a = F_k aa = F_k$, so that f_k is a well-defined map. Moreover,

$$f_k a|_{D^k_N} = (f_k|_{D^k_S})(a|_{D^k_N}) = hF_k aa|_{D^k_N} = hf_k|_{D^k_N}$$

and

$$f_k a|_{D^k_S} = (f_k|_{D^k_N})(a|_{D^k_S}) = F_k a|_{D^k_S} = hf_k|_{D^k_S},$$

so that $f_k a = hf_k$.

The construction used above is basic to the proof of our theorem. For any map $T\colon D^{2n-1}_N \to R^{2n}$ such that

$$hT = Ta \quad \text{on } S^{2n-2}, \tag{1}$$

an extension $T_*\colon S^{2n-1} \to R^{2n}$ of T may be defined by setting $T_* = hTa$ on D^{2n-1}_S. If $T_*x = T_*ax$ for some $x \in D^{2n-1}_N$ then

$$Tx = T_*x = T_*ax = (hTa)ax = hTx,$$

so that Tx is a fixed point of h. The map $T^*\colon S^{2n-1} \to S^{2n-1}$ given by $T^*x = (T_*ax - T_*x)/\|T_*ax - T_*x\|$ is thus well-defined if h leaves no point of $TD^{2n-1}_N \subset R^{2n}$ fixed.

Suppose now that $hx \neq x$ for each $x \in R^{2n}$, so that we can construct T_* and T^* as above for any T satisfying (1).

We can extend the map f of Lemma 1 to a map $F\colon D^{2n-1}_N \to R^{2n}$. Both $F\colon D^{2n-1}_N \to R^{2n}$ and $hF\colon D^{2n-1}_N \to R^{2n}$ then satisfy (1).

LEMMA 2. $d(hF)^* = dF^*$.

Proof. For any $g \in \mathcal{H}(R^{2n})$ we can define a map $\lambda_g\colon S^{2n-1} \to S^{2n-1}$ by

$$\lambda_g x = (gF_* ax - gF_* x)/\|gF_* ax - gF_* x\|.$$

The map λ_g depends continuously on g, so that $d\lambda_g = d\lambda_{\mathrm{id}} = dF^*$ for all $g \in S\mathcal{H}(R^{2n})$. But $\lambda_h = (hF)^*$, since $hF_* = (hF)_*$.

LEMMA 3. $d(hF)^* = -dF^*$.

Proof. Consider the homotopy $G\colon D^{2n-1}_N \times [0, 1] \to R^{2n}$ defined by $G_t x = t(Farx) + (1-t)(hFx)$. For each $t \in [0, 1]$ we have $G_t|S^{2n-2} = fa = hf$, so that G_t satisfies (1). The maps $(hF)^* = G^*_0$ and $F^*ar = (Far)^* = G^*_1$ are thus connected by the homotopy G^*_t, so that $d(hF)^* = d(F^*ar) = -dF^*$.

LEMMA 4. $dF^* \neq 0$.

Proof. Since $F^*a = aF^*$, this follows from one of the Borsuk-Ulam theorems [1].

The assumption that h had no fixed points has led to an impasse in Lemmas 2, 3, and 4, and must therefore be false. In fact, we used only the assumption that h fixed no point of $G(D^{2n-1}_N \times [0, 1])$, whence the strengthened conclusion of the remark.

§ 4. Proof of corollary. Let h and Σ be as in the corollary, and suppose that h leaves no point of R^{2n} fixed. Since Σ is differentiable, we may identify a neighborhood $U \subset R^{2n}$ of Σ with the product $\Sigma \times B^2$ so that $x \times (0, 0) = x$ for $x \in \Sigma$. For $\varepsilon \in [0, 1]$, let

$$U_\varepsilon = \{(x, t) \in U\colon x \in \Sigma,\ t \in B^2,\ \|t\| \leqslant \varepsilon\}.$$

If $\mu\colon B^2 \to B^2$ is a homeomorphism which is the identity on S^1, we define a homeomorphism $\mu^*\colon R^{2n}, \Sigma \to R^{2n}, \Sigma$ by $\mu^* y = y$ for $y \in R^{2n} - U$ and $\mu^*(x, t) = (x, \mu t)$ for $x \in \Sigma$, $t \in B^2$. We may assume that $(x \times B^2) \cap h(x \times B^2) = \emptyset$ for each $x \in \Sigma$, so that $h\mu^*$ has no fixed points.

Let $a\colon \Sigma \to \Sigma$ be a fixed-point-free involution. By the hypotheses of the corollary the composition $h^{-1}a\colon \Sigma \to \Sigma$ is stable, and hence isotopic to the identity on Σ (see [2], [3]). Let $\lambda\colon \Sigma \times [0, 1] \to \Sigma$ be an isotopy such that $\lambda(x, 0) = h^{-1}ax$ and $\lambda(x, t) = x$ for $t \in [\frac{1}{2}, 1]$, and choose $\mu\colon B^2 \to B^2$ as above so that $h\mu^* U_\varepsilon \subset \operatorname{int} U_\varepsilon$ for each $\varepsilon \in [0, \frac{1}{2}]$. We define a homeomorphism $h'\colon R^{2n}, \Sigma \to R^{2n}, \Sigma$ by $h'y = hy$ for $y \in R^{2n} - U$ and $h'(x, t) = h\mu^*\big(\lambda(x, t), t\big)$ for $x \in \Sigma$, $t \in B^2$. It is clear that h' is isotopic to h and hence stable, and that $h'|_\Sigma = a$. For $\varepsilon \in [0, \frac{1}{2}]$ we have $h'U_\varepsilon = h\mu^* U_\varepsilon \subset \operatorname{int} U_\varepsilon$, so that h' has no fixed points in $U_{1/2}$, and h' has no fixed points outside $U_{1/2}$ because $h\mu^*$ hasn't any. The homeomorphism h' is therefore a counter-example to our theorem.

References

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Ultrafilters over measurable cardinals

by

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0. Definitions. The notation and terminology in this paper is that of the most recent set-theoretic literature. For less well-known items we urge the reader to consult A. Mathias (1969). We shall now define our fundamental notions. Unless otherwise mentioned, all the ultrafilters discussed are assumed to be nonprincipal, $\varkappa$-complete over a fixed measurable cardinal $\varkappa$.

0.1. DEFINITION. Given two ultrafilters D, U, we say $D \leqslant U$ if there is a function $f\colon \varkappa \to \varkappa$ so that

$$x \in D \longleftrightarrow f^{-1}(x) \in U .$$

In this case we also denote

$$D = f^*(U) .$$

If $D \leqslant U$ and $U \leqslant D$ we say: *D is isomorphic to U*: In symbols, $D \cong U$.

For more on this order, see K. Kunen [2] and J. Ketonen [1]. The above definition is due to H. J. Keisler.

0.2. DEFINITION. Given an ultrafilter D, functions $f, g\colon \varkappa \to \varkappa$ we say: *f, g are isomorphic* (mod D), in symbols $f \sim g$, if there is a one-to-one function φ so that

$$f = \varphi \circ g \pmod{D} .$$

In this case $f_*(D) \cong g_*(D)$. Another way of describing the above situation is to describe f and g in terms of the partitions $\{f^{-1}(\{a\})|\ a \in \varkappa\}$, $\{g^{-1}(\{a\})|\ a \in \varkappa\}$ they induce. Then $f \sim g$ if and only if there is a set $X \in D$ and a permutation of the labels of the g-partitioning so that the ath part of the f-partitioning intersected with $X =$ ath part of the permuted g-partitioning intersected with X for every $a < \varkappa$.

The following notions are extensions of the concepts of W. Rudin [1956]:

0.3. DEFINITION. If D an ultrafilter. $f\colon \varkappa \to \varkappa$, then D is an *f-P-point*