Thus, moreover, it commutes up to homotopy, i.e. $\boldsymbol{i}$ is associated with the inclusion i. ■

By 3.1, we get

3.2. *If* $\boldsymbol{r}$ *is a retraction (a deformational retraction), then* $\boldsymbol{r}$ *is a shape retraction (a shape deformational retraction).* ■

Now, let us establish the main result:

3.3. THEOREM. *For two compact metric spaces* X, Y *to be of the same shape it is necessary and sufficient that both* X *and* Y *be imbeddable in some compactum* Z *as its fundamental deformation retracts.*

Proof. The sufficiency is obvious. Let us prove the necessity. Take two compacta X, Y and let $\boldsymbol{X}$, $\boldsymbol{Y}$ be inclusion-ANR-sequences associated with X, Y respectively. If $\mathrm{Sh}\,X = \mathrm{Sh}\,Y$, then, by the theorem due to Mardešić and Segal ([5]), $\boldsymbol{X} \simeq \boldsymbol{Y}$. Thus, by 2.2, $\boldsymbol{X}$, $\boldsymbol{Y}$ are (up to isomorphism) deformation retracts of some ANR-sequence $\boldsymbol{Z}$. Hence, by 3.2, $\boldsymbol{X}$, $\boldsymbol{Y}$ are (up to isomorphism) shape deformation retracts of $\boldsymbol{Z}$. Let $Z \underset{\mathrm{Df}}{=} \varprojlim \boldsymbol{Z}$. By the Mardešić Theorem 11 [3], both X and Y can be imbedded in the compactum Z as its fundamental deformation retracts. ■

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Čech homology for movable compacta

by

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Introduction. This paper is devoted to an investigation of properties relating to movability of compacta. Movability is a shape invariant introduced, for metric compacta, by K. Borsuk [3] in 1969. Subsequently, S. Mardešić and J. Segal gave alternative definitions of shape and of movability, using inverse sequences of ANR's. These are equivalent in the metric case to Borsuk's definitions, and extend the concept to non-metric compacta ([12], [13], [14]).

The purpose of this paper is to use the ANR-sequence approach to answer a question in [3], as to whether the Čech homology sequence for a pair of movable metric compacta is necessarily exact. We formulate in a natural way a definition of *movable pairs* of metric compacta, and, using ANR-sequences, we show that such pairs do have exact homology (Theorem 1), but that pairs of movable compacta may not (Theorem 2). The construction of a counter-example yields, as well, a new example of a non-movable compactum. We further describe a method of obtaining a certain useful class of movable compacta.

The notation used here is that of [7], [12], and [13]. All topological spaces considered are Hausdorff compacta.

1. Movability of compacta and of pairs. Suppose $\boldsymbol{X} = \{X_\alpha, p_{\alpha\beta}\}_{\alpha\in I}$ is an inverse system of ANR's, where I is closure-finite; that is, each $\alpha \in I$ has but a finite number of predecessors. If $X = \varprojlim X_\alpha$, then X is said to be *associated* with the compactum X. If $\boldsymbol{Y} = \{Y_\gamma, q_{\gamma\delta}\}_{\gamma\in I'}$ is an ANR-system also, we define a *map of* ANR-*systems* $\boldsymbol{f}$: $\boldsymbol{X} \to \boldsymbol{Y}$ to be a pair consisting of an order-preserving function f: $I' \to I$ and a collection of maps f_γ: $X_{f(\gamma)} \to Y_\gamma$ such that for any $\delta > \gamma$ in I', $f_\gamma \circ p_{f(\gamma),f(\delta)} \simeq q_{\gamma\delta} \circ f_\delta$. The identity $\mathrm{id}_{\boldsymbol{X}}$: $\boldsymbol{X} \to \boldsymbol{X}$ is given by $\mathrm{id}(\alpha) = \alpha$, for all $\alpha \in I$, and $\mathrm{id}_\alpha = \mathrm{id}_{X_\alpha}$.

Every compactum X has an associated ANR-system of cardinality no greater than the weight of the topology on X ([13], § 5, Theorem 7).

(*) These results form a portion of the author's Ph. D. thesis, written under the supervision of Professor Jack Segal at the University of Washington.

In particular, every metric compactum has an associated ANR-sequence $X = (X_n, p_n)$, where p_n is a map from X_n into X_{n-1} for each $n = 2, 3, 4, \ldots$

DEFINITION 1.1. An ANR-system $\{X_\alpha, p_{\alpha\beta}\}_{\alpha\in I}$ is *movable* provided that for every $\alpha \in I$ there exists $\alpha' \geqslant \alpha$ in I such that for any $\beta \geqslant \alpha'$ there is a map $s_{\beta\alpha'}\colon X_{\alpha'} \to X_\beta$ satisfying

$$p_{\alpha\beta} \circ s_{\beta\alpha'} \simeq p_{\alpha\alpha'}.$$

DEFINITION 1.2. A compactum X is *movable* if there exists a movable ANR-system X associated with X.

It is shown in [14] that movability is a shape invariant, and that if a compactum X is movable, then so is every ANR-system associated with X.

DEFINITION 1.3. A pair (X, A) is *movable* if there is an associated ANR-pair system $\{(X_\alpha, A_\alpha), p_{\alpha\beta}\}_{\alpha\in I}$ which is movable. That is, for all $\alpha \in I$ there exists $\alpha' \geqslant \alpha$ such that for each $\beta \geqslant \alpha$ there is a pair map $s_{\beta\alpha'}\colon (X_{\alpha'}, A_{\alpha'}) \to (X_\beta, A_\beta)$ satisfying the condition $p_{\alpha\beta} \circ s_{\beta\alpha'} \simeq p_{\alpha\alpha'}$.

Note that if (X, A) is movable, so are X and A. That the converse is not true is shown by Theorems 1 and 2 below. The shape of pairs is defined analogously to shape of compacta [13]; movability of pairs is then a shape invariant.

LEMMA 1.4. *A metric compactum X (or a pair of metric compacta (X, A)) is movable if and only if it has an associated ANR-sequence (X_n, p_n) (resp. $((X_n, A_n), p_n)$), together with a collection of maps $s_{n+1}\colon X_{n+1} \to X_{n+2}$ (resp. $s_{n+1}\colon (X_{n+1}, A_{n+1}) \to (X_n, A_n)$), $n = 1, 2, 3, \ldots$, such that $p_{n,n+2} \circ s_{n+1} \simeq p_{n+1}$ for all n.* ■

We conclude this section with a statement of the main theorems of this paper.

THEOREM 1. *The Čech homology sequence for a movable pair of metric compacta is exact.*

THEOREM 2. *The Čech homology sequence for a pair of movable metric compacta is not necessarily exact.*

The proof of Theorem 1 requires some algebraic preliminaries, which occupy § 2. The proof itself, along with the construction of an example to prove Theorem 2, are in § 3.

2. Exactness of inverse limits. Let R be a ring; and for each $n = 1, 2, \ldots$ let S_n be an exact sequence of R-modules,

$$S_n\colon \ \ldots \to G_n^r \to G_n^{r+1} \to G_n^{r+2} \to \ldots$$

Suppose that for each $n = 2, 3, 4, \ldots$ there is a morphism π_n in the category of exact sequences, $\pi_n\colon S_n \to S_{n-1}$, that is $\pi_n = \{p_n^r\colon G_n^r \to G_{n-1}^r\}$, and the usual commutivity holds. Then (S_n, π_n) forms an inverse sequence of long exact sequences.

LEMMA 2.1. *Let (S_n, π_n) be an inverse sequence of long exact sequences $S_n\colon \ldots \to G_n^r \to G_n^{r+1} \to \ldots$ of R-modules, where $\pi_n = \{p_n^r\colon G_n^r \to G_{n-1}^r\}$ for $n = 2, 3, \ldots$ Suppose that for each r, (G_n^r, p_n^r) satisfies the following condition:*

(1) *For every n, there exists $j \geqslant n$ such that for all $i \geqslant j$ $p_{n,i}(G_i) = p_{n,j}(G_j)$.*

Then the sequence

$$\ldots \to \varprojlim G_n^r \to \varprojlim G_n^{r+1} \to \varprojlim G_n^{r+2} \to \ldots$$

is also exact.

To prove this lemma, we will use the concept of the "derived functor" $\varprojlim^1$. This subject has been developed in some generality by Roos [15], Grothendieck [8], and others. An outline of the existence and basic properties of $\varprojlim^1$ for arbitrary inverse systems in suitable abelian categories (i.e. with enough projectives and products) can be found in [5], p. 82 et. seq., and in [11], p. 389.

However, we are only concerned with inverse sequences in the category of R-modules. In this setting, we can define $\varprojlim^1$ explicitly, and in fact the device is principally of notational advantage here. This approach is easier than applying Roos's equivalent, but more general and unwieldy characterization of $\varprojlim^1$.

The methods below were suggested to the author by Professor Albrecht Dold of the Universität Heidelberg.

Construction and basic properties of $\varprojlim^1$ for sequences. Let (G_i, p_i) be an inverse sequence of R-modules. Denote by $\sigma\colon \prod_{i=1}^\infty G_i \to \prod_{i=1}^\infty G_i$ the "shift homomorphism" $\sigma((x_i)) = (p_{i+1}(x_{i+1}))$. We can construct a cochain complex $\mathcal{G}$ from (G_i, p_i) as follows: Let $\mathcal{G}^0 = \prod_{i=1}^\infty G_i$, $\mathcal{G}^1 = \prod_{i=1}^\infty G_i$, and $\mathcal{G}^p = 0$ for $p \geqslant 2$. Let $\delta^0 = (\sigma - \mathrm{id})\colon \mathcal{G}^0 \to \mathcal{G}^1$, and let $\delta^p = 0\colon \mathcal{G}^p \to \mathcal{G}^{p+1}$ for all $p \geqslant 1$. Then clearly $\delta^{p+1} \circ \delta^p = 0$ for all p, so $\mathcal{G} = \{\mathcal{G}^p, \delta^p\}$ is a cochain complex. Consider the cohomology groups of $\mathcal{G}$. Since $H^p(\mathcal{G}) = \ker\delta^p/\mathrm{im}\,\delta^{p-1}$ we have

$$H^0(\mathcal{G}) = \ker(\sigma - \mathrm{id})$$

$$(2) \qquad = \Big\{(x_i) \in \prod G_i \mid p_{i+1}(x_{i+1}) - x_i = 0\Big\} = \varprojlim (G_i, p_i), \quad \text{and}$$

$$H^p(\mathcal{G}) = 0 \quad \text{for all } p \geqslant 2.$$

DEFINITION 2.2. $H^1(\mathcal{G}) = \prod G_i/\mathrm{im}(\sigma - \mathrm{id})$ is called the *first derived functor* of (G_i, p_i), and is written $\varprojlim{}^1 G_i$.

The use of $\varprojlim{}^1 G_i$ to prove exactness of limit sequences is shown by the following lemma:

LEMMA 2.3. *For $i = 1, 2, 3, \ldots$, let S_i be a short exact sequence S_i: $0 \to G_i \to H_i \to I_i \to 0$ of R-modules. If (S_i, π_i) is an inverse sequence, then*

$$(3)\qquad 0 \to \varprojlim G_i \to \varprojlim H_i \to \varprojlim I_i \to \varprojlim{}^1 G_i \to \varprojlim{}^1 H_i \to \varprojlim{}^1 I_i \to 0$$

is also exact.

Proof. Since S_i is exact for all i, so is

$$0 \to \prod_{i=1}^{\infty} G_i \to \prod_{i=1}^{\infty} H_i \to \prod_{i=1}^{\infty} I_i \to 0,$$

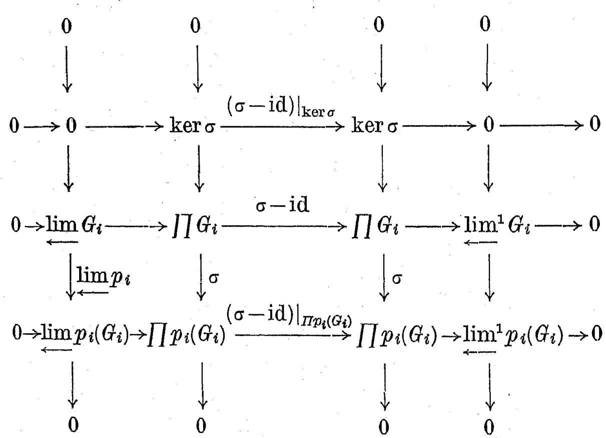
so if we construct cochain complexes $\mathcal{G}$, $\mathcal{H}$, and $\mathcal{I}$ from the inverse sequences (G_i), (H_i), and (I_i) respectively, as above, we see that

$$(4)\qquad 0 \to \mathcal{G} \to \mathcal{H} \to \mathcal{I} \to 0 \text{ is exact}.$$

By (2), the sequence (3) is just the long cohomology sequence of (4) and so (3) is exact (see e.g. [11], p. 45). ■

LEMMA 2.4. *If (G_i, p_i) is an inverse sequence of R-modules, then so is $(p_{i+1}(G_{i+1}), p_i|_{\mathrm{im}\, p_{i+1}})$, and $\varprojlim{}^1 G_i \cong \varprojlim{}^1 p_{i+1}(G_{i+1})$.*

Proof.



Consider the diagram above. All non-labeled maps are the usual injections and projections.

All squares are easily seen to commute. $(\sigma - \mathrm{id})|_{\ker\sigma} = -\mathrm{id}|_{\ker\sigma}$, so the top row is exact. The other two rows are exact by the definition of $\varprojlim G_i = \ker(\sigma - \mathrm{id})$, and $\varprojlim{}^1 G_i = \prod G_i/\ker(\sigma - \mathrm{id})$. The left column is exact because $\varprojlim p_i$ is an isomorphism, while the second and third columns are clearly exact. Thus the diagram is a 3×4 commutative diagram with all rows and all but the last column exact, and so the last column is exact as well. ■

The next lemma is from [15] and follows from the universality of the functors $\varprojlim$ and $\varprojlim{}^1$. It can also be proved, rather tediously, directly from the definitions given above.

LEMMA 2.5. *Let ν: $N \to N$ be a monotone increasing function of natural numbers, so that $(G_{\nu(i)})$ is a cofinal subsequence of (G_i). Then*

$$\varprojlim G_{\nu(i)} \cong \varprojlim G_i, \quad \text{and} \quad \varprojlim{}^1 G_{\nu(i)} \cong \varprojlim{}^1 G_i. \ ■$$

The next properly of $\varprojlim{}^1$ is derived directly from the definition.

LEMMA 2.6. *If (G_i, p_i) is an inverse sequence of R-modules with p_{i+1}: $G_{i+1} \to G_i$ epic for all i, then $\varprojlim{}^1 G_i = 0$.* ■

For ease of handling, let us adopt the following notational convention: If $(G_{\nu(i)})$ is a cofinal subsequence of (G_i, p_i), we can write the connecting homomorphisms of $(G_{\nu(i)})$ as q_{i+1}: $G_{\nu(i+1)} \to G_{\nu(i)}$; that is,

$$q_{i+1} = p_{\nu(i),\nu(i+1)} = p_{\nu(i)+1} \circ \ldots \circ p_{\nu(i+1)} \quad \text{for all } i = 1, 2, 3, \ldots$$

The following lemma is needed to connect condition (1) with Lemmas 2.4–2.6. Although the lemma is stated as an equivalence, only the implication we are going to use is proved here. The proof of the converse implication is not difficult.

LEMMA 2.7. *For any inverse sequence (G_i, p_i) of R-modules, condition (1), that for every n there exists $j(n) \geqslant n$ with $p_{n,i}(G_i) = p_{n,j(n)}(G_{j(n)})$ for all $i \geqslant j(n)$, is equivalent to the following so called Mittag–Lefler condition:*

There exists a cofinal subsequence $(G_{\nu(i)}, q_i)$ of (G_i, p_i) such that the inverse sequence $(q_i(G_{\nu(i+1)}), q_i|_{\mathrm{im}\, q_{i+1}})$ has epic connecting homomorphisms.

Proof. Suppose (1) is true, and for each n pick a particular $j(n)$ fulfilling the requirements above. Let $\nu(0) = 1$, $\nu(i+1) = j(\nu(i))$ for all $i = 0, 1, 2, \ldots$ Then $(G_{\nu(i)})_{i=1}^{\infty}$ is a cofinal subsequence of (G_i), and for all $i \geqslant 1$, $\nu(i+2) \geqslant j(\nu(i))$, so

$$q_{i+1} \circ q_{i+2}(G_{\nu(i+2)}) = p_{\nu(i),\nu(i+2)}(G_{\nu(i+2)}) = p_{\nu(i),\nu(i+1)}(G_{\nu(i+1)}) = q_{i+1}(G_{\nu(i+1)}). \ ■$$

LEMMA 2.8. *Suppose (S_i, π_i) is an inverse sequence of short exact sequences S_i: $0 \to G_i \to H_i \to I_i \to 0$ of R-modules, where $\pi_i = \{p_i^G, p_i^H, p_i^I\}$ is* *a morphism of short exact sequences for each $i = 2, 3, 4, \ldots$ If the inverse sequence (G_i, p_i^G) satisfies* (1), *then* $0 \to \varprojlim G_i \to \varprojlim H_i \to \varprojlim I_i \to 0$ *is exact.*

Proof. By Lemma 2.7, if (G_i, p_i^G) satisfies (1), then there is a cofinal subsequence $(G_{\nu(i)}, q_i)$ such that $(q_{i+1}(G_{\nu(i+1)}), q_i|_{\operatorname{im} q_{i+1}})$ has epic connecting homomorphisms. Hence by Lemma 2.6, $\varprojlim^1 q_{i+1}(G_{\nu(i+1)}) = 0$. By Lemma 2.4, $\varprojlim^1 q_{i+1}(G_{\nu(i+1)}) \cong \varprojlim^1 G_{\nu(i)}$, and by Lemma 2.5, $\varprojlim^1 G_{\nu(i)} \cong \varprojlim^1 G_i$. Thus $\varprojlim^1 G_i = 0$ and so $0 \to \varprojlim G_i \to \varprojlim H_i \to \varprojlim I_i \to 0$ is exact by Lemma 2.3. ■

Proof of Lemma 2.1. We need only to extend the last lemma to long exact sequences. Suppose (S_i, π_i) is an inverse sequence of long exact sequences

$$S_i\colon \ \ldots \to G_i^{r-1} \to G_i^r \to G_i^{r+1} \to \ldots,$$

where $\pi_i = \{\ldots p_i^{r-1}, p_i^r, p_i^{r+1}, \ldots\}$ in the commutative diagram

$$\begin{array}{ccccccc}
\ldots \to G_i^{r-1} & \xrightarrow{\alpha_i} & G_i^r & \xrightarrow{\beta_i} & G_i^{r+1} \to \ldots \\
\downarrow p_i^{r-1} & & \downarrow p_i^r & & \downarrow p_i^{r+1} \\
\ldots \to G_{i-1}^{r-1} & \xrightarrow{\alpha_{i-1}} & G_{i-1}^r & \xrightarrow{\beta_{i-1}} & G_{i-1}^{r+1} \to \ldots
\end{array}$$

and suppose further that for each r, (G_i^r, p_i^r) satisfies condition (1). Consider the associated diagram of short exact sequences:

$$\begin{array}{ccccccc}
0 \to \ker \beta_i & \longrightarrow & G_i^r & \longrightarrow & \operatorname{im} \beta_i & \to 0 \\
\downarrow p_i^r|_{\ker \beta_i} & & \downarrow p_i^r & & \downarrow p_i^{r+1} & \\
0 \to \ker \beta_{i-1} & \longrightarrow & G_{i-1}^r & \longrightarrow & \operatorname{im} \beta_{i-1} & \to 0
\end{array} \tag{5}$$

Picking j sufficiently large so that for $i \geqslant j$, $p_{n,i}^{r-1}(G_i^{r-1}) = p_{n,j}^{r-1}(G_j^{r-1})$, we see that the inverse sequence $(\ker \beta_i, p_i^r|_{\ker \beta_i})$ satisfies (1). Thus by Lemma 2.8, the following sequences are exact:

$$0 \to \varprojlim(\ker \beta_i) \to \varprojlim G_i^r \xrightarrow{\varprojlim \beta_i} \varprojlim(\operatorname{im} \beta_i) \to 0, \tag{6}$$

$$0 \to \varprojlim(\ker \alpha_i) \to \varprojlim G_i^{r-1} \xrightarrow{\varprojlim \alpha_i} \varprojlim(\operatorname{im} \alpha_i) \to 0. \tag{7}$$

From (6), $\ker(\varprojlim \beta_i) = \varprojlim(\ker \beta_i)$, and, because each S_i is exact at G_i^r, we know that $\varprojlim(\ker \beta_i) = \varprojlim(\operatorname{im} \alpha_i)$. Finally, from (7) we have $\varprojlim(\operatorname{im} \alpha_i) = \operatorname{im}(\varprojlim \alpha_i)$, so we see that $\ker(\varprojlim \beta_i) = \operatorname{im}(\varprojlim \alpha_i)$. Therefore the limit sequence $\ldots \to \varprojlim G_i^r \to \varprojlim G_i^{r+1} \to \ldots$ is exact. ■

3. Proofs of Theorems 1 and 2.

Proof of Theorem 1. Let $(X, A) = \varprojlim(X_i, A_i)$, where (X_i, A_i) is a pair of ANR's for every $i = 1, 2, \ldots$ If the sequence $((X_i, A_i))$ is movable, then for each $q = 0, 1, 2, \ldots$ and each $n = 1, 2, \ldots$ there exists $j \geqslant n$ such that for all $i \geqslant j$,

$$\begin{aligned}
p_{n,i*}(H_q(A_i)) &= p_{n,j*}(H_q(A_j)), \\
p_{n,i*}(H_q(X_i)) &= p_{n,j*}(H_q(X_j)), \quad \text{and} \\
p_{n,i*}(H_q(X_i, A_i)) &= p_{n,j*}(H_q(X_j, A_j)).
\end{aligned}$$

Thus the inverse sequence of homology sequences

$$\begin{array}{ccccccccc}
\ldots H_q(A_i) & \to & H_q(X_i) & \to & H_q(X_i, A_i) & \to & H_{q-1}(A_i) & \to \ldots \\
\downarrow p_{n,i*} & & \downarrow p_{n,i*} & & \downarrow p_{n,i*} & & \downarrow p_{n,i*} & \\
\ldots H_q(A_n) & \to & H_q(X_n) & \to & H_q(X_n, A_n) & \to & H_{q-1}(A_n) & \to \ldots
\end{array}$$

satisfies (1) of section 2, and so by Lemma 2.1,

$$\ldots \to \varprojlim H_q(A_i) \to \varprojlim H_q(X_i) \to \varprojlim H_q(X_i, A_i) \to \ldots$$

is exact. Finally, continuity of Čech theory gives the desired result. ■

The rest of this section is devoted to constructing a pair of movable metric compacta, the homology sequence of which fails to be exact. In addition to answering Borsuk's question, this construction yields a new non-movable continuum, and shows how to construct movable compacta (or continua) from non-movable ones, using inverse limits.

Throughout this discussion, D will designate the unit disc in the complex plane, and S^1 will denote the unit circle. We first get a non-movable compactum X as the inverse limit of an inverse sequence (X_n) of ANR's:

For each $n = 1, 2, 3, \ldots$, attach D to S^1 by a map $f_n\colon \operatorname{Bd} D \to S^1$ defined by $f(e^{i\theta}) = e^{2^n i\theta}$, and take $X_n = D \cup_{f_n} S^1$, $A_n = S^1 \subset X_n$. X_n is thus a generalized real projective plane.

Now consider the map $g\colon D \to D$ defined by $g(re^{i\theta}) = re^{2i\theta}$. Let $h\colon S^1 \to S^1$ be the identity map. Then for $n = 2, 3, 4, \ldots$, $h \circ f_n = f_{n-1} \circ g|_{\operatorname{Bd} D}$, so a map

$$p_n\colon X_n = D \cup_{f_n} S^1 \to D \cup_{f_{n-1}} S^1 = X_{n-1}$$

is induced by g and h (see e.g. [6], p. 129). The restriction of p_n to A_n is simply h, so we can think of p_n as a map of pairs, $p_n\colon (X_n, A_n) \to (X_{n-1}, A_{n-1})$ for each $n = 2, 3, \ldots$ Let $X = \varprojlim X_n$.

Each X_n is constructed as a CW complex, and using this construction it is easily seen that

(1) $H_1(X_n) \cong Z/\langle 2^n \rangle$, the cyclic group of order 2^n, and $p_{n*}\colon H_1(X_n) \to H_1(X_{n-1})$ is the natural projection of $Z/\langle 2^n \rangle$ onto $Z/\langle 2^{n-1} \rangle$, for all $n \geqslant 2$.

If X were movable, we would get, for each n, an index $j \geqslant n$ and $s_j\colon X_j \to X_{j+1}$ such that $(p_{n,j+1*}) \circ (s_{j*}) = (p_{n,j*})$. This violates the structures of the groups involved, so X is not a movable compactum.

However, we can construct a movable compactum $\hat{X}_n$ from X as follows. For each n, let $\hat{X}_n$ be the disjoint union $X_1 + X_2 + \ldots + X_n$, and let $\hat{p}_{n+1}\colon \hat{X}_{n+1} \to \hat{X}_n$ be defined by

$$\hat{p}_{n+1}(x) = \begin{cases} x, & \text{if } x \in X_1 + \ldots + X_n \subset \hat{X}_{n+1}, \\ p_{n+1}(x), & \text{if } x \in X_{n+1} \subset \hat{X}_{n+1}. \end{cases}$$

Now for each n we let $s_n\colon \hat{X}_n \to \hat{X}_{n+1}$ be the inclusion map: $X_1 + \ldots + X_n \to X_1 + \ldots + X_n + X_{n+1}$. Then $p_{n+1} \circ s_n = \mathrm{id}_{\hat{X}_n}$, so $(\hat{X}_n, \hat{p}_n)$ is a movable sequence. Hence $\hat{X} = \varprojlim \hat{X}_n$ is a movable compactum by $A = \varprojlim A_n$ is just S^1, which is movable, so $(\hat{X}, A)$ is a pair of movable compacta. It fails to be a movable pair, however: Since $A_n \subset X_n \subset \hat{X}_n$ for each n, any map from the pair $(\hat{X}_n, A_n)$ to $(\hat{X}_{n+1}, A_{n+1})$ would map the component X_n of $\hat{X}_n$ into X_{n+1} in $\hat{X}_{n+1}$, so X would be movable.

Geometrically, we can think of X as being embedded in $\hat{X}$ in such a way that X is the set limit of copies of the X_n's, also embedded in $\hat{X}$. That is, every neighborhood of X in $\hat{X}$ contains all but perhaps finitely many of the X_n's.

Note. A movable *continuum* X' can be constructed in similar fashion, by selecting a point $(x_n) \in \varprojlim X_n$, taking X_n' to be the one-point union of $X_1, \ldots, X_n$, where each X_i is attached at x_i, $i = 1, 2, \ldots, n$, and then defining p_{n+1}', s_n' as $\hat{p}_{n+1}$, s_n were defined. (X', A) is then a pair of movable continua.

Proof of Theorem 2. We will show that the long homology sequence is not exact for the pair $(\hat{X}, A)$. Because Čech theory is continuous, we need only to show that

(2) $$\ldots \varprojlim H_2(\hat{X}_n, A_n) \to \varprojlim H_1(A_n) \to \varprojlim H_1(\hat{X}_n) \to \varprojlim H_1(\hat{X}_n, A_n) \ldots$$

fails to be exact. We will show (2) is not exact at $\varprojlim H_1(\hat{X}_n)$.

Consider the diagram

(3) $$\begin{array}{ccccccc} \ldots H_2(\hat{X}_n, A_n) & \xrightarrow{\delta_n} & H_1(A_n) & \xrightarrow{i_*} & H_1(\hat{X}_n) & \xrightarrow{j_*} & H_1(\hat{X}_n, A_n) \ldots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \ldots H_2(\hat{X}_{n-1}, A_{n-1}) & \longrightarrow & H_1(A_{n-1}) & \longrightarrow & H_1(\hat{X}_{n-1}) & \longrightarrow & H_1(\hat{X}_{n-1}, A_{n-1}) \ldots \end{array}$$

Now because $\hat{X}_n$ is the disjoint union of $X_1, \ldots, X_n$, $H_1(\hat{X}_n) \cong H_1(X_n) \oplus \oplus H_1(\hat{X}_{n-1})$ and i_* is, in fact, into the first summand $H_1(X_n)$. Similarly, since $\hat{X}_n = X_n + \hat{X}_{n-1}$ and $A_n \subset X_n \subset \hat{X}_n$, we have that $H_*(\hat{X}_n, A_n) \cong H_*(X_n, A_n) \oplus H_*(\hat{X}_{n-1})$. Furthermore, in (3), $j_*\colon H_1(\hat{X}_n) \to H_1(\hat{X}_n, A_n)$ takes the summand $H_1(X_n)$ into $H_1(X_n, A_n)$ and $H_1(\hat{X}_{n-1})$ isomorphicaly onto $H_1(\hat{X}_{n-1}) \subset H_1(\hat{X}_n, A_n)$. Also $H_2(\hat{X}_{n-1}) \subset \ker \delta_n$.

These direct sums are preserved by the connecting homomorphisms induced by the connecting maps $\hat{p}_n\colon (\hat{X}_n, A_n) \to (\hat{X}_{n-1}, A_n)$ for all $n = 2, 3, \ldots$, so in the limit sequence (2) we get that

$$\varprojlim H_1(\hat{X}_n) \cong \varprojlim H_1(X_n) \oplus \varprojlim H_1(\hat{X}_{n-1}),$$

$$\varprojlim H_*(\hat{X}_n, A_n) \cong \varprojlim H_*(X_n, A_n) \oplus \varprojlim H_*(X_{n-1}),$$

and the limit maps preserve these sums. That is $\varprojlim H_2(\hat{X}_{n-1}) \subset \ker(\varprojlim \delta)$, $\varprojlim i_*$ takes $\varprojlim H_1(A_n)$ into the summand $\varprojlim H_1(X_n)$ of $\varprojlim H_1(\hat{X}_n)$, $\varprojlim j_*$ takes $\varprojlim H_1(X_n)$ into $\varprojlim H_1(X_n, A_n)$, and takes $\varprojlim H_1(\hat{X}_{n-1})$ isomorphicaly to $\varprojlim H_1(\hat{X}_{n-1})$ in $\varprojlim H_1(\hat{X}_n, A_n)$. Thus the exactness of (2) at $\varprojlim H_1(\hat{X}_n)$ is equivalent to exactness of

(4) $$\ldots \varprojlim H_2(X_n, A_n) \xrightarrow{\varprojlim \delta_n} \varprojlim H_1(A_n) \xrightarrow{\varprojlim i_*} \varprojlim H_1(X_n) \xrightarrow{\varprojlim j_*} \varprojlim H_1(X_n, A_n) \ldots$$

at $\varprojlim H_1(X_n)$. To show (4) is not exact, we compute the groups involved:

Consider the associated diagram

$$\begin{array}{ccccccc} H_2(X_n, A_n) & \xrightarrow{\delta_n} & H_1(A_n) & \xrightarrow{i_*} & H_1(X_n) & \longrightarrow & H_1(X_n, A_n) \\ \downarrow{\scriptstyle \alpha_n} & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \beta_n} \\ H_2(X_{n-1}, A_{n-1}) & \longrightarrow & H_1(A_{n-1}) & \longrightarrow & H_1(X_{n-1}) & \longrightarrow & H_1(X_{n-1}, A_{n-1}), \end{array}$$

where α_n and β_n are the homomorphisms induced by $p_n\colon (X_n, A_n) \to (X_{n-1}, A_{n-1})$. Because A_n is a deformation retract of a neighborhood in X_n,

$$H_*(X_n, A_n) \cong \tilde{H}_*(X_n/A_n) \cong \tilde{H}_*(D/S^1) \cong \tilde{H}_*(S^2),$$

where $\tilde{H}_*$ is the reduced homology. Thus $H_2(X_n, D_n) \cong \mathbf{Z}$. Also, $p_n: (X_n, A_n) \to (X_{n-1}, A_{n-1})$ induces a map $\alpha_n': S^2 \approx X_n/A_n \to X_{n-1}/A_{n-1} \approx S^2$ of degree 2, so $\alpha: \mathbf{Z} \to \mathbf{Z}$ is multiplication by 2. Hence $\varprojlim H_2(X_n, A_n) = 0$.

For each $n \geqslant 2$, $A_n \cong S^1$, and $p_n: A_n \to A_{n-1}$ is of degree 1, so the induced map β is an isomorphism. Thus $\varprojlim H_1(A_n) \cong \mathbf{Z}$.

Recall (see (1)) that $\varprojlim H_1(X_n)$ is the inverse limit of a sequence of groups $\mathbf{Z}/\langle 2^n \rangle$ with the natural projections. This limit is well known — it is the group of dyadic integers, $\Delta(2)$. (See e.g. [10].) Thus (4) becomes

$$0 \to \mathbf{Z} \to \Delta(2) \to 0,$$

which is not exact at $\Delta(2)$. Therefore (2) is not exact. ■

4. Open questions. The following question remains unanswered:

(4.1) Is the Čech homology sequence for a movable pair of non-metric compacta exact?

In attempting to extend the method of proof of Theorem 1 to this question, we run into problems with Lemma 2.6. This lemma guarantees exactness in the limit of short exact sequences, if the indexing set for the inverse system forms a sequence and if the connecting homomorphisms are epic. The inverse system of exact sequences comes from the ANR-system associated with the movable compact pair. The problem is that non-metric compacta are characterized by the fact that their associated ANR-systems can admit no cofinal subsequences.

In [9], L. Henkin gives an example of an inverse system $\{E_\alpha, p_{\alpha\beta}\}_{\alpha \in I}$ built on any indexing set I, such that all the $p_{\alpha\beta}$ are onto, and yet such that if I has no cofinal subsequence then $\varprojlim E_\alpha = \emptyset$. (This example also appears as a problem in [4], p. 134.) From this we can easily construct an inverse system $\{S_\alpha: 0 \to A_\alpha \to B_\alpha \to C_\alpha \to 0, \pi_{\alpha\beta}\}_{\alpha \in I}$ of short exact sequences of abelian groups, such that for all $\alpha, \beta \in I$ with $\alpha \geqslant \beta$, $\pi_{\alpha\beta}: S_\beta \to S_\alpha$ is an epimorphism, yet $0 \to \varprojlim A_\alpha \to \varprojlim \beta_\alpha \to \varprojlim C_\alpha \to 0$ is exact only if I has a cofinal subsequence. Thus Lemma 2.6 cannot be extended to handle indexing sets of ANR-systems associated with non-metrizable compacta. The following question also remains open:

(4.2) Let X be a movable compactum. Does X necessarily have an associated ANR-system $\mathbf{X}$, each connecting map of which admits a right homotopy inverse?

It should be noted that if the answer to (4.2) is affirmative, then the proof of Theorem 1 can be shortened considerably; it then depends upon Lemma 2.6 alone.

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